Finite Element Methods (version 7)

More modern (and somewhat more complicated) methods for approximate solutions than finite difference methods (FDM) are the finite element methods (FEM). We first work in one dimension with an ODE, before looking at a PDE. We study how to approximate the simple equation

$$\frac{du}{dx} + u = x, \qquad u(0) = 1, \qquad 0 < x < 1.$$

The idea is not to look for the solution u exactly, but look for the solution in some smaller class of functions. We define a vector space of *trial functions*, let us call it S. This vector space is the span of some *basis functions* $\{\phi_1(x), \ldots, \phi_n(x)\}$. So any element u of S is given as

$$u(x) = \sum_{j=1}^{n} a_j \phi_j(x) = a_1\phi_1(x) + a_2\phi_2(x) + \cdots + a_n\phi_n(x)$$

for some constants $a_1, a_2, \ldots, a_n$. For example, we could pick polynomials of degree $n-1$. Then we could let our basis be $\phi_j(x) = x^{j-1}$. Our space S would be composed of functions of the form

$$u(x) = a_1 + a_2 x + a_3 x^2 + \cdots + a_n x^{n-1}.$$

The trick is to now find the a_j in such a way so that we get "as close to the real u as possible."

There are several ways to do this, and we use the so-called *Galerkin method*. The idea is to minimize so called "energy" and it is essentially like a "least squares" approximation. First let us figure out what is the thing that we want to minimize, and that's the so-called *residual*:

$$R(x) = u'(x) + u(x) - x.$$

If R is zero, we have the precise solution. Of course, that will not be possible to achieve with S. Since we have n constants to choose, we should be able to pick n conditions (equations) to satisfy. The Galerkin method says that what we want is to set for (essentially) all j.

$$\int_0^1 R(x)\phi_j(x)\,dx = 0.$$

We also need to impose boundary conditions. In our model case there is only one boundary condition.

Let us do the example for the polynomial with $n = 3$ (degree 2). Our trial space S is

$$u(x) = a_1 + a_2 x + a_3 x^2.$$

As $u(0) = 1$, we set $a_1 = 1$. We need two more conditions for a_2 and a_3. We find the residual

$$R(x) = u'(x) + u(x) - x = a_2 + 2a_3 x + a_1 + a_2 x + a_3 x^2 - x = (a_2 + 1) + (2a_3 + a_2 - 1)x + a_3 x^2$$

We impose the conditions first for ϕ_2:

$$\begin{aligned} 0 = \int_0^1 R(x)\phi_2(x)\,dx &= \int_0^1 \big((a_2+1) + (2a_3 + a_2 - 1)x + a_3x^2\big)x\,dx \\ &= (a_2+1)\int_0^1 x\,dx + (2a_3+a_2-1)\int_0^1 x^2\,dx + a_3\int_0^1 x^3\,dx \\ &= \frac{1}{2}(a_2+1) + \frac{1}{3}(2a_3+a_2-1) + \frac{1}{4}a_3. \end{aligned}$$

So $\frac{5}{6}a_2 + \frac{11}{12}a_3 = -\frac{1}{6}$.

The second equation we get is

$$
\begin{aligned}
0 &= \int_0^1 R(x)\phi_3(x)\,dx = \int_0^1 \big((a_2+1) + (2a_3 + a_2 - 1)x + a_3 x^2\big)x^2\,dx \\
&= (a_2+1)\int_0^1 x^2\,dx + (2a_3 + a_2 - 1)\int_0^1 x^3\,dx + a_3\int_0^1 x^4\,dx \\
&= \frac{1}{3}(a_2+1) + \frac{1}{4}(2a_3 + a_2 - 1) + \frac{1}{5}a_3.
\end{aligned}
$$

