Chapter 8

Several variables and partial derivatives

8.1 Vector spaces, linear mappings, and convexity

Note: 2–3 lectures

8.1.1 Vector spaces

The euclidean space $\mathbb{R}^n$ has already made an appearance in the metric space chapter. In this chapter, we will extend the differential calculus we created for one variable to several variables. The key idea in differential calculus is to approximate functions by lines and linear functions. In several variables we must introduce a little bit of linear algebra before we can move on. So let us start with vector spaces and linear functions on vector spaces.

While it is common to use $\vec{x}$ or the bold $\mathbf{x}$ for elements of $\mathbb{R}^n$, especially in the applied sciences, we use just plain x, which is common in mathematics. That is, $v \in \mathbb{R}^n$ is a *vector*, which means $v = (v_1, v_2, \ldots, v_n)$ is an n-tuple of real numbers.*

It is common to write vectors as *column vectors*, that is, $n \times 1$ matrices:

$$v = (v_1, v_2, \ldots, v_n) = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}.$$

We will do so when convenient. We call real numbers *scalars* to distinguish them from vectors.

The set $\mathbb{R}^n$ has a so-called *vector space* structure defined on it. However, even though we will be looking at functions defined on $\mathbb{R}^n$, not all spaces we wish to deal with are equal to $\mathbb{R}^n$. Therefore, let us define the abstract notion of the vector space.

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*Subscripts are used for many purposes, so sometimes we may have several vectors which may also be identified by subscript such as a finite or infinite sequence of vectors $y_1, y_2, \ldots$.

Definition 8.1.1. Let X be a set together with operations of addition, $+ \colon X \times X \to X$, and multiplication, $\cdot \colon \mathbb{R} \times X \to X$, (we usually write ax instead of $a \cdot x$). X is called a *vector space* (or a *real vector space*) if the following conditions are satisfied:

(i) (Addition is associative) If $u, v, w \in X$, then $u + (v + w) = (u + v) + w$.

(ii) (Addition is commutative) If $u, v \in X$, then $u + v = v + u$.

(iii) (Additive identity) There is a $0 \in X$ such that $v + 0 = v$ for all $v \in X$.

(iv) (Additive inverse) For every $v \in X$, there is a $-v \in X$, such that $v + (-v) = 0$.

(v) (Distributive law) If $a \in \mathbb{R}$, $u, v \in X$, then $a(u + v) = au + av$.

(vi) (Distributive law) If $a, b \in \mathbb{R}$, $v \in X$, then $(a + b)v = av + bv$.

(vii) (Multiplication is associative) If $a, b \in \mathbb{R}$, $v \in X$, then $(ab)v = a(bv)$.

(viii) (Multiplicative identity) $1v = v$ for all $v \in X$.

Elements of a vector space are usually called *vectors*, even if they are not elements of $\mathbb{R}^n$ (vectors in the "traditional" sense).

If $Y \subset X$ is a subset that is a vector space itself with the same operations, then Y is called a *subspace* or *vector subspace* of X.

Example 8.1.2: An example vector space is $\mathbb{R}^n$, where addition and multiplication by a constant is done componentwise: if $a \in \mathbb{R}$, $v = (v_1, v_2, \ldots, v_n) \in \mathbb{R}^n$, and $w = (w_1, w_2, \ldots, w_n) \in \mathbb{R}^n$, then

$$
\begin{aligned}
&v + w := (v_1, v_2, \ldots, v_n) + (w_1, w_2, \ldots, w_n) = (v_1 + w_1, v_2 + w_2, \ldots, v_n + w_n), \\
&av := a(v_1, v_2, \ldots, v_n) = (av_1, av_2, \ldots, av_n).
\end{aligned}
$$

In this book we mostly deal with vector spaces that can be often regarded as subsets of $\mathbb{R}^n$, but there are other vector spaces useful in analysis. Let us give a couple of examples.

Example 8.1.3: A trivial example of a vector space (the smallest one in fact) is just $X = \{0\}$. The operations are defined in the obvious way. You always need a zero vector to exist, so all vector spaces are nonempty sets.

Example 8.1.4: The space $C([0,1],\mathbb{R})$ of continuous functions on the interval $[0,1]$ is a vector space. For two functions f and g in $C([0,1],\mathbb{R})$ and $a \in \mathbb{R}$ we make the obvious definitions of $f + g$ and af:

$$
(f + g)(x) := f(x) + g(x), \qquad (af)(x) := a\bigl(f(x)\bigr).
$$

The 0 is the function that is identically zero. We leave it as an exercise to check that all the vector space conditions are satisfied.

Example 8.1.5: The space of polynomials $c_0 + c_1 t + c_2 t^2 + \cdots + c_m t^m$ is a vector space, let us denote it by $\mathbb{R}[t]$ (coefficients are real and the variable is t). The operations can be defined in the same way as for functions above. Suppose we have two polynomials, one of degree m and one of degree n. Assume $n \geq m$ for simplicity. Then

$$(c_0 + c_1 t + c_2 t^2 + \cdots + c_m t^m) + (d_0 + d_1 t + d_2 t^2 + \cdots + d_n t^n) = \\ (c_0 + d_0) + (c_1 + d_1)t + (c_2 + d_2)t^2 + \cdots + (c_m + d_m)t^m + d_{m+1}t^{m+1} + \cdots + d_n t^n$$

and

$$a(c_0 + c_1 t + c_2 t^2 + \cdots + c_m t^m) = (ac_0) + (ac_1)t + (ac_2)t^2 + \cdots + (ac_m)t^m.$$

Despite what it looks like, $\mathbb{R}[t]$ is not equivalent to $\mathbb{R}^n$ for any n. In particular it is not "finite dimensional", we will make this notion precise in just a little bit. One can make a finite dimensional vector subspace by restricting the degree. For example, if we say $\mathscr{P}_n$ is the space of polynomials of degree n or less, then we have a finite dimensional vector space.

