Chapter 9

One dimensional integrals in several variables

9.1 Differentiation under the integral

Note: less than 1 lecture

Let $f(x,y)$ be a function of two variables and define

$$g(y) := \int_a^b f(x,y)\, dx$$

Suppose that f is differentiable in y. The question we ask is when can we simply "differentiate under the integral", that is, when is it true that g is differentiable and its derivative

$$g'(y) \stackrel{?}{=} \int_a^b \frac{\partial f}{\partial y}(x,y)\, dx.$$

Differentiation is a limit and therefore we are really asking when do the two limitting operations of integration and differentiation commute. As we have seen, this is not always possible. In particular, the first question we would face is the integrability of $\frac{\partial f}{\partial y}$.

Let us prove a simple, but the most useful version of this theorem.

Theorem 9.1.1 (Leibniz integral rule)**.** *Suppose $f\colon [a,b]\times[c,d] \to \mathbb{R}$ is a continuous function, such that $\frac{\partial f}{\partial y}$ exists for all $(x,y) \in [a,b]\times[c,d]$ and is continuous. Define*

$$g(y) := \int_a^b f(x,y)\, dx.$$

Then $g\colon [c,d] \to \mathbb{R}$ is differentiable and

$$g'(y) = \int_a^b \frac{\partial f}{\partial y}(x,y)\, dx.$$

Note that the continuity requirements for f and $\frac{\partial f}{\partial y}$ can be weakened but not dropped outright. The main point is for $\frac{\partial f}{\partial y}$ to exist and be continuous for a small interval in the y direction. In applications, the $[c,d]$ can be made a very small interval around the point where you need to differentiate.

Proof. Fix $y \in [c,d]$ and let $\varepsilon > 0$ be given. As $\frac{\partial f}{\partial y}$ is continuous on $[a,b] \times [c,d]$ it is uniformly continuous. In particular, there exists $\delta > 0$ such that whenever $y_1 \in [c,d]$ with $|y_1 - y| < \delta$ and all $x \in [a,b]$ we have

$$\left| \frac{\partial f}{\partial y}(x,y_1) - \frac{\partial f}{\partial y}(x,y) \right| < \varepsilon.$$

Now suppose h is such that $y+h \in [c,d]$ and $|h| < \delta$. Fix x for a moment and apply mean value theorem to find a y_1 between y and $y+h$ such that

$$\frac{f(x,y+h) - f(x,y)}{h} = \frac{\partial f}{\partial y}(x,y_1).$$

If $|h| < \delta$ then

$$\left| \frac{f(x,y+h) - f(x,y)}{h} - \frac{\partial f}{\partial y}(x,y) \right| = \left| \frac{\partial f}{\partial y}(x,y_1) - \frac{\partial f}{\partial y}(x,y) \right| < \varepsilon.$$

This argument worked for every $x \in [a,b]$. Therefore, as a function of x

$$x \mapsto \frac{f(x,y+h) - f(x,y)}{h} \qquad \text{converges uniformly to} \qquad x \mapsto \frac{\partial f}{\partial y}(x,y) \qquad \text{as } h \to 0.$$

We only defined uniform convergence for sequences although the idea is the same. If you wish you can replace h with $1/n$ above and let $n \to \infty$.

Now consider the difference quotient

$$\frac{g(y+h) - g(y)}{h} = \frac{\int_a^b f(x,y+h)\,dx - \int_a^b f(x,y)\,dx}{h} = \int_a^b \frac{f(x,y+h) - f(x,y)}{h}\,dx.$$

Uniform convergence can be taken underneath the integral and therefore

$$\lim_{h \to 0} \frac{g(y+h) - g(y)}{h} = \int_a^b \lim_{h \to 0} \frac{f(x,y+h) - f(x,y)}{h}\,dx = \int_a^b \frac{\partial f}{\partial y}(x,y)\,dx.$$

□

Example 9.1.2: Let

$$f(y) = \int_0^1 \sin(x^2 - y^2)\,dx.$$

Then

$$f'(y) = \int_0^1 -2y\cos(x^2 - y^2)\,dx.$$

Example 9.1.3: Suppose we start with

$$\int_0^1 \frac{x-1}{\ln(x)}\, dx.$$

The function under the integral can be checked to be continuous, and in fact extends to be continuous on $[0,1]$, and hence the integral exists. Trouble is finding it. Introduce a parameter y and define a function.

$$f(y) := \int_0^1 \frac{x^y-1}{\ln(x)}\, dx.$$

Again it can be checked that $\frac{x^y-1}{\ln(x)}$ is a continuous function (that is extends to be continuous) of x and y for $(x,y) \in [0,1]\times[0,1]$. Therefore f is a continuous function of on $[0,1]$. In particular $f(0) = 0$. For any $\varepsilon > 0$, the y derivative of the integrand, that is x^y is continuous on $[0,1]\times[\varepsilon,1]$. Therefore, for $y > 0$ we can differentiate under the integral sign

$$f'(y) = \int_0^1 \frac{\ln(x)x^y}{\ln(x)}\, dx = \int_0^1 x^y\, dx = \frac{1}{y+1}.$$

We need to figure out $f(1)$, knowing $f'(y) = \frac{1}{y+1}$ and $f(0) = 0$. By elementary calculus we find $f(1) = \int_0^1 f'(y)\, dy = \ln(2)$. Therefore

$$\int_0^1 \frac{x-1}{\ln(x)}\, dx = \ln(2).$$

Exercise* 9.1.1*: *Prove the two statements that were asserted in the example.*
a) Prove $\frac{x-1}{\ln(x)}$ *extends to a continuous function of* $[0,1]$.
b) Prove $\frac{x^y-1}{\ln(x)}$ *extends to be a continuous function on* $[0,1]\times[0,1]$.