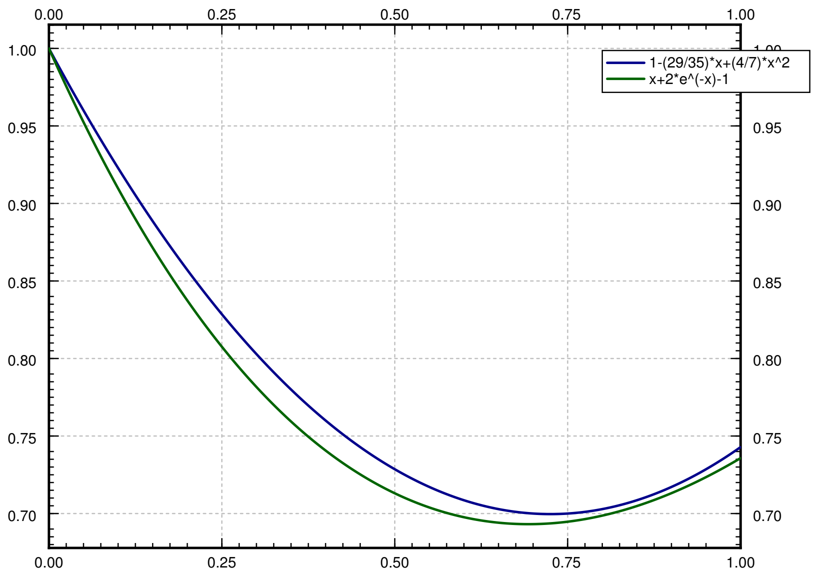
So $\frac{7}{12}a_2 + \frac{7}{10}a_3 = -\frac{1}{12}$. We solve to get $a_2 = \frac{-29}{35}$ and $a_3 = \frac{4}{7}$. Therefore our approximation is

$$u_{approx}(x) = 1 - \frac{29}{35}x + \frac{4}{7}x^2.$$

The real solution is

$$u(x) = x + 2e^{-x} - 1.$$

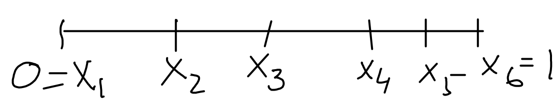
Let us plot the two functions to see that they are quite close:



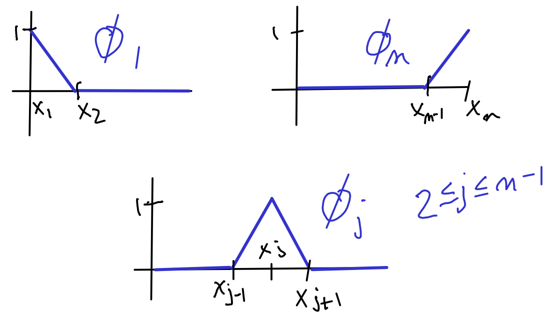
You may be wondering where are those *elements* from the title of the method. Well, we have really done things only in some sense for “one-element”. Usually one does not want to do the approximation in some large class like polynomials for the entire domain. We wish to use simpler functions that only make a difference locally. There are many reasons for this, and one of the biggest ones is that if we use a class such as polynomials of high degree, then the resulting linear system is very hard to solve if it is too big.

Let us look at the simplest types of basis functions that would commonly be encountered. Often what are used are so called piecewise linear functions. First, let us split the domain into intervals, or so-called *elements*: Write the interval $[0, 1]$ as a series of subintervals, by picking points, which are the so-called *nodes*: $x_1, x_2, \ldots, x_n$, with $x_1 = 0$ and $x_n = 1$.

$m = 6$



We define the basis functions as "peaks" at the nodes.



$$\phi_1(x) = \begin{cases} \frac{x_2 - x}{x_2 - x_1} & \text{if } x_1 \le x < x_2 \\ 0 & \text{else} \end{cases} \qquad \phi_n(x) = \begin{cases} \frac{x - x_{n-1}}{x_n - x_{n-1}} & \text{if } x_{n-1} \le x \le x_n \\ 0 & \text{else} \end{cases}$$