The space $\mathbb{R}[t]$ can be thought of as a subspace of $C(\mathbb{R},\mathbb{R})$. If we restrict the range of t to $[0,1]$, $\mathbb{R}[t]$ can be identified with a subspace of $C([0,1],\mathbb{R})$.

It is often better to think of even simpler "finite dimensional" vector spaces using the abstract notion rather than always $\mathbb{R}^n$. It is possible to use other fields than $\mathbb{R}$ in the definition (for example it is common to use the complex numbers $\mathbb{C}$), but let us stick with the real numbers*.

8.1.2 Linear combinations and dimension

Definition 8.1.6. Suppose X is a vector space, $x_1, x_2, \ldots, x_k \in X$ are vectors, and $a_1, a_2, \ldots, a_k \in \mathbb{R}$ are scalars. Then

$$a_1 x_1 + a_2 x_2 + \cdots + a_k x_k$$

is called a *linear combination* of the vectors $x_1, x_2, \ldots, x_k$.

If $Y \subset X$ is a set then the *span* of Y, or in notation $\operatorname{span}(Y)$, is the set of all linear combinations of some finite number of elements of Y. We also say Y *spans* $\operatorname{span}(Y)$.

Example 8.1.7: Let $Y := \{(1,1)\} \subset \mathbb{R}^2$. Then

$$\operatorname{span}(Y) = \{(x,x) \in \mathbb{R}^2 : x \in \mathbb{R}\}.$$

That is, $\operatorname{span}(Y)$ is the line through the origin and the point $(1,1)$.

Example 8.1.8: Let $Y := \{(1,1),(0,1)\} \subset \mathbb{R}^2$. Then

$$\operatorname{span}(Y) = \mathbb{R}^2,$$

as any point $(x,y) \in \mathbb{R}^2$ can be written as a linear combination

$$(x,y) = x(1,1) + (y-x)(0,1).$$

*If you want a very funky vector space over a different field, $\mathbb{R}$ itself is a vector space over the rational numbers.

A sum of two linear combinations is again a linear combination, and a scalar multiple of a linear combination is a linear combination, which proves the following proposition.

Proposition 8.1.9. *Let X be a vector space. For any $Y \subset X$, the set* $\operatorname{span}(Y)$ *is a vector space itself. That is,* $\operatorname{span}(Y)$ *is a subspace of X.*

If Y is already a vector space then $\operatorname{span}(Y) = Y$.

Definition 8.1.10. A set of vectors $\{x_1, x_2, \ldots, x_k\} \subset X$ is *linearly independent*, if the only solution to

$$a_1 x_1 + a_2 x_2 + \cdots + a_k x_k = 0 \tag{8.1}$$

is the trivial solution $a_1 = a_2 = \cdots = a_k = 0$. A set that is not linearly independent, is *linearly dependent*.

A linearly independent set B of vectors such that $\operatorname{span}(B) = X$ is called a *basis* of X. For example the set Y of the two vectors in Example 8.1.8 is a basis of $\mathbb{R}^2$.

If a vector space X contains a linearly independent set of d vectors, but no linearly independent set of $d+1$ vectors then we say the *dimension* or $\dim X := d$. If for all $d \in \mathbb{N}$ the vector space X contains a set of d linearly independent vectors, we say X is infinite dimensional and write $\dim X := \infty$.

Clearly for the trivial vector space, $\dim \{0\} = 0$. We will see in a moment that any vector space that is a subset of $\mathbb{R}^n$ has a finite dimension, and that dimension is less than or equal to n.

If a set is linearly dependent, then one of the vectors is a linear combination of the others. In other words, in (8.1) if $a_j \neq 0$, then we can solve for x_j

$$x_j = \frac{a_1}{a_j} x_1 + \cdots + \frac{a_{j-1}}{a_j} x_{j-1} + \frac{a_{j+1}}{a_j} x_{j+1} + \cdots + \frac{a_k}{a_k} x_k.$$

Clearly then the vector x_j has at least two different representations as linear combinations of $\{x_1, x_2, \ldots, x_k\}$.

Proposition 8.1.11. *If $B = \{x_1, x_2, \ldots, x_k\}$ is a basis of a vector space X, then every point $y \in X$ has a unique representation of the form*

$$y = \sum_{j=1}^{k} a_j x_j$$

for some scalars $a_1, a_2, \ldots, a_k$.

Proof. Every $y \in X$ is a linear combination of elements of B since X is the span of B. For uniqueness suppose

$$y = \sum_{j=1}^{k} a_j x_j = \sum_{j=1}^{k} b_j x_j,$$

then

$$\sum_{j=1}^{k} (a_j - b_j) x_j = 0.$$

By linear independence of the basis $a_j = b_j$ for all j. □

For $\mathbb{R}^n$ we define

$$e_1 := (1,0,0,\ldots,0), \quad e_2 := (0,1,0,\ldots,0), \quad \ldots, \quad e_n := (0,0,0,\ldots,1),$$

and call this the *standard basis* of $\mathbb{R}^n$. We use the same letters e_j for any $\mathbb{R}^n$, and which space $\mathbb{R}^n$ we are working in is understood from context. A direct computation shows that $\{e_1, e_2, \ldots, e_n\}$ is really a basis of $\mathbb{R}^n$; it spans $\mathbb{R}^n$ and is linearly independent. In fact,

$$x = (x_1, x_2, \ldots, x_n) = \sum_{j=1}^{n} x_j e_j.$$

Proposition 8.1.12. *Let X be a vector space and d a nonnegative integer.*

(i) If X is spanned by d vectors, then $\dim X \leq d$.

(ii) $\dim X = d$ *if and only if X has a basis of d vectors (and so every basis has d vectors).*

(iii) In particular, $\dim \mathbb{R}^n = n$.

(iv) If $Y \subset X$ is a vector subspace and $\dim X = d$*, then* $\dim Y \leq d$.

