VI Workshop on Geometric Analysis of PDE and Several Complex Variables

Singular set of a Levi-flat hypersurface is Levi-flat

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Abstract:

We study the singular set of a singular Levi-flat real-analytic hypersurface. We prove that the singular set of such a hypersurface is Levi-flat in the appropriate sense. We also show that if the singular set is small enough, then the Levi-foliation extends to a singular codimension one holomorphic foliation of a neighborhood of the hypersurface.

1 Definitions

Let $M \subset \mathbb{C}^n$ be a (smooth) real submanifold. At $p \in M$ write

$$T_p^{(1,0)} M = T_p^{(1,0)} \mathbb{C}^n \cap (T_p M \otimes \mathbb{C}) .$$

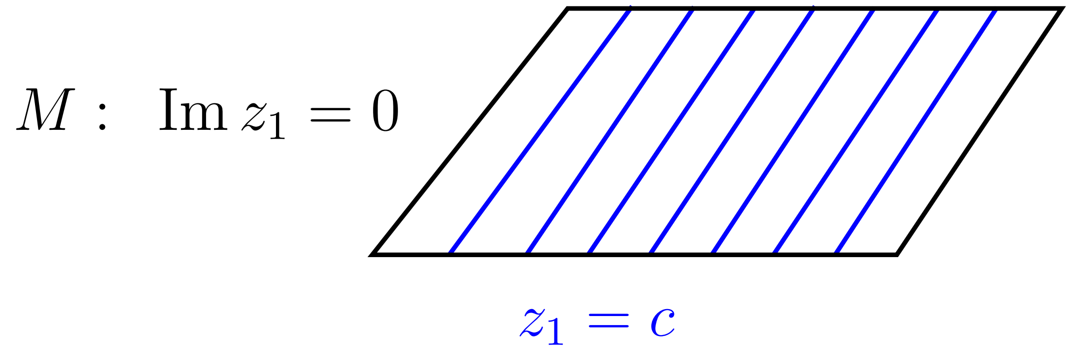
If the dimension of $T_p^{(1,0)} M$ is constant, then M is a *CR-manifold*. A (real-analytic) CR submanifold is *generic* if no nontrivial holomorphic function vanishes identically on it.

A real smooth hypersurface $M \subset \mathbb{C}^n$ defined by $r = 0$ (with $dr \neq 0$) is said to be *Levi-flat* if it is pseudoconvex from both sides:

$$\sum_{j=1,k=1}^{n} \frac{\partial^2 r}{\partial z_j \partial \bar{z}_k} w_j \bar{w}_k = 0$$

for all $w = (w_1, \ldots, w_n) \in T_p^{(1,0)} M$. That is, the *Levi-form* vanishes identically.

A real-analytic smooth hypersurface M is Levi-flat if and only if it can be written in local coordinates as $\operatorname{Im} z_1 = 0$. From this equation, we can see that M is locally foliated by complex submanifolds of complex dimension $n - 1$ defined by $z_1 = c$ for $c \in \mathbb{R}$.



We wish to extend this definition to higher codimensions. Let us say that a real-analytic CR manifold $M \subset \mathbb{C}^n$ is *Levi-flat* if in suitable local coordinates we can write its defining equations as

$$\operatorname{Im} z_1 = \cdots = \operatorname{Im} z_j = 0$$

and

$$z_{j+1} = \cdots = z_k = 0$$

for some j and k (where we interpret $j = 0$ and $k = 0$ appropriately).

Let $U \subset \mathbb{C}^n$ be an open set and let $H \subset \mathbb{C}^n$ be a real subvariety (a set locally defined by the vanishing of real-analytic functions) of dimension $2n - 1$.

Define H^* to be the set of points of H near which H is a real-analytic submanifold of dimension $2n-1$. We say H is *Levi-flat* if H^* is Levi-flat.

For any set X define X_s to be the set of points of X near which X is not a real-analytic submanifold.

2 Motivation

Singular Levi-flat hypersurfaces occur naturally, for example as invariant sets of a singular codimension one holomorphic foliation. A singular holomorphic codimension one foliation is locally given by a holomorphic one-form ω such that

$$\omega \wedge d\omega = 0.$$

The set where $\omega = 0$ is the singular set.

A real-analytic hypersurface (possibly singular) that is an invariant set of the foliation (is a union of leaves) is necessarily a Levi-flat hypersurface.