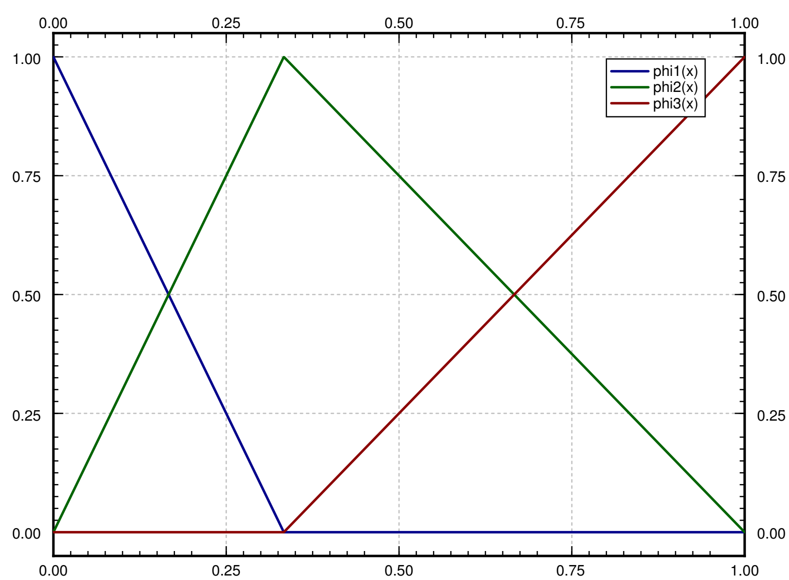
$$\phi_j(x) = \begin{cases} \frac{x - x_{j-1}}{x_j - x_{j-1}} & \text{if } x_{j-1} \le x < x_j \\ \frac{x_{j+1} - x}{x_{j+1} - x_j} & \text{if } x_j \le x < x_{j+1} \\ 0 & \text{else} \end{cases} \qquad \text{for } 2 \le j \le n-1:$$

The functions are set up so that $\phi_j(x_j) = 1$ and $\phi_j(x_k) = 0$ for all $k \neq j$. The resulting linear system of equations is "nice" because each ϕ_j only interacts with two elements, so the resulting equations (the matrix to be inverted) have many zeros for large n. This niceness is not so visible for the small example we do next.

Example: still $u' + u = x$, $u(0) = 1$. Take 3 nodes, $x_1 = 0$, $x_2 = \frac{1}{3}$, and $x_3 = 1$. We do not have to divide the domain evenly, this is one of the advantages of FEM.

$$\phi_1(x) = \begin{cases} \frac{(1/3) - x}{1/3} = 1 - 3x & \text{if } 0 \le x < \frac{1}{3}, \\ 0 & \text{if } \frac{1}{3} \le x \le 1, \end{cases} \qquad \phi_2(x) = \begin{cases} \frac{x}{1/3} = 3x & \text{if } 0 \le x < \frac{1}{3}, \\ \frac{1-x}{1-(1/3)} = \frac{3}{2}(1-x) & \text{if } \frac{1}{3} \le x \le 1, \end{cases}$$

$$\phi_3(x) = \begin{cases} 0 & \text{if } 0 \le x < \frac{1}{3}, \\ \frac{x - (1/3)}{1 - (1/3)} = \frac{3}{2}\left(x - \frac{1}{3}\right) & \text{if } \frac{1}{3} \le x \le 1. \end{cases}$$



Let us differentiate. We do not really want to think of the derivatives at the nodes as they are undefined there, but those are just a few points.

$$\phi_1'(x) = \begin{cases} -3 & \text{if } 0 < x < \frac{1}{3}, \\ 0 & \text{if } \frac{1}{3} < x < 1, \end{cases} \qquad \phi_2'(x) = \begin{cases} 3 & \text{if } 0 < x < \frac{1}{3}, \\ \frac{-3}{2} & \text{if } \frac{1}{3} < x < 1, \end{cases}$$

$$\phi_3'(x) = \begin{cases} 0 & \text{if } 0 < x < \frac{1}{3}, \\ \frac{3}{2} & \text{if } \frac{1}{3} < x < 1. \end{cases}$$

Our trial space is composed of functions of the form

$$u(x) = a_1\phi_1(x) + a_2\phi_2(x) + a_3\phi_3(x).$$

The boundary condition $u(0) = 1$ gives $a_1 = 1$ (the only way to satisfy this condition). The other two conditions give a_2 and a_3:

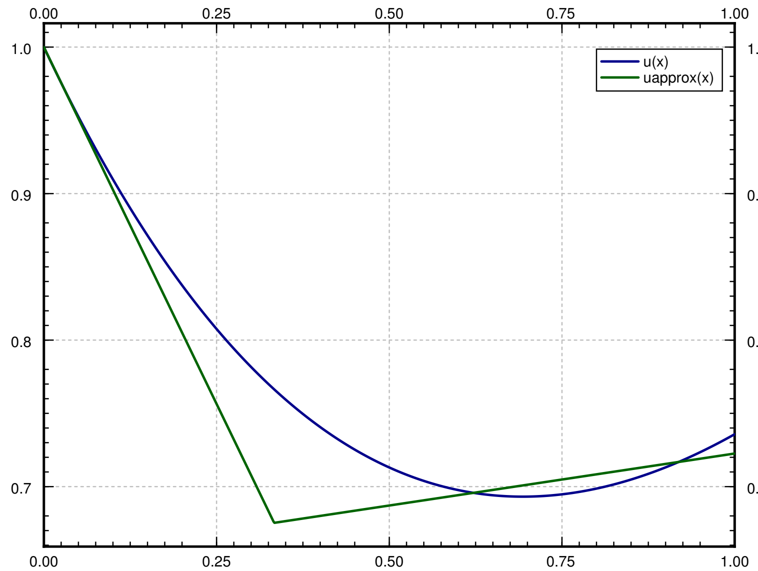
$$\begin{aligned}
0 &= \int_0^1 R(x)\phi_2(x)\,dx = \int_0^1 \big(u'(x) + u(x) - x\big)\phi_2\,dx \\
&= a_1 \int_0^1 \big(\phi_1'(x) + \phi_1(x)\big)\phi_2(x)\,dx + a_2 \int_0^1 \big(\phi_2'(x) + \phi_2(x)\big)\phi_2(x)\,dx \\
&\quad + a_3 \int_0^1 \big(\phi_3'(x) + \phi_3(x)\big)\phi_2(x)\,dx + \int_0^1 (-x)\phi_2(x)\,dx \\
&= a_1 \int_0^{1/3} (-3 + 1 - 3x)3x\,dx + a_2 \left(\int_0^{1/3} (3 + 3x)3x\,dx + \int_{1/3}^1 \left(\frac{-3}{2} + \frac{3}{2}(1-x) \right) \frac{3}{2}(1-x)\,dx \right) \\
&\quad + a_3 \int_{1/3}^1 \left(\frac{3}{2} + \frac{3}{2}\left(x - \frac{1}{3}\right) \right) \frac{3}{2}(1-x)\,dx + \int_0^{1/3} (-x)3x\,dx + \int_{1/3}^1 (-x)\frac{3}{2}(1-x)\,dx \\
&= \frac{-4}{9}a_1 + \left(\frac{11}{18} - \frac{5}{18} \right) a_2 + \frac{11}{18}a_3 - \frac{1}{27} - \frac{5}{27}.
\end{aligned}$$

So as $a_1 = 1$, the condition says $\frac{6}{18}a_2 + \frac{11}{18}a_3 = \frac{2}{3}$.

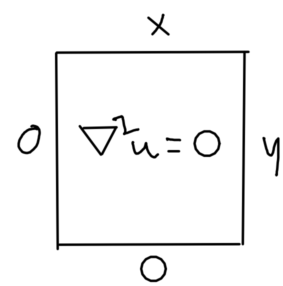
$$\begin{aligned}
0 &= \int_0^1 R(x)\phi_3(x)\,dx = \int_0^1 \big(u'(x) + u(x) - x\big)\phi_3\,dx \\
&= a_1 \int_0^1 \big(\phi_1'(x) + \phi_1(x)\big)\phi_3(x)\,dx + a_2 \int_0^1 \big(\phi_2'(x) + \phi_2(x)\big)\phi_3(x)\,dx \\
&\quad + a_3 \int_0^1 \big(\phi_3'(x) + \phi_3(x)\big)\phi_3(x)\,dx + \int_0^1 (-x)\phi_3(x)\,dx \\
&= a_2 \left(\int_{1/3}^1 \left(\frac{-3}{2} + \frac{3}{2}(1-x) \right) \frac{3}{2}\left(x - \frac{1}{3}\right)\,dx \right) \\
&\quad + a_3 \int_{1/3}^1 \left(\frac{3}{2} + \frac{3}{2}\left(x - \frac{1}{3}\right) \right) \frac{3}{2}\left(x - \frac{1}{3}\right)\,dx + \int_{1/3}^1 (-x)\frac{3}{2}\left(x - \frac{1}{3}\right)\,dx \\
&= \frac{-7}{18}a_2 + \frac{13}{18}a_3 - \frac{7}{27}.
\end{aligned}$$

So $\frac{-7}{18}a_2 + \frac{13}{18}a_3 = \frac{7}{27}$. Solving we get $a_2 = \frac{314}{465}$ and $a_3 = \frac{112}{155}$. Hence the approximate solution is

$$u_{approx}(x) = \phi_1(x) + \frac{314}{465}\phi_2(x) + \frac{112}{155}\phi_3(x).$$

