9.3 Path independence

Note: ??? lectures

9.3.1 Path independent integrals

Let $U \subset \mathbb{R}^n$ be a set and ω a one-form defined on U, The integral of ω is said to be *path independent* if for any two points $x, y \in U$ and any two piecewise smooth paths $\gamma\colon [a,b] \to U$ and $\beta\colon [c,d] \to U$ such that $\gamma(a) = \beta(c) = x$ and $\gamma(b) = \beta(d) = y$ we have

$$\int_\gamma \omega = \int_\beta \omega.$$

In this case we simply write

$$\int_x^y \omega = \int_\gamma \omega = \int_\beta \omega.$$

Not every one-form gives a path independent integral. In fact, most do not.

Example 9.3.1: Let $\gamma\colon [0,1] \to \mathbb{R}^2$ be the path $\gamma(t) = (t,0)$ going from $(0,0)$ to $(1,0)$. Let $\beta\colon [0,1] \to \mathbb{R}^2$ be the path $\beta(t) = \bigl(t, (1-t)t\bigr)$ also going between the same points. Then

$$\int_\gamma y\,dx = \int_0^1 \gamma_2(t)\gamma_1'(t)\,dt \int_0^1 0(1)dt = 0,$$
$$\int_\beta y\,dx = \int_0^1 \beta_2(t)\beta_1'(t)\,dt \int_0^1 (1-t)t(1)dt = \frac{1}{6}.$$

So the integral of $y\,dx$ is not path independent. In particular, $\int_{(0,0)}^{(1,0)} y\,dx$ does not make sense.

Definition 9.3.2. Let $U \subset \mathbb{R}^n$ be an open set and $f\colon U \to \mathbb{R}$ a continuously differentiable function. Then the one-form

$$df := \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 + \cdots + \frac{\partial f}{\partial x_n} dx_n$$

is called the *total derivative* of f.

An open set $U \subset \mathbb{R}^n$ is said to be *path connected** if for every two points x and y in U, there exists a piecewise smooth path starting at x and ending at y.

We will leave as an exercise that every connected open set is path connected.

*Normally only a continuous path is used in this definition, but for open sets two two definitions are equivalent. See the exercises.

Proposition 9.3.3. *Let $U \subset \mathbb{R}^n$ be a path connected open set and ω a one-form defined on U. Then*

$$\int_x^y \omega$$

is path independent (for all $x, y \in U$) if and only if there exists a continuously differentiable $f : U \to \mathbb{R}$ such that $\omega = df$.

In fact, if such an f exists, then for any two points $x, y \in U$

$$\int_x^y \omega = f(y) - f(x).$$

In other words if we fix x_0, then $f(x) = C + \int_{x_0}^x \omega$.

Proof. First suppose that the integral is path independent. Pick $x_0 \in U$ and define

$$f(x) = \int_{x_0}^x \omega.$$

Let e_j be an arbitrary standard basis vector. Compute

$$\frac{f(x+he_j) - f(x)}{h} = \frac{1}{h}\left(\int_{x_0}^{x+he_j} \omega - \int_{x_0}^{x} \omega\right) = \frac{1}{h}\int_{x}^{x+he_j} \omega,$$

which follows by Proposition 9.2.10 and path indepdendence as $\int_{x_0}^{x+he_j} \omega = \int_{x_0}^{x} \omega + \int_{x}^{x+he_j} \omega$.

Write $\omega = \omega_1 dx_1 + \omega_2 dx_2 + \cdots + \omega_n dx_n$. Now pick the simplest path possible from x to $x + he_j$, that is $\gamma(t) = x + the_j$ for $t \in [0,1]$. Notice that $\gamma'(t)$ has only a simple nonzero component and that is the jth component which is h. Therefore

$$\frac{1}{h}\int_{x}^{x+he_j} \omega = \frac{1}{h}\int_0^1 \omega_j(x+the_j)h\,dt = \int_0^1 \omega_j(x+the_j)\,dt.$$

We wish to take the limit as $h \to 0$. The function ω_j is continuous. So given $\varepsilon > 0$, h can be small enough so that $\left|\omega(x) - \omega_j(x+the_j)\right| < \varepsilon$. Therefore for such small h we find that $\left|\int_0^1 \omega_j(x+the_j)\,dt - \omega(x)\right| < \varepsilon$. That is

$$\lim_{h\to 0} \frac{f(x+he_j) - f(x)}{h} = \omega_j(x),$$

which is what we wanted that is $df = \omega$. As ω_j are continuous for all j, we find that f has continuous partial derivatives and therefore is continuously differentiable.