Another example: One cannot divide the projective space into pseudoconvex domains by a nonsingular hypersurface. The hypersurface must then be Levi-flat.

Theorem (Lins Neto [8]). *If $H \subset \mathbb{CP}^n$ $(n \geq 3)$ is a Levi-flat real subvariety of dimension $2n-1$ then H is singular (that is H_s is nonempty).*

On the other hand one can divide the projective space into pseudoconvex domains by a singular hypersurface.

One can prove that if H is a singular Levi-flat hypersurface, then $\overline{H^*}$ (but perhaps not H) divides its ambient space into pseudoconvex parts.

This lemma follows from the work of Fornæss (a weaker version of this lemma appeared in Burns-Gong [2])

Lemma 1 (L. [7]). *Let $H \subset U \subset \mathbb{C}^n$ be a singular Levi-flat hypersurface. If $p \in \overline{H^*} \cap U$, then there exists a neighborhood V of p and a complex subvariety $X \subset V$ of dimension $n-1$ such that $X \subset \overline{H^*}$ and $p \in X$.*

In light of this lemma, a singular real-analytic hypersurface H is Levi-flat if and only if $\overline{H^*}$ is pseudoconvex from both sides.

Singular Levi-flat hypersurfaces can arise in solutions of the Levi-flat Plateau problem see [3] and [5] (and the references within). For example, the Levi-flat hypersurface with boundary defined by $\operatorname{Im}(z_1^2+z_2^2+z_3^2) = 0$, $|z_1|^2 + |z_2|^2 + |z_3|^2 = 1$ has a singularity at the origin.

Finally, a singular Levi-flat hypersurface can arise when trying to "flatten" a nowhere minimal CR singular manifold. For example (see [4]) if M is an algebraic codimension 2 nowhere minimal submanifold then it is contained in a possibly singular Levi-flat hypersurface.

3 Result

Theorem 1 (L. [7]). *Let $U \subset \mathbb{C}^n$ be an open set and let $H \subset U$ be a Levi-flat real-subvariety of dimension $2n-1$. Then the singular set $(\overline{H^*} \cap U)_s$ is Levi-flat near points where it is a CR real-analytic submanifold.*

Furthermore, if $(\overline{H^*} \cap U)_s$ is a generic submanifold, then $(\overline{H^*} \cap U)_s$ is a generic Levi-flat manifold of dimension $2n-2$.

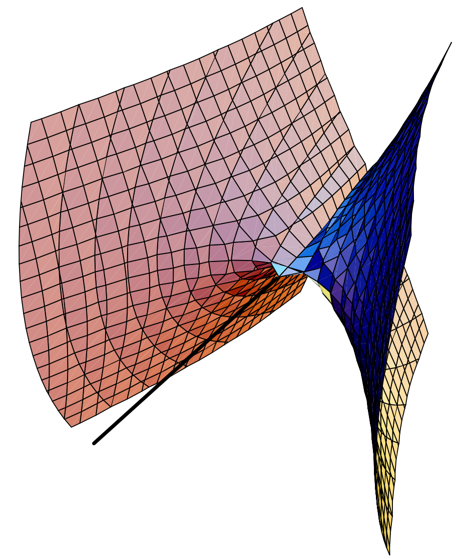
In particular, if $\overline{H^*} \cap U = H$, then H_s is Levi-flat near points where it is CR. Also note that our general definition of a Levi-flat manifold does include complex analytic manifolds. Examples show that the hypotheses of the theorem are necessary and the conclusion is optimal given the hypotheses.

4 Examples

The reason for looking only at $(\overline{H^*} \cap U)_s$ is that real-analytic varieties can be somewhat pathological.