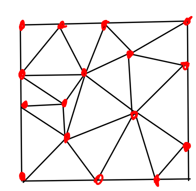


Let us move to two dimensions. Instead of the abstract details let us simply solve a specific two dimensional example:

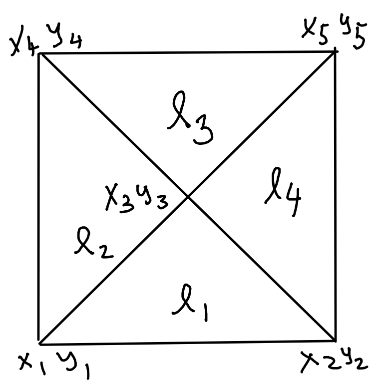


PDE: $\nabla^2 u = 0$, $0 < x, y < 1$
BC: $u(x, 0) = 0$, $u(0, y) = 0$, $u(x, 1) = x$, $u(1, y) = y$.

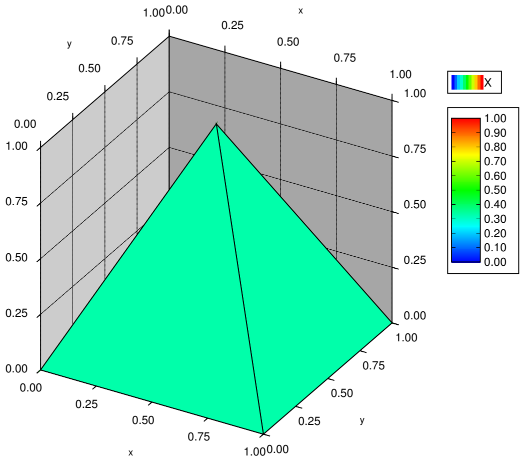
We divide the domain into elements. We use triangular elements and piecewise linear basis functions. Normally we would divide the domain into many triangles, where the vertices of the triangles are the nodes as in the picture to the right. The nodes are marked in red.

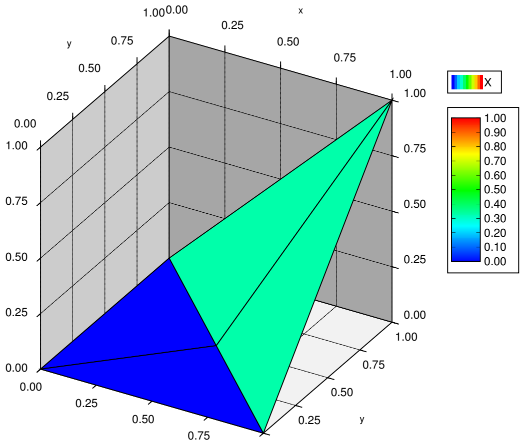


Let us consider a simple situation. Take the nodes $(x_1, y_1) = (0, 0)$, $(x_2, y_2) = (1, 0)$, $(x_3, y_3) = (1/2, 1/2)$, $(x_4, y_4) = (0, 1)$, $(x_5, y_5) = (1, 1)$. We have 4 triangular elements, labeled e_1, e_2, e_3, e_4 in the following picture:

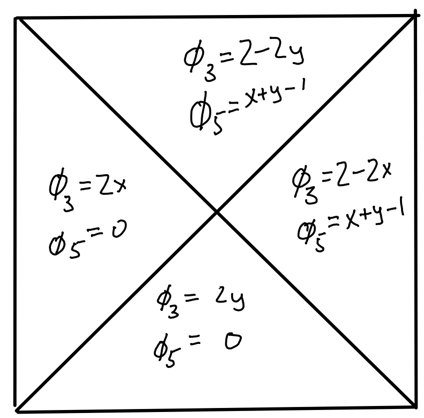


We define the basis functions $\phi_1, \phi_2, \phi_3, \phi_4, \phi_5$. We arrange things so that $\phi_j(x_j, y_j) = 1$, and $\phi_j(x_k, y_k) = 0$ if $k \neq j$. For example, on the next picture we have ϕ_3 and ϕ_5.