(v) If $\dim X = d$ *and a set T of d vectors spans X, then T is linearly independent.*

(vi) If $\dim X = d$ *and a set T of m vectors is linearly independent, then there is a set S of $d - m$ vectors such that $T \cup S$ is a basis of X.*

Proof. Let us start with (i). Suppose $S = \{x_1, x_2, \ldots, x_d\}$ spans X, and $T = \{y_1, y_2, \ldots, y_m\}$ is a set of linearly independent vectors of X. We wish to show that $m \leq d$. Write

$$y_1 = \sum_{k=1}^{d} a_{k,1} x_k,$$

for some numbers $a_{1,1}, a_{2,1}, \ldots, a_{d,1}$, which we can do as S spans X. One of the $a_{k,1}$ is nonzero (otherwise y_1 would be zero), so suppose without loss of generality that this is $a_{1,1}$. Then we can solve

$$x_1 = \frac{1}{a_{1,1}} y_1 - \sum_{k=2}^{d} \frac{a_{k,1}}{a_{1,1}} x_k.$$

In particular $\{y_1, x_2, \ldots, x_d\}$ span X, since x_1 can be obtained from $\{y_1, x_2, \ldots, x_d\}$. Therefore, there are some numbers for some numbers $a_{1,2}, a_{2,2}, \ldots, a_{d,2}$, such that

$$y_2 = a_{1,2} y_1 + \sum_{k=2}^{d} a_{k,2} x_k.$$

As T is linearly independent, we must have that one of the $a_{k,2}$ for $k \geq 2$ must be nonzero. Without loss of generality suppose $a_{2,2} \neq 0$. Proceed to solve for

$$x_2 = \frac{1}{a_{2,2}} y_2 - \frac{a_{1,2}}{a_{2,2}} y_1 - \sum_{k=3}^{d} \frac{a_{k,2}}{a_{2,2}} x_k.$$

In particular $\{y_1, y_2, x_3, \ldots, x_d\}$ spans X.

We continue this procedure. If $m < d$, then we are done. So suppose $m \geq d$. After d steps we obtain that $\{y_1, y_2, \ldots, y_d\}$ spans X. Any other vector v in X is a linear combination of $\{y_1, y_2, \ldots, y_d\}$, and hence cannot be in T as T is linearly independent. So $m = d$.

Let us look at (ii). First notice that if we have a set T of k linearly independent vectors that do not span X, then we can always choose a vector $v \in X \setminus \operatorname{span}(T)$. The set $T \cup \{v\}$ is linearly independent (exercise). If $\dim X = d$, then there must exist some linearly independent set of d vectors T, and it must span X, otherwise we could choose a larger set of linearly independent vectors. So we have a basis of d vectors. On the other hand if we have a basis of d vectors, it is linearly independent and spans X. By (i) we know there is no set of $d+1$ linearly independent vectors, so dimension must be d.

For (iii) notice that $\{e_1, e_2, \ldots, e_n\}$ is a basis of $\mathbb{R}^n$.

To see (iv), suppose Y is a vector space and $Y \subset X$, where $\dim X = d$. As X cannot contain $d+1$ linearly independent vectors, neither can Y.

For (v) suppose T is a set of m vectors that is linearly dependent and spans X. Then one of the vectors is a linear combination of the others. Therefore if we remove it from T we obtain a set of $m-1$ vectors that still span X and hence $\dim X \leq m-1$ by (i).

For (vi) suppose $T = \{x_1, x_2, \ldots, x_m\}$ is a linearly independent set. We follow the procedure above in the proof of (ii) to keep adding vectors while keeping the set linearly independent. As the dimension is d we can add a vector exactly $d-m$ times. □

8.1.3 Linear mappings

A function $f\colon X \to Y$, when Y is not $\mathbb{R}$, is often called a *mapping* or a *map* rather than a *function*.

Definition 8.1.13. A mapping $A\colon X \to Y$ of vector spaces X and Y is *linear* (or a *linear transformation*) if for every $a \in \mathbb{R}$ and $x, y \in X$ we have

$$A(ax) = aA(x), \qquad \text{and} \qquad A(x+y) = A(x) + A(y).$$

We usually write Ax instead of $A(x)$ if A is linear.

If A is one-to-one an onto then we say A is *invertible* and we denote the inverse by A^{-1}.

If $A\colon X \to X$ is linear then we say A is a *linear operator* on X.

We write $L(X,Y)$ for the set of all linear transformations from X to Y, and just $L(X)$ for the set of linear operators on X. If $a \in \mathbb{R}$ and $A, B \in L(X,Y)$, define the transformations aA and $A+B$ by

$$(aA)(x) := aAx, \qquad (A+B)(x) := Ax + Bx.$$

If $A \in L(Y,Z)$ and $B \in L(X,Y)$, define the transformation AB as the composition $A \circ B$, that is,

$$ABx := A(Bx).$$

Finally denote by $I \in L(X)$ the *identity*: the linear operator such that $Ix = x$ for all x.

It is not hard to see that $aA \in L(X,Y)$ and $A+B \in L(X,Y)$, and that $AB \in L(X,Z)$. In particular, $L(X,Y)$ is a vector space. As the set $L(X)$ is not only a vector space, but also admits a product, it is often called an *algebra*.

An immediate consequence of the definition of a linear mapping is that if A is linear then $A0 = 0$.