Even if a variety is irreducible, it can have components of smaller dimension. For example, the hypersurface

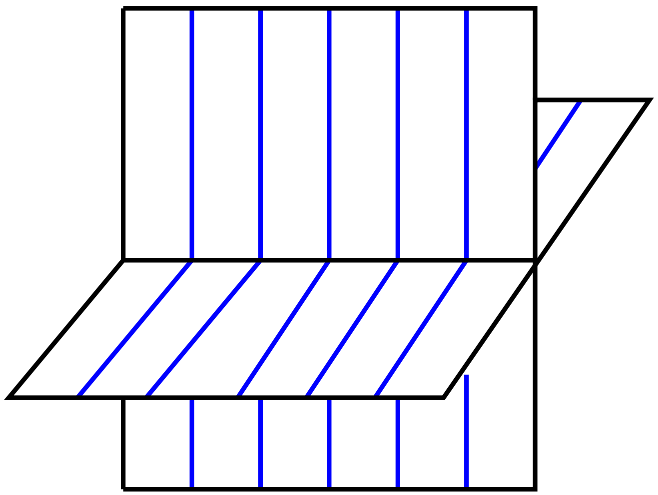
$$(\operatorname{Im} w)^2 = 4((\operatorname{Im} z)^2 + \operatorname{Re} w)(\operatorname{Im} z)^2$$



has this property, see Brunella [1]. The singularity (and the extra component) is a generic Levi-flat ($\operatorname{Im} z = \operatorname{Im} w = 0$). Note that this H is not pseudoconvex from both sides (but $\overline{H^*}$ is of course).

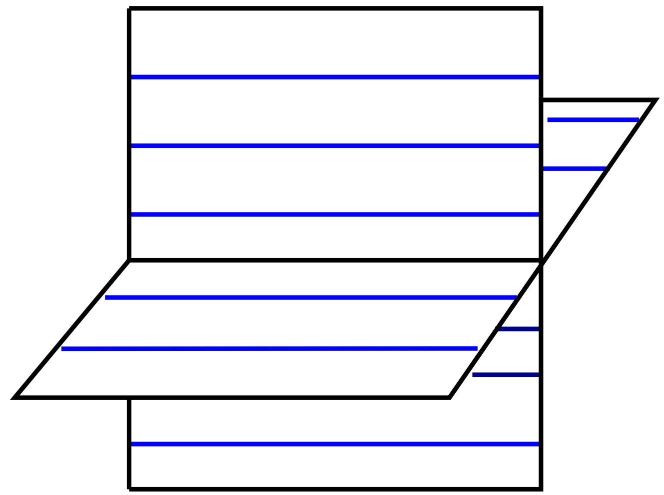
Finally we are only assuming that H^* is Levi-flat, so the topological closure of H^* is the natural object to study from the point of view of analysis.

Let us look at examples of the different singularities we can encounter. First let us look at an intersection of two nonsingular Levi-flats. We obtain two different types. Either a generic Levi-flat of real dimension $2n-2$, or a complex manifold of complex dimension $n-1$. The behavior depends on whether the Levi-foliations of the two manifolds intersect transversally or not.



$(\operatorname{Im} z_1)(\operatorname{Im} z_2) = 0$

$H_s = \mathbb{R}^2 \times \mathbb{C}^{n-2}$



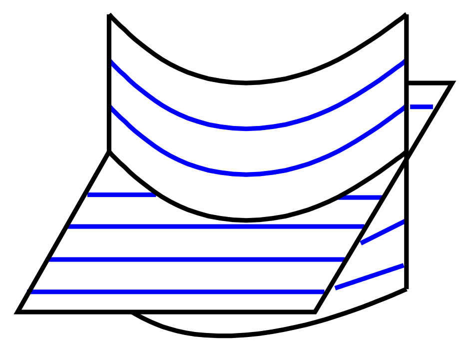
$\operatorname{Im}(z_1^2) = 0$

$H_s = \mathbb{C}^{n-1}$

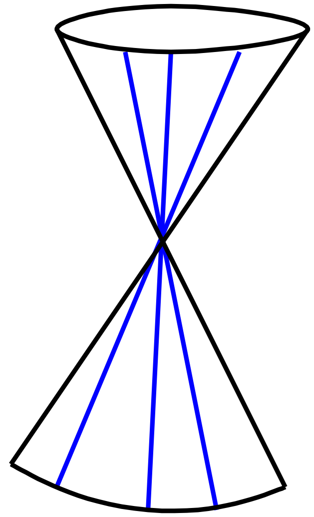
Both behaviors could happen at the same time and we could get a non-CR singular set. The singular set of

$$\big(\operatorname{Re}(z_2 - z_1^2)\big)\big(\operatorname{Im} z_2\big) = 0$$

is defined by $z_2 = \operatorname{Re} z_1^2$, which is not a CR submanifold at the origin.