For the other direction suppose f exists such that $df = \omega$. Suppose we take a smooth path $\gamma \colon [a,b] \to U$ such that $\gamma(a) = x$ and $\gamma(b) = y$, then

$$\begin{aligned}
\int_\gamma df &= \int_a^b \left(\frac{\partial f}{\partial x_1}\big(\gamma(t)\big)\gamma_1'(t) + \frac{\partial f}{\partial x_2}\big(\gamma(t)\big)\gamma_2'(t) + \cdots + \frac{\partial f}{\partial x_n}\big(\gamma(t)\big)\gamma_n'(t)\right) dt \\
&= \int_a^b \frac{d}{dt}\left[f\big(\gamma(t)\big)\right] dt \\
&= f(y) - f(x).
\end{aligned}$$

The value of the integral only depends on x and y, not the path taken. Therefore the integral is path independent. We leave checking this for a piecewise smooth path as an exercise to the reader. $\square$

Proposition 9.3.4. *Let $U \subset \mathbb{R}^n$ be a path connected open set and ω a 1-form defined on U. Then $\omega = df$ for some continuously differentiable $f \colon U \colon \mathbb{R}$ if and only if*

$$\int_\gamma \omega = 0$$

for every piecewise smooth closed path $\gamma \colon [a,b] \to U$.

Proof. Suppose first that $\omega = df$ and let γ be a piecewise smooth closed path. Then we from above we have that

$$\int_\gamma \omega = f\bigl(\gamma(b)\bigr) - f\bigl(\gamma(a)\bigr) = 0,$$

because $\gamma(a) = \gamma(b)$ for a closed path.

Now suppose that for every piecewise smooth closed path γ, $\int_\gamma \omega = 0$. Let x, y be two points in U and let $\alpha \colon [0,1] \to U$ and $\beta \colon [0,1] \to U$ be two piecewise smooth paths with $\alpha(0) = \beta(0) = x$ and $\alpha(1) = \beta(1) = y$. Then let $\gamma \colon [0,2] \to U$ be defined by

$$\gamma(t) := \begin{cases} \alpha(t) & \text{if } t \in [0,1], \\ \beta(2-t) & \text{if } t \in (1,2]. \end{cases}$$

This is a piecewise smooth closed path and so

$$0 = \int_\gamma \omega = \int_\alpha \omega - \int_\beta \omega.$$

This follows first by Proposition 9.2.10, and then noticing that the second part is β travelled backwards so that we get minus the β integral. Thus the integral of ω on U is path independent. $\square$

There is a local criterion, that is a differential equation, that guarantees path independence. That is, under the right condition there exists an *antiderivative* f whose total derivative is the given one form ω. However, since the criterion is local, we only get the result locally. We can define the antiderivative in any so-called *simply connected* domain, which informally is a domain where any path between two points can be "continuously deformed" into any other path between those two points. To make matters simple, the usual way this result is proved is for so-called star-shaped domains.

Definition 9.3.5. Let $U \subset \mathbb{R}^n$ be an open set and $x_0 \in U$. We say U is a *star shaped domain* with respect to x_0 if for any other point $x \in U$, the line segment between x_0 and x is in U, that is, if $(1-t)x_0 + tx \in U$ for all $t \in [0,1]$. If we say simply *star shaped* then U is star shaped with respect to some $x_0 \in U$.

Notice the difference between star shaped and convex. A convex domain is star shaped, but a star shaped domain need not be convex.