We define these piecewise



$$\phi_3(x,y) = \begin{cases} 2y & \text{if } x > y \text{ and } x \le 1-y, \\ 2x & \text{if } x \le y \text{ and } x \le 1-y, \\ 2-2y & \text{if } x \le y \text{ and } x > 1-y, \\ 2-2x & \text{if } x > y \text{ and } x > 1-y. \end{cases}$$

Similarly

$$\phi_5(x,y) = \begin{cases} 0 & \text{if } x \le 1-y, \\ x+y-1 & \text{if } x > 1-y. \end{cases}$$

We write our approximate solution as

$$u(x,y) = \sum_{j=1}^{5} a_j \phi_j(x,y).$$

We will not bother with defining ϕ_1, ϕ_2, and ϕ_4, because the boundary value is 0 at the 3 corners corresponding to the nodes number 1, 2, and 4. The boundary value is 1 at the fifth node. Therefore we have that $a_1 = a_2 = a_4 = 0$ and $a_5 = 1$. We have

$$u(x,y) = a_3\phi_3(x,y) + \phi_5(x,y).$$

Let Ω be the unit square $0 < x, y < 1$. The one condition we need to solve for is

$$\iint_\Omega (\nabla^2 u)\phi_3 \, dA = 0.$$

As u is composed of piecewise linear functions, computing second derivatives involves the delta function. While that is certainly possible, it is best to avoid via the use of Green's theorem. It is really integration by parts, since we will apply the divergence, $\nabla\cdot$, to the product $(\nabla u)\phi_3$. That is,

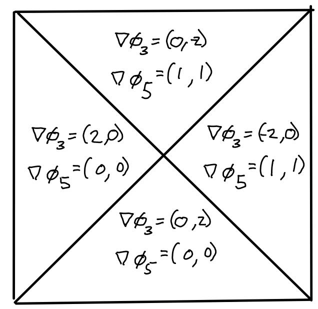
$$\nabla \cdot \big((\nabla u)\phi_3\big) = (\nabla^2 u)\phi_3 + \nabla u \cdot \nabla \phi_3.$$

Applying Green's (or Divergence theorem) theorem gets us

$$\iint_\Omega (\nabla^2 u)\phi_3 \, dA = \int_{\partial\Omega} (\nabla u \cdot \vec{n})\phi_3 \, ds - \iint_\Omega \nabla u \cdot \nabla \phi_3 \, dA,$$

where $\partial\Omega$ is the boundary of the unit square, and $\vec{n}$ is the outer unit normal. Because ϕ_3 is zero on the boundary, the boundary term just disappears. We are left with the second term.

$$\begin{aligned}0 &= \iint_\Omega (\nabla^2 u)\phi_3\, dA = -\iint_\Omega \nabla u \cdot \nabla\phi_3\, dA \\ &= -a_3 \iint_\Omega \nabla\phi_3 \cdot \nabla\phi_3\, dA - \iint_\Omega \nabla\phi_5 \cdot \nabla\phi_3\, dA.\end{aligned}$$



The gradients are constant vectors. In e_1, $\nabla\phi_3 = \langle 0, 2\rangle$, in e_2, $\nabla\phi_3 = \langle 2, 0\rangle$, in e_3, $\nabla\phi_3 = \langle 0, -2\rangle$, in e_4, $\nabla\phi_3 = \langle -2, 0\rangle$. Similarly In e_1, $\nabla\phi_5 = \langle 0, 0\rangle$, in e_2, $\nabla\phi_5 = \langle 0, 0\rangle$, in e_3, $\nabla\phi_5 = \langle 1, 1\rangle$, in e_4, $\nabla\phi_5 = \langle 1, 1\rangle$.