9.1.1 Exercises

Exercise* 9.1.2*: *Suppose* $h\colon \mathbb{R}\to\mathbb{R}$ *is a continuous function. Suppose that* $g\colon \mathbb{R}\to\mathbb{R}$ *is which is continuously differentiable and compactly supported. That is there exists some* $M > 0$ *such that* $g(x) = 0$ *whenever* $|x| \geq M$*. Define*

$$f(x) := \int_{-\infty}^{\infty} h(y)g(x-y)\, dy.$$

Show that f *is differentiable.*

Exercise* 9.1.3*: *Suppose* f *is an infinitely differentiable function (all derivatives exist) such that* $f(0) = 0$*. Then show that there exists another infinitely differentiable function* $g(x)$ *such that* $f(x) = xg(x)$*. Finally show that if* $f'(0) \neq 0$ *then* $g(0) \neq 0$*. Hint: first write* $f(x) = \int_0^x f'(s)ds$ *and then rewrite the integral to go from* 0 *to 1.*

Exercise* 9.1.4*: *Compute $\int_0^1 e^{tx}\,dx$. Derive the formula for $\int_0^1 x^n e^x\,dx$ not using itnegration by parts, but by differentiation underneath the integral.*

Exercise* 9.1.5*: *Let $U \subset \mathbb{R}^n$ be an open set and suppose $f(x, y_1, y_2, \ldots, y_n)$ is a continuous function defined on $[0,1] \times U \subset \mathbb{R}^{n+1}$. Suppose $\frac{\partial f}{\partial y_1}, \frac{\partial f}{\partial y_2}, \ldots, \frac{\partial f}{\partial y_n}$ exist and are continuous on $[0,1] \times U$. Then prove that $F \colon U \to \mathbb{R}$ defined by*

$$F(y_1, y_2, \ldots, y_n) := \int_0^1 f(x, y_1, y_2, \ldots, y_n)\,dx$$

is continuously differentiable.

9.2 Path integrals

Note: ??? lectures

9.2.1 Piecewise smooth paths

Definition 9.2.1. A continuously differentiable function $\gamma\colon [a,b] \to \mathbb{R}^n$ is called a *smooth path* or a *continuously differentiable path** if γ is continuously differentiable and $\gamma'(t) \neq 0$ for all $t \in [a,b]$.

The function γ is called a *piecewise smooth path* or a *piecewise continuously differentiable path* if there exist finitely many points $t_0 = a < t_1 < t_2 < \cdots < t_k = b$ such that the restriction of the function $\gamma|_{[t_{j-1},t_j]}$ is smooth path.

We say γ is a *simple path* if $\gamma|_{(a,b)}$ is a one-to-one function. A γ is a *closed path* if $\gamma(a) = \gamma(b)$, that is if the path starts and ends in the same point.

Since γ is a function of one variable, we have seen before that treating $\gamma'(t)$ as a matrix is equivalent to treating it as a vector since it is an $n \times 1$ matrix, that is, a column vector. In fact, by an exercise, even the operator norm of $\gamma'(t)$ is equal to the euclidean norm. Therefore, we will write $\gamma'(t)$ as a vector as is usual.

Generally, it is the direct image $\gamma\bigl([a,b]\bigr)$ that is what we are interested in. We will informally talk about a curve, by which we will generally mean the set $\gamma\bigl([a,b]\bigr)$.

Example 9.2.2: Let $\gamma\colon [0,4] \to \mathbb{R}^2$ be defined by

$$\gamma(t) := \begin{cases} (t,0) & \text{if } t \in [0,1], \\ (1,t-1) & \text{if } t \in (1,2], \\ (3-t,1) & \text{if } t \in (2,3], \\ (0,4-t) & \text{if } t \in (3,4]. \end{cases}$$

Then the reader can check that the path is the unit square traversed counterclockwise. We can check that for example $\gamma|_{[1,2]}(t) = (1,t-1)$ and therefore $(\gamma|_{[1,2]})'(t) = (0,1) \neq 0$. It is good to notice at this point that $(\gamma|_{[1,2]})'(1) = (0,1)$, $(\gamma|_{[0,1]})'(1) = (1,0)$, and $\gamma'(1)$ does not exist. That is, at the corners γ is of course not differentiable, even though the restrictions are differentiable and the derivative depends on which restriction you take.

Example 9.2.3: The condition that $\gamma'(t) \neq 0$ means that the image of γ has no "corners" where γ is smooth. For example, the function

$$\gamma(t) := \begin{cases} (t^2,0) & \text{if } t < 0, \\ (0,t^2) & \text{if } t \geq 0. \end{cases}$$

*Note that the word "smooth" is used sometimes of continuously differentiable, sometimes for infinitely differentiable in the literature.

It is left for the reader to check that γ is continuously differentiable, yet the image $\gamma(\mathbb{R}) = \{(x,y) \in \mathbb{R}^2 : (x,y) = (-s,0) \text{ or } (x,y) = (0,s) \text{ for some } s \geq 0\ \}$ has a "corner" at the origin.

Example 9.2.4: A graph of a continuously differentiable function $f\colon [a,b] \to \mathbb{R}$ is a smooth path. That is, define $\gamma\colon [a,b] \to \mathbb{R}^2$ by

$$\gamma(t) := \bigl(t, f(t)\bigr).$$

Then $\gamma'(t) = \bigl(1, f'(t)\bigr)$, which is never zero.

There are other ways of parametrizing the path. That is, having a different path with the same image. For example, the function that takes t to $(1-t)a+tb$, takes the interval $[0,1]$ to $[a,b]$. So let $\alpha\colon [0,1] \to \mathbb{R}^2$ be defined by

$$\alpha(t) := \bigl((1-t)a+tb, f((1-t)a+tb)\bigr).$$

Then $\alpha'(t) = \bigl(b-a, (b-a)f'((1-t)a+tb)\bigr)$, which is never zero. Furthermore as sets $\alpha\bigl([0,1]\bigr) = \gamma\bigl([a,b]\bigr) = \{(x,y) \in \mathbb{R}^2 : x \in [a,b] \text{ and } f(x) = y\}$, which is just the graph of f.

The last example leads us to a definition.