And of course not all singularities are simple intersections. For example, the singularity could be the intersection of the leaves.



$$|z_1|^2 - |z_2|^2 = 0$$

Leaves are $z_1 = e^{i\theta} z_2$

Burns and Gong [2] classified the quadratic Levi-flats. Some of them we have already seen.

Normal form in $\mathbb{C}^n$	Singular set
$(\operatorname{Im} z_1)^2 = 0$	empty
$\operatorname{Im}(z_1^2 + z_2^2 + \cdots + z_k^2) = 0$	$\mathbb{C}^{n-k}$
$z_1^2 + 2\lambda z_1 \bar{z}_1 + \bar{z}_1^2 = 0 \ \big(\lambda \in (0,1)\big)$	$\mathbb{C}^{n-1}$
$(\operatorname{Im} z_1)(\operatorname{Im} z_2) = 0$	$\mathbb{R}^2 \times \mathbb{C}^{n-2}$
$\lvert z_1\rvert^2 - \lvert z_2\rvert^2 = 0$	$\mathbb{C}^{n-2}$

Notice that the singularity can be a complex variety of any dimension.

Some singular Levi-flats can be defined by the set where a holomorphic function is real valued,

$$H = \{z : \operatorname{Im} f(z) = 0\}.$$

Then (Burns-Gong [2]) the singular set is a complex analytic subvariety.

Not every Levi-flat hypersurface can be given as the vanishing set of a holomorphic function, e.g. the standard cusp in $\mathbb{C}$ (Burns-Gong [2]). However, this hypersurface is of course the pullback of a curve in $\mathbb{C}$ via a holomorphic map.

The hypersurface given by the set of $(z_1, z_2, z_3) \in \mathbb{C}^3$ that give a real solution t to

$$z_1 t^2 + z_2 t + z_3 = 0$$

cannot be defined by $\operatorname{Im} f(z) = 0$, and in fact, it cannot even be given by a pullback of a one dimensional curve in $\mathbb{C}$ via a meromorphic function (see [6]).

Note that also the Brunella example above cannot be define via $\operatorname{Im} f(z) = 0$ as it has a generic Levi-flat singular set.

5 Singular holomorphic foliations

It is natural to ask if the Levi-foliation on H^* extends to a holomorphic foliation of a neighbourhood of H. If H is nonsingular the answer is *yes*: Write H as $\operatorname{Im} z_1 = 0$ and then take $\omega = dz_1$. When H is singular the answer is *no* in general, see the example above by Brunella and [1] for more information.

When the singularity is small, however, the foliations always extends.

Theorem 2 (L. [7])**.** *Let $U \subset \mathbb{C}^n$ be an open set and let $H \subset U$ be a Levi-flat real-subvariety of dimension $2n-1$ that is irreducible as a germ at $p \in \overline{H^*} \cap U$. If either*

(i) $\dim H_s = 2n-4$ *and H is not leaf-degenerate at p, or*

(ii) $\dim H_s < 2n-4$*,*

then there exists a neighborhood U' of p, and a nontrivial holomorphic one form ω defined in U', such that $\omega \wedge d\omega = 0$ and such that the leaves of the Levi-foliation of $H^* \cap U'$ are integral submanifolds of ω.

Leaf-degenerate at p means that there are infinitely many complex (singular) hypersurfaces through p contained in H (e.g. $|z_1|^2 - |z_2|^2 = 0$).

Sketch of proof of Theorem 2: First we define a foliation in a neighbourhood of H^*. Then by using the complex hypersurface in H through p (Lemma 1) and the fact that the singularity is small we can apply a similar technique as that used by Lins Neto [8] to define a form ω in a neighbourhood of p.

Sketch of proof of Theorem 1: First we refine the technique of [4] to prove the theorem for generic singular sets. Then we use Theorem 2 to extend the foliation and prove that the singular set must be Levi-flat if the foliation extends. Finally we handle the remaining cases (leaf-degenerate and singular set of dimension $2n-3$).

6 Acknowledgements

The author would like to acknowledge Peter Ebenfelt for suggesting the study of this problem when the author was still a graduate student, and also for many conversations on the topic. The author would also like to thank Xianghong Gong, John P. D'Angelo, Salah Baouendi, and Linda Rothschild for fruitful discussions on topics related to this research and suggestions on the manuscript.

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See the papers above for a more complete set of references.