Proposition 8.1.14. *If $A \in L(X,Y)$ is invertible, then A^{-1} is linear.*

Proof. Let $a \in \mathbb{R}$ and $y \in Y$. As A is onto, then there is an x such that $y = Ax$, and further as it is also one-to-one $A^{-1}(Az) = z$ for all $z \in X$. So

$$A^{-1}(ay) = A^{-1}(aAx) = A^{-1}\bigl(A(ax)\bigr) = ax = aA^{-1}(y).$$

Similarly let $y_1, y_2 \in Y$, and $x_1, x_2 \in X$ such that $Ax_1 = y_1$ and $Ax_2 = y_2$, then

$$A^{-1}(y_1+y_2) = A^{-1}(Ax_1+Ax_2) = A^{-1}\bigl(A(x_1+x_2)\bigr) = x_1+x_2 = A^{-1}(y_1)+A^{-1}(y_2). \qquad \square$$

Proposition 8.1.15. *If $A \in L(X,Y)$ is linear then it is completely determined by its values on a basis of X. Furthermore, if B is a basis of X, then any function $\widetilde{A} \colon B \to Y$ extends to a linear function on X.*

We will only prove this proposition for finite dimensional spaces, as we do not need infinite dimensional spaces. For infinite dimensional spaces, the proof is essentially the same, but a little trickier to write, so let us stick with finitely many dimensions.

Proof. Let $\{x_1, x_2, \ldots, x_n\}$ be a basis and suppose $Ax_j = y_j$. Every $x \in X$ has a unique representation

$$x = \sum_{j=1}^{n} b_j x_j$$

for some numbers $b_1, b_2, \ldots, b_n$. By linearity

$$Ax = A\sum_{j=1}^{n} b_j x_j = \sum_{j=1}^{n} b_j Ax_j = \sum_{j=1}^{n} b_j y_j.$$

The "furthermore" follows by setting $y_j := \widetilde{A}(x_j)$, and defining the extension as $Ax := \sum_{j=1}^{n} b_j y_j$. The function is well defined by uniqueness of the representation of x. We leave it to the reader to check that A is linear. $\square$

The next proposition only works for finite dimensional vector spaces. It is a special case of the so called rank-nullity theorem from linear algebra.

Proposition 8.1.16. *If X is a finite dimensional vector space and $A \in L(X)$, then A is one-to-one if and only if it is onto.*

Proof. Let $\{x_1, x_2, \ldots, x_n\}$ be a basis for X. Suppose A is one-to-one. Now suppose

$$\sum_{j=1}^{n} c_j A x_j = A \sum_{j=1}^{n} c_j x_j = 0.$$

As A is one-to-one, the only vector that is taken to 0 is 0 itself. Hence,

$$0 = \sum_{j=1}^{n} c_j x_j$$

and $c_j = 0$ for all j. So $\{Ax_1, Ax_2, \ldots, Ax_n\}$ is a linearly independent set. By Proposition 8.1.12 and the fact that the dimension is n, we have that $\{Ax_1, Ax_2, \ldots, Ax_n\}$ span X. Any point $x \in X$ can be written as

$$x = \sum_{j=1}^{n} a_j A x_j = A \sum_{j=1}^{n} a_j x_j,$$

so A is onto.

Now suppose A is onto. As A is determined by the action on the basis we see that every element of X has to be in the span of $\{Ax_1, Ax_2, \ldots, Ax_n\}$. Suppose

$$A \sum_{j=1}^{n} c_j x_j = \sum_{j=1}^{n} c_j A x_j = 0.$$

By Proposition 8.1.12 as $\{Ax_1, Ax_2, \ldots, Ax_n\}$ span X, the set is independent, and hence $c_j = 0$ for all j. In other words if $Ax = 0$, then $x = 0$. This means that A is one-to-one: If $Ax = Ay$, then $A(x - y) = 0$ and so $x = y$. $\square$

We leave the proof of the next proposition as an exercise.

Proposition 8.1.17. *If X and Y are finite dimensional vector spaces, then $L(X,Y)$ is also finite dimensional.*

Finally let us note that we often identify a finite dimensional vector space X of dimension n with $\mathbb{R}^n$, provided we fix a basis $\{x_1, x_2, \ldots, x_n\}$ in X. That is, we define a bijective linear map $A \in L(X, \mathbb{R}^n)$ by $Ax_j = e_j$, where $\{e_1, e_2, \ldots, e_n\}$. Then we have the correspondence

$$\sum_{j=1}^{n} c_j x_j \in X \quad \overset{A}{\mapsto} \quad (c_1, c_2, \ldots, c_n) \in \mathbb{R}^n.$$

8.1.4 Convexity

A subset U of a vector space is *convex* if whenever $x, y \in U$, the line segment from x to y lies in U. That is, if the *convex combination* $(1-t)x + ty$ is in U for all $t \in [0,1]$. See Figure 8.1.

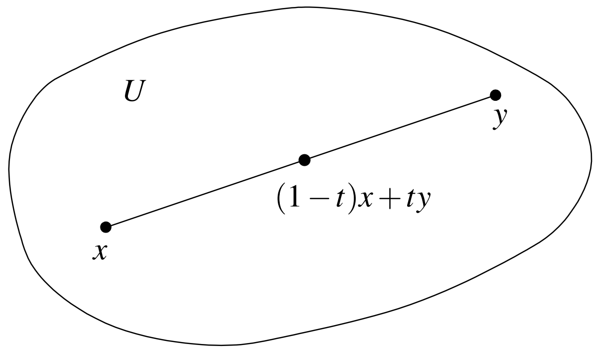

Figure 8.1: Convexity.

Note that in $\mathbb{R}$, every connected interval is convex. In $\mathbb{R}^2$ (or higher dimensions) there are lots of nonconvex connected sets. For example the set $\mathbb{R}^2 \setminus \{0\}$ is not convex but it is connected. To see this simply take any $x \in \mathbb{R}^2 \setminus \{0\}$ and let $y := -x$. Then $(1/2)x + (1/2)y = 0$, which is not in the set. On the other hand, the ball $B(x,r) \subset \mathbb{R}^n$ (using the standard metric on $\mathbb{R}^n$) is always convex by the triangle inequality.

Exercise* 8.1.1*: *Show that in $\mathbb{R}^n$ any ball $B(x,r)$ for $x \in \mathbb{R}^n$ and $r > 0$ is convex.*

Example 8.1.18: Any subspace V of a vector space X is convex.