Definition 9.2.5. Let $\gamma\colon [a,b] \to \mathbb{R}^n$ be a piecewise smooth path and $h\colon [c,d] \to [a,b]$ a continuously differentiable function such that $h'(t) \neq 0$ for all $t \in [c,d]$. Then the composition $\gamma \circ h$ is called a *smooth reparametrization* of γ. If $h'(t) < 0$ for $t \in [c,d]$, then h is said to *reverses orientation*. If h does not reverse orientation then h is said to *preserve orientation*.

A reparametrization is another path for the same set. That is, $(\gamma \circ h)\bigl([c,d]\bigr) = \gamma\bigl([a,b]\bigr)$.

Let us remark that since the function h' is continuous and $h'(t) \neq 0$ for all $t \in [c,d]$, then if $h'(t) < 0$ for one $t \in [c,d]$, then $h'(t) < 0$ for all $t \in [c,d]$ by the intermediate value theorem. That is, $h'(t)$ has the same sign at every $t \in [c,d]$.

Proposition 9.2.6. *If $\gamma\colon [a,b] \to \mathbb{R}^n$ is a piecewise smooth path, and $\gamma \circ h\colon [c,d] \to \mathbb{R}^n$ is a smooth reparametrization, then $\gamma \circ h$ is a piecewise smooth path.*

Proof. If $h\colon [c,d] \to [a,b]$ gives a smooth reparametrization, then as $h'(t)$ has the same sign for all $t \in [c,d]$. Hence it is a bijective mapping with a continuously differentiable inverse. Suppose $t_0 = a < t_1 < t_2 < \cdots < t_k = b$ is the partition from the definition of piecewise smooth for γ.

Suppose first that h preserves orientation, that is h is strictly increasing. Let $s_j := h^{-1}(t_j)$. Then $s_0 = c < s_1 < s_2 < \cdots < s_k = d$. For $t \in [s_{j-1}, s_j]$ notice that $h(t) \in [t_{j-1}, t_j]$ and so

$$(\gamma \circ h)|_{[s_{j-1},s_j]}(t) = \gamma|_{[t_{j-1},t_j]}\bigl(h(t)\bigr).$$

The function $(\gamma \circ h)|_{[s_{j-1},s_j]}$ is therefore continuously differentiable and by the chain rule

$$\bigl((\gamma \circ h)|_{[s_{j-1},s_j]}\bigr)'(t) = \bigl(\gamma|_{[t_{j-1},t_j]}\bigr)'\bigl(h(t)\bigr)h'(t) \neq 0.$$

Therefore $\gamma \circ h$ is a piecewise smooth path. $\square$

One need not have the reparametrization be smooth at all points, it really only needs to be again "piecewise smooth," but the above definition will suffice for us and it keeps matters simpler.

9.2.2 Path integral of a one-form

Definition 9.2.7. If $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ are our coordinates, and given n real-valued continuous functions $f_1, f_2, \ldots, f_n$ defined on some set $S \subset \mathbb{R}^n$ we define a so called one-form:

$$\omega = f_1 dx_1 + f_2 dx_2 + \cdots f_n dx_n.$$

That is, we could represent ω as a function from S to $\mathbb{R}^n$.

Example 9.2.8: For example,

$$\omega(x,y) = \frac{-y}{x^2+y^2} dx + \frac{x}{x^2+y^2} dy$$

is a one-form defined on $\mathbb{R}^2 \setminus \{(0,0)\}$.

Definition 9.2.9. Let $\gamma\colon [a,b] \to \mathbb{R}^n$ be a smooth path and

$$\omega = f_1 dx_1 + f_2 dx_2 + \cdots f_n dx_n,$$

a one-form defined on the direct image $\gamma\bigl([a,b]\bigr)$. Let $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n)$ be the components of γ. Then define

$$\begin{aligned} \int_\gamma \omega &:= \int_a^b \Bigl(f_1\bigl(\gamma(t)\bigr) \gamma_1'(t) + f_2\bigl(\gamma(t)\bigr) \gamma_2'(t) + \cdots + f_n\bigl(\gamma(t)\bigr) \gamma_n'(t) \Bigr)\, dt \\ &= \int_a^b \left(\sum_{j=1}^n f_j\bigl(\gamma(t)\bigr) \gamma_j'(t) \right) dt. \end{aligned}$$

If γ is piecewise smooth with the corresponding partition $t_0 = a < t_1 < t_2 < \ldots < t_k = b$, then each $\gamma|_{[t_{j-1},t_j]}$ is a smooth path and we define

$$\int_\gamma \omega := \int_{\gamma|_{[t_0,t_1]}} \omega + \int_{\gamma|_{[t_1,t_2]}} \omega + \cdots + \int_{\gamma|_{[t_{n-1},t_n]}} \omega.$$

The notation makes sense from the formula you remember from calculus, let us state it somewhat informally: if $x_j(t) = \gamma_j(t)$, then $dx_j = \gamma_j'(t) dt$.

Example 9.2.10: Let the one-form ω and the path $\gamma\colon [0,2\pi] \to \mathbb{R}^2$ be defined by

$$\omega(x,y) := \frac{-y}{x^2+y^2} dx + \frac{x}{x^2+y^2} dy, \qquad \gamma(t) := \bigl(\cos(t), \sin(t)\bigr).$$

Then

$$\begin{aligned} \int_\gamma \omega &= \int_0^{2\pi} \left(\frac{-\sin(t)}{\bigl(\cos(t)\bigr)^2 + \bigl(\sin(t)\bigr)^2} \bigl(-\sin(t)\bigr) + \frac{\cos(t)}{\bigl(\cos(t)\bigr)^2 + \bigl(\sin(t)\bigr)^2} \bigl(\cos(t)\bigr) \right) dt \\ &= \int_0^{2\pi} 1 \, dt = 2\pi. \end{aligned}$$