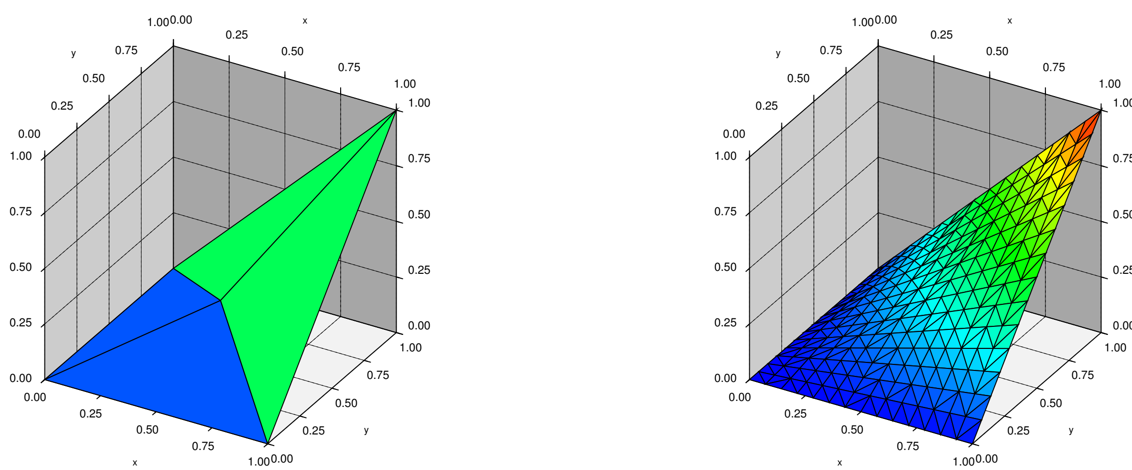
Therefore

$$\begin{aligned}0 &= -a_3 \iint_\Omega \nabla\phi_3 \cdot \nabla\phi_3\, dA - \iint_\Omega \nabla\phi_5 \cdot \nabla\phi_3\, dA \\ &= -a_3 \left(\iint_{e_1} \nabla\phi_3 \cdot \nabla\phi_3\, dA + \iint_{e_2} \nabla\phi_3 \cdot \nabla\phi_3\, dA + \iint_{e_3} \nabla\phi_3 \cdot \nabla\phi_3\, dA + \iint_{e_4} \nabla\phi_3 \cdot \nabla\phi_3\, dA \right) \\ &\quad - \left(\iint_{e_1} \nabla\phi_5 \cdot \nabla\phi_3\, dA + \iint_{e_2} \nabla\phi_5 \cdot \nabla\phi_3\, dA + \iint_{e_3} \nabla\phi_5 \cdot \nabla\phi_3\, dA + \iint_{e_4} \nabla\phi_5 \cdot \nabla\phi_3\, dA \right) \\ &= -a_3 \left(\iint_{e_1} \langle 0,2\rangle \cdot \langle 0,2\rangle\, dA + \iint_{e_2} \langle 2,0\rangle \cdot \langle 2,0\rangle\, dA + \iint_{e_3} \langle 0,-2\rangle \cdot \langle 0,-2\rangle\, dA + \iint_{e_4} \langle -2,0\rangle \cdot \langle -2,0\rangle\, dA \right) \\ &\quad - \left(\iint_{e_1} \langle 0,0\rangle \cdot \langle 0,2\rangle\, dA + \iint_{e_2} \langle 0,0\rangle \cdot \langle 2,0\rangle\, dA + \iint_{e_3} \langle 1,1\rangle \cdot \langle 0,-2\rangle\, dA + \iint_{e_4} \langle 1,1\rangle \cdot \langle -2,0\rangle\, dA \right) \\ &= -4a_3 \left(\iint_{e_1} dA + \iint_{e_2} dA + \iint_{e_3} dA + \iint_{e_4} dA \right) + 2 \left(\iint_{e_3} dA + \iint_{e_4} dA \right) \\ &= -4a_3 + 1.\end{aligned}$$

So $a_3 = \frac{1}{4}$ and hence our approximate solution is

$$u(x,y) = \frac{1}{4}\phi_3(x,y) + \phi_5(x,y).$$

Here are the plots of this approximate and the real solution:



Exercises:

1) Using piecewise linear basis functions, solve $u' + u = 1$, $u(0) = 3$, on $0 < x < 1$, with nodes at $x_1 = 0$, $x_2 = \frac{1}{2}$, $x_3 = 1$.

2) Using polynomial trial functions with degree going up to 3, solve $u' + u = x$, $u(0) = 1$ on $0 < x < 1$.

3) Using piecewise linear basis functions, solve $u'' + u = 0$, $u(0) = 0$, $u(1) = 1$, on $0 < x < 1$, with $x_1 = 0$, $x_2 = \frac{1}{2}$, $x_3 = 1$. Be careful, you are doing second order derivatives of piecwise linear functions, you should do integration by parts if you do not want to deal with delta functions.

4) Do the 2-dimensional example with $\nabla^2 u = 1$ instead and same boundary conditions.

5) In the 2-dimensional example, divide elements e_3 and e_4 exactly in half to add two new elements (that is, add another two nodes on the sides). Write out the formulas for all the basis functions ϕ.

6) Triangulate the square with 8 triangles with at least two internal nodes so that at no node do more than 4 elements meet.