Example 8.1.19: A somewhat more complicated example is given by the following. Let $C([0,1],\mathbb{R})$ be the vector space of continuous real valued functions on $\mathbb{R}$. Let $X \subset C([0,1],\mathbb{R})$ be the set of those f such

$$\int_0^1 f(x)\,dx \leq 1 \qquad \text{and} \qquad f(x) \geq 0 \text{ for all } x \in [0,1].$$

Then X is convex. Take $t \in [0,1]$ and note that if $f, g \in X$ then $t f(x) + (1-t)g(x) \geq 0$ for all x. Furthermore

$$\int_0^1 \bigl(t f(x) + (1-t)g(x)\bigr)\,dx = t \int_0^1 f(x)\,dx + (1-t)\int_0^1 g(x)\,dx \leq 1.$$

Note that X is not a subspace of $C([0,1],\mathbb{R})$.

Proposition 8.1.20. *The intersection two convex sets is convex. In fact, if $\{C_\lambda\}_{\lambda \in I}$ is an arbitrary collection of convex sets, then*

$$C := \bigcap_{\lambda \in I} C_\lambda$$

is convex.

Proof. If $x, y \in C$, then $x, y \in C_\lambda$ for all $\lambda \in I$, and hence if $t \in [0,1]$, then $tx + (1-t)y \in C_\lambda$ for all $\lambda \in I$. Therefore $tx + (1-t)y \in C$ and C is convex. □

Proposition 8.1.21. *Let $T \colon V \to W$ be a linear mapping between two vector spaces and let $C \subset V$ be a convex set. Then $T(C)$ is convex.*

Proof. Take any two points $p, q \in T(C)$. Pick $x, y \in C$ such that $Tx = p$ and $Ty = q$. As C is convex then for all $t \in [0,1]$ we have $tx + (1-t)y \in C$, so

$$tp + (1-t)q = tTx + (1-t)Ty = T\big(tx + (1-t)y\big) \in T(C).$$ □

For completeness, a very useful construction is the *convex hull*. Given any set $S \subset V$ of a vector space, define the convex hull of S, by

$$\operatorname{co}(S) := \bigcap \{C \subset V : S \subset C, \text{ and } C \text{ is convex}\}.$$

That is, the convex hull is the smallest convex set containing S. By a proposition above, the intersection of convex sets is convex and hence, the convex hull is convex.

Example 8.1.22: The convex hull of 0 and 1 in $\mathbb{R}$ is $[0,1]$. Proof: Any convex set containing 0 and 1 must contain $[0,1]$. The set $[0,1]$ is convex, therefore it must be the convex hull.

8.1.5 Exercises

Exercise* 8.1.2*: *Verify that $\mathbb{R}^n$ is a vector space.*

Exercise* 8.1.3*: *Let X be a vector space. Prove that a finite set of vectors $\{x_1, \ldots, x_n\} \subset X$ is linearly independent if and only if for every $j = 1, 2, \ldots, n$*

$$\operatorname{span}(\{x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n\}) \subsetneq \operatorname{span}(\{x_1, \ldots, x_n\}).$$

That is, the span of the set with one vector removed is strictly smaller.

Exercise* 8.1.4** (Challenging):*** *Prove that $C([0,1], \mathbb{R})$ is an infinite dimensional vector space where the operations are defined in the obvious way: $s = f + g$ and $m = fg$ are defined as $s(x) := f(x) + g(x)$ and $m(x) := f(x)g(x)$. Hint: for the dimension, think of functions that are only nonzero on the interval $(1/n+1, 1/n)$.*

Exercise* 8.1.5*: *Let $k \colon [0,1]^2 \to \mathbb{R}$ be continuous. Show that $L \colon C([0,1], \mathbb{R}) \to C([0,1], \mathbb{R})$ defined by*

$$Lf(y) := \int_0^1 k(x,y) f(x)\, dx$$

is a linear operator. That is, show that L is well defined (that Lf is continuous), and that L is linear.

Exercise* 8.1.6*: *Let $\mathscr{P}_n$ be the vector space of polynomials in one variable of degree n or less. Show that $\mathscr{P}_n$ is a vector space of dimension $n+1$.*

Exercise* 8.1.7*: *Let $\mathbb{R}[t]$ be the vector space of polynomials in one variable t. Let $D\colon \mathbb{R}[t] \to \mathbb{R}[t]$ be the derivative operator (derivative in t). Show that D is a linear operator.*

Exercise* 8.1.8*: *Let us show that Proposition 8.1.16 only works in finite dimensions. Take $\mathbb{R}[t]$ and define the operator $A\colon \mathbb{R}[t] \to \mathbb{R}[t]$ by $A\big(P(t)\big) = tP(t)$. Show that A is linear and one-to-one, but show that it is not onto.*

Exercise* 8.1.9*: *Finish the proof of Proposition 8.1.15 in finite dimensional case. That is, suppose, $\{x_1, x_2, \ldots x_n\}$ is a basis of X, $\{y_1, y_2, \ldots y_n\} \subset Y$ and we define a function*

$$Ax := \sum_{j=1}^{n} b_j y_j, \qquad \text{if} \quad x = \sum_{j=1}^{n} b_j x_j.$$

Then prove that $A\colon X \to Y$ is linear.

Exercise* 8.1.10*: *Prove Proposition 8.1.17. Hint: A linear operator is determined by its action on a basis. So given two bases $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_m\}$ for X and Y respectively, consider the linear operators A_{jk} that send $A_{jk} x_j = y_k$, and $A_{jk} x_\ell = 0$ if $\ell \neq j$.*

Exercise* 8.1.11** (Easy):*** *Suppose X and Y are vector spaces and $A \in L(X,Y)$ is a linear operator.*
a) Show that the nullspace $N := \{x \in X : Ax = 0\}$ is a vectorspace.
b) Show that the range $R := \{y \in Y : Ax = y \text{ for some } x \in X\}$ is a vectorspace.