Next, let us parametrize the same curve as $\alpha \colon [0,1] \to \mathbb{R}^2$ defined by $\alpha(t) := \big(\cos(2\pi t), \sin(2\pi t)\big)$, that is α is a smooth reparametrization of γ. Then

$$\begin{aligned}\int_\alpha \omega &= \int_0^1 \Bigg(\frac{-\sin(2\pi t)}{\big(\cos(2\pi t)\big)^2 + \big(\sin(2\pi t)\big)^2} \big(-2\pi \sin(2\pi t)\big) \\ &\qquad + \frac{\cos(2\pi t)}{\big(\cos(2\pi t)\big)^2 + \big(\sin(2\pi t)\big)^2} \big(2\pi \cos(2\pi t)\big) \Bigg)\, dt \\ &= \int_0^1 2\pi \, dt = 2\pi.\end{aligned}$$

Now let us reparametrize with $\beta \colon [0,2\pi] \to \mathbb{R}^2$ as $\beta(t) := \big(\cos(-t), \sin(-t)\big)$. Then

$$\begin{aligned}\int_\beta \omega &= \int_0^{2\pi} \left(\frac{-\sin(-t)}{\big(\cos(-t)\big)^2 + \big(\sin(-t)\big)^2} \big(\sin(-t)\big) + \frac{\cos(-t)}{\big(\cos(-t)\big)^2 + \big(\sin(-t)\big)^2} \big(-\cos(-t)\big) \right) dt \\ &= \int_0^{2\pi} (-1)\, dt = -2\pi.\end{aligned}$$

Now, α was an orientation preserving reparametrization of γ, and the integral was the same. On the other hand β is an orientation reversing reparametrization and the integral was minus the original.

The previous example is not a fluke. The path integral does not depend on the parametrization of the curve, the only thing that matters is the direction in which the curve is traversed.

Proposition 9.2.11. *Let $\gamma \colon [a,b] \to \mathbb{R}^n$ be a piecewise smooth path and $\gamma \circ h \colon [c,d] \to \mathbb{R}^n$ a smooth reparametrization. Suppose ω is a one-form defined on the set $\gamma\big([a,b]\big)$. Then*

$$\int_{\gamma \circ h} \omega = \begin{cases} \int_\gamma \omega & \text{if } h \text{ preserves orientation,} \\ -\int_\gamma \omega & \text{if } h \text{ reverses orientation.} \end{cases}$$

Proof. Write the one form as $\omega = f_1 dx_1 + f_2 dx_2 + \cdots + f_n dx_n$. Suppose first that h is orientation preserving. Using the definition of the path integral and the change of variables formula for the Riemann integral,

$$\begin{aligned}\int_\gamma \omega &= \int_a^b \left(\sum_{j=1}^n f_j\big(\gamma(t)\big) \gamma_j'(t) \right) dt \\ &= \int_c^d \left(\sum_{j=1}^n f_j\Big(\gamma\big(h(\tau)\big)\Big) \gamma_j'\big(h(\tau)\big) \right) d\tau \\ &= \int_c^d \left(\sum_{j=1}^n f_j\Big(\gamma\big(h(\tau)\big)\Big) (\gamma_j \circ h)'(\tau) \right) d\tau \\ &= \int_{\gamma \circ h} \omega.\end{aligned}$$

If h is orientation reversing it will swap the order of the limits on the integral introducing a minus sign. The details, along with finishing the proof for piecewise smooth paths is left to the reader as Exercise 9.2.1. □

Due to this proposition, if we have a a set $\Gamma \subset \mathbb{R}^n$ which is the image of a simple piecewise smooth path $\gamma\big([a,b]\big)$, then if we somehow indicate the orientation, that is, which direction we traverse the curve, in other words where we start and where we finish, then we can just write

$$\int_\Gamma \omega,$$

without mentioning the specific γ. Recall that *simple* means that γ restricted to (a,b) is one-to-one, that is, it is one-to-one except perhaps at the endpoints. We can also often relax the simple path condition a little bit. That is, as long as $\gamma\colon [a,b] \to \mathbb{R}^n$ is one-to-one except at finitely many points. That is, there are only finitely many points $p \in \mathbb{R}^n$ such that $\gamma^{-1}(p)$ is more than one point. See the exercises. The issue about the injectivity problem is illustrated by the following example:

Example 9.2.12: Suppose $\gamma\colon [0,2\pi] \to \mathbb{R}^2$ is given by $\gamma(t) := \big(\cos(t), \sin(t)\big)$ and $\beta : [0,2\pi] \to \mathbb{R}^2$ is given by $\beta(t) := \big(\cos(2t), \sin(2t)\big)$. Notice that $\gamma\big([0,2\pi]\big) = \beta\big([0,2\pi]\big)$, and we travel around the same curve, the unit circle. But γ goes around the unit circle once in the counter clockwise direction, and β goes around the unit circle twice (in the same direction). Then

$$\begin{aligned}
\int_\gamma -y\,dx + x\,dy &= \int_0^{2\pi} \Big(\big(-\sin(t)\big)\big(-\sin(t)\big) + \cos(t)\cos(t)\Big)dt = 2\pi, \\
\int_\beta -y\,dx + x\,dy &= \int_0^{2\pi} \Big(\big(-\sin(2t)\big)\big(-2\sin(2t)\big) + \cos(t)\big(2\cos(t)\big)\Big)dt = 4\pi.
\end{aligned}$$

Furthermore, for a simple closed path, it does not even matter where we start the parametrization. See the exercises.

Paths can be cut up or concatenated as follows. The proof is a direct application of the additivity of the Riemann integral, and is left as an exercise.

Proposition 9.2.13. *Let $\gamma\colon [a,c] \to \mathbb{R}^n$ be a piecewise smooth path. For some $b \in (a,c)$, define the piecewise smooth paths $\alpha = \gamma|_{[a,b]}$ and $\beta = \gamma|_{[b,c]}$. For a one-form ω defined on the curve defined by γ we have*

$$\int_\gamma \omega = \int_\alpha \omega + \int_\beta \omega.$$

9.2.3 Line integral of a function

Sometimes we wish to simply integrate a function against the so called arc length measure.