Theorem 9.3.6 (Poincarè lemma)**.** *Let $U \subset \mathbb{R}^n$ be a star shaped domain and ω a continuously differentiable one-form defined on U. That is, if*

$$\omega = \omega_1 dx_1 + \omega_2 dx_2 + \cdots + \omega_n dx_n,$$

then $\omega_1, \omega_2, \ldots, \omega_n$ are continuously differentiable functions. Suppose that for every j and k

$$\frac{\partial \omega_j}{\partial x_k} = \frac{\partial \omega_k}{\partial x_j},$$

then there exists a twice continuously differentiable function $f \colon U \to \mathbb{R}$ such that $df = \omega$.

The condition on the derivatives of ω is precisely the condition that the second partial derivatives commute. That is, if $df = \omega$, then

$$\frac{\partial \omega_j}{\partial x_k} = \frac{\partial^2 f}{\partial x_k \partial x_j}.$$

Proof. Suppose U is star shaped with respect to $y = (y_1, y_2, \ldots, y_n) \in U$.

Given $x = (x_1, x_2, \ldots, x_n) \in U$, define the path $\gamma \colon [0,1] \to U$ as $\gamma(t) = (1-t)y + tx$, so $\gamma'(t) = y - x$. Then let

$$f(x) = \int_\gamma \omega = \int_0^1 \left(\sum_{k=1}^n \omega_k\big((1-t)y+tx\big)(y_k - x_k) \right) dt$$

Now we can differentiate in x_j under the integral. We can do that since everything, including the partials themselves are continuous.

$$\begin{aligned}
\frac{\partial f}{\partial x_j}(x) &= \int_0^1 \left(\left(\sum_{k=1}^n \frac{\partial \omega_k}{\partial x_j}\big((1-t)y+tx\big)t(y_k - x_k) \right) - \omega_j\big((1-t)y+tx\big) \right) dt \\
&= \int_0^1 \left(\left(\sum_{k=1}^n \frac{\partial \omega_j}{\partial x_k}\big((1-t)y+tx\big)t(y_k - x_k) \right) - \omega_j\big((1-t)y+tx\big) \right) dt \\
&= \int_0^1 \frac{d}{dt} \left[t\,\omega_j\big((1-t)y+tx\big) \right] dt \\
&= \omega_j(x).
\end{aligned}$$

And this is precisely what we wanted. □

Example 9.3.7: Without some hypothesis on U the theorem is not true. Let

$$\omega(x,y) = \frac{-y}{x^2+y^2} dx + \frac{x}{x^2+y^2} dy$$

be defined on $\mathbb{R}^2 \setminus \{0\}$. It is easy to see that

$$\frac{\partial}{\partial y}\left[\frac{-y}{x^2+y^2}\right] = \frac{\partial}{\partial x}\left[\frac{x}{x^2+y^2}\right].$$

However, there is no $f\colon \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$ such that $df = \omega$. We saw in if we integrate from $(1,0)$ to $(1,0)$ along the unit circle, that is $\gamma(t) = \big(\cos(t), \sin(t)\big)$ for $t \in [0,2\pi]$ we got 2π and not 0 as it should be if the integral is path independent or in other words if there would exist an f such that $df = \omega$.

9.3.2 Vector fields

A common object to integrate is a so-called vector field. That is an assignment of a vector at each point of a domain.

Definition 9.3.8. Let $U \subset \mathbb{R}^n$ be a set. A continuous function $v\colon U \to \mathbb{R}^n$ is called a *vector field*. Write $v = (v_1, v_2, \ldots, v_n)$

Given a smooth path $\gamma\colon [a,b] \to \mathbb{R}^n$ with $\gamma\big([a,b]\big) \subset U$ we define the path integral of the vectorfield v as

$$\int_\gamma v \cdot d\gamma := \int_a^b v\big(\gamma(t)\big) \cdot \gamma'(t)\, dt,$$

where the dot in the definition is the standard dot product. Again the definition of a piecewise smooth path is done by integrating over each smooth interval and adding the result.

If we unravel the definition we find that

$$\int_\gamma v \cdot d\gamma = \int_\gamma v_1 dx_1 + v_2 dx_2 + \cdots + v_n dx_n.$$

Therefore what we know about integration of one-forms carries over to the integration of vector fields. For example path independence for integration of vector fields is simply that

$$\int_x^y v \cdot d\gamma$$

is path independent (so for any γ) if and only if $v = \nabla f$, that is the gradient of a function. The function f is then called the *potential* for v.