Exercise* 8.1.12** (Easy):*** *Show by example that a union of convex sets need not be convex.*

Exercise* 8.1.13*: *Compute the convex hull of the set of 3 points $\{(0,0),(0,1),(1,1)\}$ in $\mathbb{R}^2$.*

Exercise* 8.1.14*: *Show that the set $\{(x,y) \in \mathbb{R}^2 : y > x^2\}$ is a convex set.*

Exercise* 8.1.15*: *Show that every convex set in $\mathbb{R}^n$ is connected using the standard topology on $\mathbb{R}^n$.*

Exercise* 8.1.16*: *Suppose $K \subset \mathbb{R}^2$ is a convex set such that the only point of the form $(x,0)$ in K is the point $(0,0)$. Further suppose that there $(0,1) \in K$ and $(1,1) \in K$. Then show that if $(x,y) \in K$ then $y > 0$ unless $x = 0$.*

8.2 Analysis with vector spaces

Note: 2-3 lectures

8.2.1 Norms

Let us start measuring distance.

Definition 8.2.1. If X is a vector space, then we say a function $\|\cdot\|: X \to \mathbb{R}$ is a *norm* if:

(i) $\|x\| \geq 0$, with $\|x\| = 0$ if and only if $x = 0$.

(ii) $\|cx\| = |c| \, \|x\|$ for all $c \in \mathbb{R}$ and $x \in X$.

(iii) $\|x+y\| \leq \|x\| + \|y\|$ for all $x, y \in X$ (Triangle inequality).

Before defining the standard norm on $\mathbb{R}^n$, let us define the standard scalar *dot product* on $\mathbb{R}^n$. For two vectors if $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and $y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$, define

$$x \cdot y := \sum_{j=1}^{n} x_j y_j.$$

It is easy to see that the dot product is linear in each variable separately, that is, it is a linear mapping when you keep one of the variables constant. The *Euclidean norm* is then defined as

$$\|x\| := \|x\|_{\mathbb{R}^n} := \sqrt{x \cdot x} = \sqrt{(x_1)^2 + (x_2)^2 + \cdots + (x_n)^2}.$$

We normally just use $\|x\|$, but sometimes it will be necessary to emphasize that we are talking about the euclidean norm and use $\|x\|_{\mathbb{R}^n}$. It is easy to see that the Euclidean norm satisfies (i) and (ii). To prove that (iii) holds, the key inequality is the so-called Cauchy-Schwarz inequality that we have seen before. As this inequality is so important let us restate and reprove it using the notation of this chapter.

Theorem 8.2.2 (Cauchy-Schwarz inequality)**.** *Let $x, y \in \mathbb{R}^n$, then*

$$|x \cdot y| \leq \|x\| \|y\| = \sqrt{x \cdot x} \sqrt{y \cdot y},$$

with equality if and only if the vectors are scalar multiples of each other.

Proof. If $x = 0$ or $y = 0$, then the theorem holds trivially. So assume $x \neq 0$ and $y \neq 0$.

If x is a scalar multiple of y, that is $x = \lambda y$ for some $\lambda \in \mathbb{R}$, then the theorem holds with equality:

$$|\lambda y \cdot y| = |\lambda| \, |y \cdot y| = |\lambda| \, \|y\|^2 = \|\lambda y\| \|y\|.$$

Next take $x+ty$,

$$\|x+ty\|^2 = (x+ty)\cdot(x+ty) = x\cdot x + x\cdot ty + ty\cdot x + ty\cdot ty = \|x\|^2 + 2t(x\cdot y) + t^2\|y\|^2.$$

If x is not a scalar multiple of y, then $\|x+ty\|^2 > 0$ for all t. So the above polynomial in t is never zero. From elementary algebra it follows that the discriminant must be negative:

$$4(x\cdot y)^2 - 4\|x\|^2\|y\|^2 < 0,$$

or in other words $(x\cdot y)^2 < \|x\|^2\|y\|^2$. $\square$

Item (iii), the triangle inequality, follows via a simple computation:

$$\|x+y\|^2 = x\cdot x + y\cdot y + 2(x\cdot y) \leq \|x\|^2 + \|y\|^2 + 2(\|x\|\|y\|) = (\|x\| + \|y\|)^2.$$

The distance $d(x,y) := \|x-y\|$ is the standard distance function on $\mathbb{R}^n$ that we used when we talked about metric spaces.

In fact, on any vector space X, once we have a norm (any norm), we define a distance $d(x,y) := \|x-y\|$ that makes X into a metric space (an easy exercise).

Definition 8.2.3. Let $A \in L(X,Y)$. Define

$$\|A\| := \sup\{\|Ax\| : x \in X \text{ with } \|x\| = 1\}.$$

The number $\|A\|$ is called the *operator norm*. We will see below that indeed it is a norm (at least for finite dimensional spaces). Again, when necessary to emphasize which norm we are talking about, we may write it as $\|A\|_{L(X,Y)}$.

By linearity for any nonzero vector $x \in X$, $\left\|A\frac{x}{\|x\|}\right\| = \frac{\|Ax\|}{\|x\|}$. The vector $\frac{x}{\|x\|}$ is of norm 1. Therefore,

$$\|A\| = \sup\{\|Ax\| : x \in X \text{ with } \|x\| = 1\} = \sup_{\substack{x\in X\\ x\neq 0}} \frac{\|Ax\|}{\|x\|}.$$

This implies that

$$\|Ax\| \leq \|A\|\|x\|.$$

It is not hard to see from the definition that $\|A\| = 0$ if and only if $A = 0$, that is, if A takes every vector to the zero vector.