Definition 9.2.14. Suppose $\gamma\colon [a,b] \to \mathbb{R}^n$ is a smooth path, and f is a continuous function defined on the image $\gamma\big([a,b]\big)$. Then define

$$\int_\gamma f\,ds := \int_a^b f\big(\gamma(t)\big)\,\lVert\gamma'(t)\rVert\,dt.$$

The definition for a piecewise smooth path is similar as before and is left to the reader.

The geometric idea of this integral is to find the "area under the graph of a function" as we move around the path γ. The line integral of a function is also independent of the parametrization, but in this case, the orientation does not matter.

Proposition 9.2.15. *Let $\gamma\colon [a,b] \to \mathbb{R}^n$ be a piecewise smooth path and $\gamma \circ h\colon [c,d] \to \mathbb{R}^n$ a smooth reparametrization. Suppose f is a continuous function defined on the set $\gamma\big([a,b]\big)$. Then*

$$\int_{\gamma\circ h} f\,ds = \int_\gamma f\,ds.$$

Proof. Suppose first that h is orientation preserving and γ is a smooth path. Then as before

$$\begin{aligned}
\int_\gamma f\,ds &= \int_a^b f(t)\,\lVert\gamma'(t)\rVert\,dt \\
&= \int_c^d f\big(h(\tau)\big)\,\lVert\gamma'\big(h(\tau)\big)\rVert h'(\tau)\,d\tau \\
&= \int_c^d f\big(h(\tau)\big)\,\lVert\gamma'\big(h(\tau)\big)h'(\tau)\rVert\,d\tau \\
&= \int_c^d f\big(h(\tau)\big)\,\lVert(\gamma\circ h)'(\tau)\rVert\,d\tau \\
&= \int_{\gamma\circ h} f\,ds.
\end{aligned}$$

If h is orientation reversing it will swap the order of the limits on the integral but you also have to introduce a minus sign in order to have h' inside the norm. The details, along with finishing the proof for piecewise smooth paths is left to the reader as Exercise 9.2.2. $\square$

Similarly as before, because of this proposition, if γ is simple, it does not matter which parametrization we use. Therefore, if $\Gamma = \gamma\big([a,b]\big)$ we can simply write

$$\int_\Gamma f\,ds.$$

In this case we also do not need to worry about orientation, either way we get the same thing.

Example 9.2.16: Let $f(x,y) = x$. Let $C \subset \mathbb{R}^2$ be half of the unit circle for $x \geq 0$. We wish to compute

$$\int_C f\, ds.$$

Parametrize C by $\gamma\colon [-\pi/2, \pi/2] \to \mathbb{R}^2$ given by $\gamma(t) = \big(\cos(t), \sin(t)\big)$. Then $\gamma'(t) = \big(-\sin(t), \cos(t)\big)$, and

$$\int_C f\, ds = \int_\gamma f\, ds = \int_{-\pi/2}^{\pi/2} \cos(t) \sqrt{\big(-\sin(t)\big)^2 + \big(\cos(t)\big)^2}\, dt = \int_{-\pi/2}^{\pi/2} \cos(t)\, dt = 2.$$

Definition 9.2.17. Suppose $\Gamma \subset \mathbb{R}^n$ is parametrized by a simple piecewise smooth path $\gamma\colon [a,b] \to \mathbb{R}^n$, that is $\gamma\big([a,b]\big) = \Gamma$. The we define the *length* by

$$\ell(\Gamma) := \int_\Gamma ds = \int_\gamma ds = \int_a^b \lVert \gamma'(t) \rVert\, dt.$$

Example 9.2.18: Let $x, y \in \mathbb{R}^n$ be two points and write $[x,y]$ as the straight line segment between the two points x and y. We can parametrize $[x,y]$ by $\gamma(t) := (1-t)x + ty$ for t running between 0 and 1. We note that $\gamma'(t) = y - x$ and therefore we compute

$$\ell\big([x,y]\big) = \int_{[x,y]} ds = \int_0^1 \lVert y - x \rVert\, dt = \lVert y - x \rVert.$$

So the length of $[x,y]$ is the distance between x and y in the euclidean metric.

A simple piecewise smooth path $\gamma\colon [0,r] \to \mathbb{R}^n$ is said to be an *arc length parametrization* if

$$\ell\big(\gamma([0,t])\big) = \int_0^t \lVert \gamma'(\tau) \rVert\, d\tau = t.$$

You can think of such a parametrization as moving around your curve at speed 1.

9.2.4 Exercises

Exercise* 9.2.1*: *Finish the proof of Proposition 9.2.11 for a) orientation reversing reparametrizations and b) piecewise smooth paths.*

Exercise* 9.2.2*: *Finish the proof of Proposition 9.2.15 for a) orientation reversing reparametrizations and b) piecewise smooth paths.*

Exercise* 9.2.3*: *Prove Proposition 9.2.13.*

Exercise* 9.2.4*: *Suppose $\gamma\colon [a,b] \to \mathbb{R}^n$ is a piecewise smooth path, and f is a continuous function defined on the image $\gamma\big([a,b]\big)$. Provide a definition of $\int_\gamma f\, ds$.*

***Exercise* 9.2.5:** *Compute the length of the unit square from Example 9.2.2 using the given parametrization.*

***Exercise* 9.2.6:** *Suppose $\gamma \colon [0,1] \to \mathbb{R}^n$ is a piecewise smooth path, and ω is a one-form defined on the image $\gamma\bigl([a,b]\bigr)$. For $r \in [0,1]$, let $\gamma_r \colon [0,r] \to \mathbb{R}^n$ be defined as simply the restriction of γ to $[0,r]$. Show that the function $h(r) := \int_{\gamma_r} \omega$ is a continuously differentiable function on $[0,1]$.*

***Exercise* 9.2.7:** *Suppose $\alpha \colon [a,b] \to \mathbb{R}^n$ and $\beta \colon [b,c] \to \mathbb{R}^n$ are piecewise smooth paths with $\alpha(b) = \beta(b)$. Let $\gamma \colon [a,c] \to \mathbb{R}^n$ be defined by*

$$\gamma(t) := \begin{cases} \alpha(t) & \text{if } t \in [a,b], \\ \beta(t) & \text{if } t \in (b,c]. \end{cases}$$