A vector field v whose path integrals are path independent is called a *conservative vector field*. The naming comes from the fact that such vector fields arise in physical systems where a certain quantity, the energy is conserved.

9.3.3 Exercises

Exercise* 9.3.1*: *Find an $f\colon \mathbb{R}^2 \to \mathbb{R}$ such that $df = xe^{x^2+y^2}dx + ye^{x^2+y^2}dy$.*

Exercise* 9.3.2*: *Finish the proof of Proposition 9.3.3, that is, we only proved the second direction for a smooth path, not a piecewise smooth path.*

Exercise* 9.3.3*: *Show that a star shaped domain $U \subset \mathbb{R}^n$ is path connected.*

Exercise* 9.3.4*: *Show that $U := \mathbb{R}^2 \setminus \{(x,y) \in \mathbb{R}^2 : x \leq 0, y = 0\}$ is star shaped and find all points $(x_0,y_0) \in U$ such that U is star shaped with respect to (x_0,y_0).*

Exercise* 9.3.5*: *Let $\gamma\colon [a,b] \to \mathbb{R}^n$ be a simple nonclosed path (so γ is one-to-one). Suppose that ω is a continuously differentiable one-form defined on some open set V with $\gamma\big([a,b]\big) \subset V$ and $\frac{\partial \omega_j}{\partial x_k} = \frac{\partial \omega_k}{\partial x_j}$ for all j and k. Prove that there exists an open set U with $\gamma\big([a,b]\big) \subset U \subset V$ and a twice continuously differentiable function $f\colon U \to \mathbb{R}$ such that $df = \omega$.*
Hint 1: $\gamma\big([a,b]\big)$ is compact.
Hint 2: Piecing together several different functions f can be tricky, but notice that the intersection of any number of balls is always convex as balls are convex, and convex sets are in particular connected (path connected).

Exercise* 9.3.6*: *a) Show that a connected open set is path connected. Hint: Start with two points x and y in a connected set U, and let $U_x \subset U$ is the set of points that are reachable by a path from x and similarly for U_y. Show that both sets are open, since they are nonempty ($x \in U_x$ and $y \in U_y$) it must be that $U_x = U_y = U$.*
b) Prove the converse that is, a path connected set $U \subset \mathbb{R}^n$ is connected. Hint: for contradiction assume there exist two open and disjoint nonempty open sets and then assume there is a piecewise smooth (and therefore continuous) path between a point in one to a point in the other.

Exercise* 9.3.7*: *Usually path connectedness is defined using just continuous paths rather than piecewise smooth paths. Prove that the definitions are equivalent, in other words prove the following statement:*
Suppose $U \subset \mathbb{R}^n$ is such that for any $x,y \in U$, there exists a continuous function $\gamma\colon [a,b] \to U$ such that $\gamma(a) = x$ and $\gamma(b) = y$. Then U is path connected (in other words, then there exists a piecewise smooth path).

Exercise* 9.3.8** (Hard):*** *Take*

$$\omega(x,y) = \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$$

defined on $\mathbb{R}^2 \setminus \{(0,0)\}$. Let $\gamma\colon [a,b] \to \mathbb{R}^2 \setminus \{(0,0)\}$ be a closed piecewise smooth path. Let $R := \{(x,y) \in \mathbb{R}^2 : x \leq 0 \text{ and } y = 0\}$. Suppose that $R \cap \gamma\big([a,b]\big)$ is a finite set of k points. Then

$$\int_\gamma \omega = 2\pi\ell$$

for some integer ℓ with $|\ell| \leq k$.
Hint 1: First prove that for a path β that starts and end on R but does not intersect it otherwise, you find that $\int_\beta \omega$ is -2π, 0, or 2π. Hint 2: You proved above that $\mathbb{R}^2 \setminus R$ is star shaped.
Note: The number ℓ is called the winding number *it it measures how many times does γ wind around the origin in the clockwise direction.*