It is also not difficult to see the norm of the identity operator:

$$\|I\| = \sup_{\substack{x\in X\\ x\neq 0}} \frac{\|Ix\|}{\|x\|} = \sup_{\substack{x\in X\\ x\neq 0}} \frac{\|x\|}{\|x\|} = 1.$$

For finite dimensional spaces $\|A\|$ is always finite as we prove below. This also implies that A is continuous. For infinite dimensional spaces neither statement needs to be true. For a simple example, take the vector space of continuously differentiable functions on $[0,1]$ and as the norm use the uniform norm. The functions $\sin(nx)$ have norm 1, but the derivatives have norm n. So differentiation (which is a linear operator) has unbounded norm on this space. But let us stick to finite dimensional spaces now.

When we talk about finite dimensional vector space, one often thinks of $\mathbb{R}^n$, although if we have a norm, the norm might perhaps not be the standard euclidean norm. In the exercises, you can prove that every norm is in some sense "equivalent" to the euclidean norm in that the topology it generates is the same. For simplicity we only prove the following proposition for the euclidean space, and the proof for a general finite dimensional space is left as an exercise.

Proposition 8.2.4. *Let X and Y be finite dimensional vector spaces with a norm. If $A \in L(X,Y)$, then $\|A\| < \infty$, and A is uniformly continuous (Lipschitz with constant $\|A\|$).*

Proof. As we said we only prove the proposition for euclidean space so suppose that $X = \mathbb{R}^n$ and $Y = \mathbb{R}^m$ and the norm is the standard euclidean norm. The general case is left as an exercise.

Let $\{e_1, e_2, \ldots, e_n\}$ be the standard basis of $\mathbb{R}^n$. Write $x \in \mathbb{R}^n$, with $\|x\| = 1$, as

$$x = \sum_{j=1}^{n} c_j e_j.$$

Since $e_j \cdot e_\ell = 0$ whenever $j \neq \ell$ and $e_j \cdot e_j = 1$ then $c_j = x \cdot e_j$ and

$$\left|c_j\right| = \left|x \cdot e_j\right| \leq \|x\| \|e_j\| = 1.$$

Then

$$\|Ax\| = \left\| \sum_{j=1}^{n} c_j A e_j \right\| \leq \sum_{j=1}^{n} \left|c_j\right| \|Ae_j\| \leq \sum_{j=1}^{n} \|Ae_j\|.$$

The right hand side does not depend on x. We found a finite upper bound independent of x, so $\|A\| < \infty$.

Now for any vector spaces X and Y, and $A \in L(X,Y)$, suppose that $\|A\| < \infty$. For $v, w \in X$,

$$\|A(v-w)\| \leq \|A\| \|v-w\|.$$

As $\|A\| < \infty$, then this says A is Lipschitz with constant $\|A\|$. $\square$

Proposition 8.2.5. *Let X, Y, and Z be finite dimensional vector spaces with a norm.*

(i) If $A, B \in L(X,Y)$ and $c \in \mathbb{R}$, then

$$\|A+B\| \leq \|A\| + \|B\|, \qquad \|cA\| = |c| \, \|A\|.$$

In particular, the operator norm is a norm on the vector space $L(X,Y)$.

(ii) If $A \in L(X,Y)$ and $B \in L(Y,Z)$, then

$$\|BA\| \leq \|B\| \|A\|.$$

Proof. For (i), let us note that

$$\|(A+B)x\| = \|Ax+Bx\| \leq \|Ax\| + \|Bx\| \leq \|A\| \|x\| + \|B\| \|x\| = (\|A\| + \|B\|) \|x\|.$$

So $\|A+B\| \leq \|A\| + \|B\|$.

Similarly

$$\|(cA)x\| = |c| \, \|Ax\| \leq (|c| \, \|A\|) \|x\|.$$

Thus $\|cA\| \leq |c| \, \|A\|$. Next note

$$|c| \, \|Ax\| = \|cAx\| \leq \|cA\| \|x\|.$$

Hence $|c| \, \|A\| \leq \|cA\|$.

For (ii) write

$$\|BAx\| \leq \|B\| \|Ax\| \leq \|B\| \|A\| \|x\|.$$ □

As a norm defines a metric, we have a metric space topology on $L(X,Y)$, so we can talk about open/closed sets, continuity, and convergence.

Proposition 8.2.6. *Let X be a finite dimensional vector space with a norm. Let $U \subset L(X)$ be the set of invertible linear operators.*

(i) If $A \in U$ and $B \in L(X)$, and

$$\|A - B\| < \frac{1}{\|A^{-1}\|}, \tag{8.2}$$

then B is invertible.

(ii) U is open and $A \mapsto A^{-1}$ is a continuous function on U.

Let us make sense of this on a simple example. Think back to $\mathbb{R}^1$, where linear operators are just numbers a and the operator norm of a is simply $|a|$. The operator a is invertible ($a^{-1} = 1/a$) whenever $a \neq 0$. The condition $|a-b| < \frac{1}{|a^{-1}|}$ does indeed imply that b is not zero. And $a \mapsto 1/a$ is a continuous map. When $n > 1$, then there are other noninvertible operators than just zero, and in general things are a bit more difficult.

Proof. Let us prove (i). We know something about A^{-1} and something about $A - B$. These are linear operators so let us apply them to a vector.

$$A^{-1}(A-B)x = x - A^{-1}Bx.$$

Therefore,

$$\begin{aligned}\|x\| &= \|A^{-1}(A-B)x + A^{-1}Bx\| \\ &\leq \|A^{-1}\|\|A-B\|\|x\| + \|A^{-1}\|\|Bx\|.\end{aligned}$$

Now assume $x \neq 0$ and so $\|x\| \neq 0$. Using (8.2) we obtain

$$\|x\| < \|x\| + \|A^{-1}\|\|Bx\|,$$

or in other words $\|Bx\| \neq 0$ for all nonzero x, and hence $Bx \neq 0$ for all nonzero x. This is enough to see that B is one-to-one (if $Bx = By$, then $B(x-y) = 0$, so $x = y$). As B is one-to-one operator from X to X which is finite dimensional and hence is invertible.