Show that γ is a piecewise smooth path, and that if ω is a one-form defined on the curve given by γ, then

$$\int_\gamma \omega = \int_\alpha \omega + \int_\beta \omega.$$

***Exercise* 9.2.8:** *Suppose $\gamma \colon [a,b] \to \mathbb{R}^n$ and $\beta \colon [c,d] \to \mathbb{R}^n$ are two simple piecewise smooth closed paths. That is $\gamma(a) = \gamma(b)$ and $\beta(c) = \beta(d)$ and the restrictions $\gamma|_{(a,b)}$ and $\beta|_{(c,d)}$ are one-to-one. Suppose $\Gamma = \gamma\bigl([a,b]\bigr) = \beta\bigl([c,d]\bigr)$ and ω is a one-form defined on $\Gamma \subset \mathbb{R}^n$. Show that either*

$$\int_\gamma \omega = \int_\beta \omega, \qquad \text{or} \qquad \int_\gamma \omega = -\int_\beta \omega.$$

In particular, the notation $\int_\Gamma \omega$ makes sense if we indicate the direction in which the integral is evaluated.

***Exercise* 9.2.9:** *Suppose $\gamma \colon [a,b] \to \mathbb{R}^n$ and $\beta \colon [c,d] \to \mathbb{R}^n$ are two piecewise smooth paths which are one-to-one except at finitely many points. That is, there is at most finitely many points $p \in \mathbb{R}^n$ such that $\gamma^{-1}(p)$ or $\beta^{-1}(p)$ contains more than one point. Suppose $\Gamma = \gamma\bigl([a,b]\bigr) = \beta\bigl([c,d]\bigr)$ and ω is a one-form defined on $\Gamma \subset \mathbb{R}^n$. Show that either*

$$\int_\gamma \omega = \int_\beta \omega, \qquad \text{or} \qquad \int_\gamma \omega = -\int_\beta \omega.$$

In particular, the notation $\int_\Gamma \omega$ makes sense if we indicate the direction in which the integral is evaluated.

9.3 Path independence

Note: ??? lectures

9.3.1 Path independent integrals

Let $U \subset \mathbb{R}^n$ be a set and ω a one-form defined on U, and x and y two points in U. The integral of ω from x to y is said to be *path independent* if for any two piecewise smooth paths $\gamma\colon [a,b] \to \mathbb{R}^n$ and $\beta\colon [c,d] \to \mathbb{R}^n$ such that $\gamma(a) = \beta(c) = x$ and $\gamma(b) = \beta(d) = y$ we have

$$\int_\gamma \omega = \int_\beta \omega.$$

In this case we simply write

$$\int_x^y \omega = \int_\gamma \omega = \int_\beta \omega.$$

Not every one-form gives a path independent integral. In fact, most do not.

Example 9.3.1: Let $\gamma\colon [0,1] \to \mathbb{R}^2$ be the path $\gamma(t) = (t,0)$ going from $(0,0)$ to $(1,0)$. Let $\beta\colon [0,1] \to \mathbb{R}^2$ be the path $\beta(t) = \bigl(t,(1-t)t\bigr)$ also going between the same points. Then

$$\begin{aligned}
\int_\gamma y dx &= \int_0^1 \gamma_2(t)\gamma_1'(t)\,dt \int_0^1 0(1)dt = 0, \\
\int_\beta y dx &= \int_0^1 \beta_2(t)\beta_1'(t)\,dt \int_0^1 (1-t)t(1)dt = \frac{1}{6}.
\end{aligned}$$

So $\int_{(0,0)}^{(1,0)} y\,dx$ is not path independent.

Definition 9.3.2. Let $U \subset \mathbb{R}^n$ be an open set and $f\colon U \to \mathbb{R}$ a continuously differentiable function. Then the one-form

$$df := \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 + \cdots + \frac{\partial f}{\partial x_n} dx_n$$

is called the *total derivative* of f.

An open set $U \subset \mathbb{R}^n$ is said to be *path connected* if for every two points x and y in U, there exists a piecewise smooth path starting at x and ending at y.

We will leave as an exercise that every connected open set is path connected.

Proposition 9.3.3. *Let $U \subset \mathbb{R}^n$ be a path connected open set and ω a one-form defined on U. Then*

$$\int_x^y \omega$$

is path independent for all $x,y \in U$ if and only if there exists a continuously differentiable $f : U \to \mathbb{R}$ such that $\omega = df$.

In fact, if such an f exists, then for any two pints $x,y \in U$

$$\int_x^y \omega = f(y) - f(x).$$

In other words if we fix x_0, then $f(x) = C + \int_{x_0}^x \omega$.

Proof. First suppose that the integral is path independent. Pick $x_0 \in U$ and define

$$f(x) = \int_{x_0}^x \omega.$$

Let e_j be an arbitrary standard basis vector. Compute

$$\frac{f(x+he_j) - f(x)}{h} = \frac{1}{h}\left(\int_{x_0}^{x+he_j} \omega - \int_{x_0}^{x}\right) = \frac{1}{h}\int_x^{x+he_j} \omega,$$

which follows by Proposition 9.2.13 and path indepdendence as $\int_{x_0}^{x+he_j} \omega = \int_{x_0}^x \omega + \int_x^{x+he_j} \omega$.

Write $\omega = \omega_1 dx_1 + \omega_2 dx_2 + \cdots + \omega_n dx_n$. Now pick the simplest path possible from x to $x+he_j$, that is $\gamma(t) = x + the_j$ for $t \in [0,1]$. Notice that $\gamma'(t)$ has only a simple nonzero component and that is the jth component which is h. Therefore

$$\frac{1}{h}\int_x^{x+he_j} \omega = \frac{1}{h}\int_0^1 \omega_j(x+the_j)h dt = \int_0^1 \omega_j(x+the_j) dt.$$

We wish to take the limit as $h \to 0$. The function ω_j is continuous. So given $\varepsilon > 0$, h can be small enough so that $\left|\omega(x) - \omega_j(x+the_j)\right| < \varepsilon$. Therefore for such small h we find that $\left|\int_0^1 \omega_j(x+the_j)dt - \omega(x)\right| < \varepsilon$. That is

$$\lim_{h\to 0} \frac{f(x+he_j) - f(x)}{h} = \omega_j(x),$$

which is what we wanted that is $df = \omega$. As ω_j are continuous for all j, we find that f has continuous partial derivatives and therefore is continuously differentiable.

For the other direction suppose f exists such that $df = \omega$. Suppose we take a smooth path $\gamma\colon [a,b] \to U$ such that $\gamma(a) = x$ and $\gamma(b) = y$, then

$$\begin{aligned}
\int_\gamma df &= \int_a^b \left(\frac{\partial f}{\partial x_1}\big(\gamma(t)\big)\gamma_1'(t) + \frac{\partial f}{\partial x_2}\big(\gamma(t)\big)\gamma_2'(t) + \cdots + \frac{\partial f}{\partial x_n}\big(\gamma(t)\big)\gamma_n'(t)\right) dt \\
&= \int_a^b \frac{d}{dt}\left[f\big(\gamma(t)\big)\right] dt \\
&= f(y) - f(x).
\end{aligned}$$

Notice that the value of the integral only depends on x and y, not the path taken. Therefore the integral is path independent. We leave checking this for a piecewise smooth path as an exercise to the reader.

This also proves the last assertion since Then $\int_{x_0}^{x} \omega = f(x) - f(x_0)$, letting $C = f(x_0)$ finishes the proof. □

Proposition 9.3.4. *Let $U \subset \mathbb{R}^n$ be a path connected open set and ω a 1-form defined on U. Then $\omega = df$ for some continuously differentiable $f\colon U : \mathbb{R}$ if and only if*

$$\int_{\gamma} \omega = 0$$

for every closed path $\gamma\colon [a,b] \to U$.

Proof. Suppose first that $\omega = df$ and let γ be a closed path. Then we from above we have that

$$\int_{\gamma} \omega = f\big(\gamma(b)\big) - f\big(\gamma(a)\big) = 0,$$

because $\gamma(a) = \gamma(b)$ for a closed path.

Now suppose that for every piecewise smooth closed path γ, $\int_{\gamma} \omega = 0$. Let x, y be two points in U and let $\alpha\colon [0,1] \to U$ and $\beta\colon [0,1] \to U$ be two piecewise smooth paths with $\alpha(0) = \beta(0) = x$ and $\alpha(1) = \beta(1) = y$. Then let $\gamma\colon [0,2] \to U$ be defined by

$$\gamma(t) := \begin{cases} \alpha(t) & \text{if } t \in [0,1], \\ \beta(2-t) & \text{if } t \in (1,2]. \end{cases}$$

This is a piecewise smooth closed path and so

$$0 = \int_{\gamma} \omega = \int_{\alpha} \omega - \int_{\beta} \omega.$$

This follows first by Proposition 9.2.13, and then noticing that the second part is β travelled backwards so that we get minus the β integral. Thus the integral of ω on U is path independent. □

There is a local criterion, that is a differential equation, that guarantees path independence. That is, under the right condition there exists an *antiderivative* f whose total derivative is the given one form ω. However, since the criterion is local, we only get the result locally. We can define the antiderivative in any so-called *simply connected* domain, which informally is a domain with no 1-dimensional holes to go around. To make matters simple, the usual way this result is proved is for so-called star-shaped domains.

Definition 9.3.5. Let $U \subset \mathbb{R}^n$ be an open set and $x_0 \in U$. We say U is a *star shaped domain* with respect to x_0 if for any other point $x \in U$, the line segment between x_0 and x is in U, that is, if $(1-t)x_0 + tx \in U$ for all $t \in [0,1]$. If we say simply *star shaped* then U is star shaped with respect to some $x_0 \in U$.

Notice the difference between star shaped and convex. A convex domain is star shaped, but a star shaped domain need not be convex.

Theorem 9.3.6 (Poincarè lemma)**.** *Let $U \subset \mathbb{R}^n$ be a star shaped domain and ω a continuously differentiable one-form defined on U. That is, if*

$$\omega = \omega_1 dx_1 + \omega_2 dx_2 + \cdots + \omega_n dx_n,$$

then $\omega_1, \omega_2, \ldots, \omega_n$ are continuously differentiable functions. Suppose that for every j and k we have

$$\frac{\partial \omega_j}{\partial x_k} = \frac{\partial \omega_k}{\partial x_j},$$

then there exists a twice continuously differentiable function $f \colon U \to \mathbb{R}$ such that $df = \omega$.

The condition on the derivatives of ω is precisely the condition that the second partial derivatives commute. That is, if $df = \omega$, then

$$\frac{\partial \omega_j}{\partial x_k} = \frac{\partial^2 f}{\partial x_k \partial x_j}.$$

Proof. Suppose U is star shaped with respect to $y = (y_1, y_2, \ldots, y_n) \in U$.

Given $x = (x_1, x_2, \ldots, x_n) \in U$, define the path $\gamma \colon [0,1] \to U$ as $\gamma(t) = (1-t)y + tx$, so $\gamma'(t) = y - x$. Then let

$$f(x) = \int_\gamma \omega = \int_0^1 \left(\sum_{k=1}^n \omega_k\big((1-t)y+tx\big)(y_k - x_k) \right) dt$$

Now we can differentiate in x_j under the integral. We can do that since everything, including the partials themselves are continuous.

$$\begin{aligned}
\frac{\partial f}{\partial x_j}(x) &= \int_0^1 \left(\left(\sum_{k=1}^n \frac{\partial \omega_k}{\partial x_j}\big((1-t)y+tx\big) t (y_k - x_k) \right) - \omega_j\big((1-t)y+tx\big) \right) dt \\
&= \int_0^1 \left(\left(\sum_{k=1}^n \frac{\partial \omega_j}{\partial x_k}\big((1-t)y+tx\big) t (y_k - x_k) \right) - \omega_j\big((1-t)y+tx\big) \right) dt \\
&= \int_0^1 \frac{d}{dt} \left[t\omega_j\big((1-t)y+tx\big) \right] dt \\
&= \omega_j(x).
\end{aligned}$$

And this is precisely what we wanted. □

Example 9.3.7: Without some hypothesis on U the theorem is not true. Let

$$\omega(x,y) = \frac{-y}{x^2+y^2} dx + \frac{x}{x^2+y^2} dy$$

be defined on $\mathbb{R}^2 \setminus \{0\}$. It is easy to see that

$$\frac{\partial}{\partial y}\left[\frac{-y}{x^2+y^2}\right] = \frac{\partial}{\partial x}\left[\frac{x}{x^2+y^2}\right].$$

However, there is no $f\colon \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$ such that $df = \omega$. We saw in if we integrate from $(1,0)$ to $(1,0)$ along the unit circle, that is $\gamma(t) = \big(\cos(t), \sin(t)\big)$ for $t \in [0, 2\pi]$ we got 2π and not 0 as it should be if the integral is path independent or in other words if there would exist an f such that $df = \omega$.

9.3.2 Vector fields

A common object to integrate is a so-called vector field. That is an assignment of a vector at each point of a domain.

Definition 9.3.8. Let $U \subset \mathbb{R}^n$ be a set. A continuous function $v\colon U \to \mathbb{R}^n$ is called a *vector field*. Write $v = (v_1, v_2, \ldots, v_n)$

Given a smooth path $\gamma\colon [a,b] \to \mathbb{R}^n$ with $\gamma\big([a,b]\big) \subset U$ we define the path integral of the vectorfield v as

$$\int_\gamma v \cdot d\gamma := \int_a^b v\big(\gamma(t)\big) \cdot \gamma'(t)\, dt,$$

where the dot in the definition is the standard dot product. Again the definition of a piecewise smooth path is done by integrating over each smooth interval and adding the result.

If we unravel the definition we find that

$$\int_\gamma v \cdot d\gamma = \int_\gamma v_1 dx_1 + v_2 dx_2 + \cdots + v_n dx_n.$$

Therefore what we know about integration of one-forms carries over to the integration of vector fields. For example path independence for integration of vector fields is simply that

$$\int_x^y v \cdot d\gamma$$

is path independent (for any γ) if and only if $v = \nabla f$, that is the gradient of a function. The function f is then called the *potential* for v.

A vector field v whose path integrals are path independent is called a *conservative vector field*. The naming comes from the fact that such vector fields arise in physical systems where a certain quantity, the energy is conserved.

9.3.3 Exercises

Exercise* 9.3.1*: *Find an $f\colon \mathbb{R}^2 \to \mathbb{R}$ such that $df = xe^{x^2+y^2}dx + ye^{x^2+y^2}dy$.*

Exercise* 9.3.2*: *Finish the proof of Proposition 9.3.3, that is, we only proved the second direction for a smooth path, not a piecewise smooth path.*

Exercise* 9.3.3*: *Show that a star shaped domain $U \subset \mathbb{R}^n$ is path connected.*

Exercise* 9.3.4*: *Show that $U := \mathbb{R}^2 \setminus \{(x,y) \in \mathbb{R}^2 : x \leq 0, y = 0\}$ is star shaped and find all points $(x_0,y_0) \in U$ such that U is star shaped with respect to (x_0,y_0).*

Exercise* 9.3.5*: *Let $\gamma\colon [a,b] \to \mathbb{R}^n$ be a simple nonclosed path (so γ is one-to-one). Suppose that ω is a continuously differentiable one-form defined on some open set V with $\gamma\bigl([a,b]\bigr) \subset V$ and $\frac{\partial \omega_j}{\partial x_k} = \frac{\partial \omega_k}{\partial x_j}$ for all j and k. Prove that there exists an open set U with $\gamma\bigl([a,b]\bigr) \subset U \subset V$ and a twice continuously differentiable function $f\colon U \to \mathbb{R}$ such that $df = \omega$.*
Hint 1: $\gamma\bigl([a,b]\bigr)$ is compact.
Hint 2: Piecing together several different functions f can be tricky, but notice that the intersection of any number of balls is always convex as balls are convex, and convex sets are in particular connected (path connected).

Exercise* 9.3.6*: *a) Show that a connected open set is path connected. Hint: Start with two points x and y in a connected set U, and let $U_x \subset U$ is the set of points that are reachable by a path from x and similarly for U_y. Show that both sets are open, since they are nonempty ($x \in U_x$ and $y \in U_y$) it must be that $U_x = U_y = U$.*
b) Prove the converse that is, a path connected set $U \subset \mathbb{R}^n$ is connected. Hint: for contradiction assume there exist two open and disjoint nonempty open sets and then assume there is a piecewise smooth (and therefore continuous) path between a point in one to a point in the other.

Exercise* 9.3.7** (Hard):*** *Take*

$$\omega(x,y) = \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$$

defined on $\mathbb{R}^2 \setminus \{(0,0)\}$. Let $\gamma\colon [a,b] \to \mathbb{R}^2 \setminus \{(0,0)\}$ be a closed piecewise smooth path. Let $R := \{(x,y) \in \mathbb{R}^2 : x \leq 0$ and $y = 0\}$. Suppose that $R \cap \gamma\bigl([a,b]\bigr)$ is a finite set of k points. Then

$$\int_\gamma \omega = 2\pi\ell$$

for some integer ℓ with $|\ell| \leq k$.
Hint 1: First prove that for a path β that starts and end on R but does not intersect it otherwise, you find that $\int_\beta \omega$ is -2π, 0, or 2π. Hint 2: You proved above that $\mathbb{R}^2 \setminus R$ is star shaped.
Note: The number ℓ is called the winding number *it it measures how many times does γ wind around the origin in the clockwise direction.*