Let us look at (ii). Fix some $A \in U$. Let B be invertible and near A, that is $\|A-B\|\|A^{-1}\| < 1/2$. Then (8.2) is satisfied. We have shown above (using $B^{-1}y$ instead of x)

$$\|B^{-1}y\| \leq \|A^{-1}\|\|A-B\|\|B^{-1}y\| + \|A^{-1}\|\|y\| \leq 1/2\|B^{-1}y\| + \|A^{-1}\|\|y\|,$$

or

$$\|B^{-1}y\| \leq 2\|A^{-1}\|\|y\|.$$

So $\|B^{-1}\| \leq 2\|A^{-1}\|$.

Now note that

$$A^{-1}(A-B)B^{-1} = A^{-1}(AB^{-1} - I) = B^{-1} - A^{-1},$$

and

$$\|B^{-1} - A^{-1}\| = \|A^{-1}(A-B)B^{-1}\| \leq \|A^{-1}\|\|A-B\|\|B^{-1}\| \leq 2\|A^{-1}\|^2\|A-B\|.$$

Therefore, if as B tends to A, $\|B^{-1} - A^{-1}\|$ tends to 0, and therefore the inverse operation is a continuous function at A. $\square$

8.2.2 Matrices

As we previously noted, once we fix a basis in a finite dimensional vector space X, we can represent an vector of X as an n-tuple of numbers, that is a vector in $\mathbb{R}^n$. The same thing can be done with $L(X,Y)$, which brings us to matrices, which are a convenient way to represent finite-dimensional linear transformations. Suppose we have bases $\{x_1, x_2, \ldots, x_n\}$ and $\{y_1, y_2, \ldots, y_m\}$ for vector spaces X and Y respectively. A linear operator is determined by its values on the basis. Given $A \in L(X,Y)$, Ax_j is an element of Y. Therefore, define the numbers $\{a_{i,j}\}$ as follows

$$Ax_j = \sum_{i=1}^{m} a_{i,j}\, y_i,$$

and write them as a *matrix*

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{bmatrix}.$$

And we say A is an m-by-n matrix. The *columns* of the matrix are precisely the coefficients that represent Ax_j. Let us derive the familiar rule for matrix multiplication.

When

$$z = \sum_{j=1}^{n} c_j x_j,$$

then

$$Az = \sum_{j=1}^{n} c_j A x_j = \sum_{j=1}^{n} c_j \left(\sum_{i=1}^{m} a_{i,j} y_i \right) = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{i,j} c_j \right) y_i,$$

which gives rise to the familiar rule for matrix multiplication.

There is a one-to-one correspondence between matrices and linear operators in $L(X,Y)$. That is, once we fix a basis in X and in Y. If we would choose a different basis, we would get different matrices. This is important, the operator A acts on elements of X, the matrix is something that works with n-tuples of numbers, that is, vectors of $\mathbb{R}^n$.

If B is an n-by-r matrix with entries $b_{j,k}$, then the matrix for $C = AB$ is an m-by-r matrix whose i,kth entry $c_{i,k}$ is

$$c_{i,k} = \sum_{j=1}^{n} a_{i,j} b_{j,k}.$$

A way to remember it is if you order the indices as we do, that is *row,column*, and put the elements in the same order as the matrices, then it is the "middle index" that is "summed-out."

A linear mapping changing one basis to another is a square matrix in which the columns represent basis elements of the second basis in terms of the first basis. We call such a linear mapping an *change of basis*.

Now suppose all the bases are just the standard bases and $X = \mathbb{R}^n$ and $Y = \mathbb{R}^m$. If we recall the Cauchy-Schwarz inequality we note that

$$\|Az\|^2 = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{i,j} c_j \right)^2 \leq \sum_{i=1}^{m} \left(\sum_{j=1}^{n} (c_j)^2 \right) \left(\sum_{j=1}^{n} (a_{i,j})^2 \right) = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} (a_{i,j})^2 \right) \|z\|^2.$$

In other words, we have a bound on the operator norm (note that equality rarely happens)

$$\|A\| \leq \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} (a_{i,j})^2}.$$

If the entries go to zero, then $\|A\|$ goes to zero. In particular, if A if fixed and B is changing such that the entries of $A-B$ go to zero then B goes to A in operator norm. That is B goes to A in the metric space topology induced by the operator norm. We have proved the first part of:

Proposition 8.2.7. *If $f\colon S \to \mathbb{R}^{nm}$ is a continuous function for a metric space S, then taking the components of f as the entries of a matrix, f is a continuous mapping from S to $L(\mathbb{R}^n,\mathbb{R}^m)$. Conversely if $f\colon S \to L(\mathbb{R}^n,\mathbb{R}^m)$ is a continuous function then the entries of the matrix are continuous functions.*

The proof of the second part is rather easy. Take $f(x)e_j$ and note that is a continuous function to $\mathbb{R}^m$ with standard Euclidean norm: notice that $\|f(x)e_j - f(y)e_j\| = \|\big(f(x)-f(y)\big)e_j\| \leq \|f(x)-f(y)\|$. So as $x \to y$, then $\|f(x)-f(y)\| \to 0$ and so $\|f(x)e_j - f(y)e_j\| \to 0$. Such a function is continuous if and only if its components are continuous and these are the components of the jth column of the matrix $f(x)$.

8.2.3 Determinants

A certain number can be assigned to square matrices that measures how the corresponding linear mapping stretches space. In particular this number, called the determinant, can be used to test for invertibility of a matrix.

First define the symbol $\operatorname{sgn}(x)$ for a number is defined by

$$\operatorname{sgn}(x) := \begin{cases} -1 & \text{if } x < 0, \\ 0 & \text{if } x = 0, \\ 1 & \text{if } x > 0. \end{cases}$$

Suppose $\sigma = (\sigma_1,\sigma_2,\ldots,\sigma_n)$ is a *permutation* of the integers $(1,2,\ldots,n)$, that is, a reordering of $(1,2,\ldots,n)$. Any permutation can be obtained by a sequence of transpositions (switchings of two elements). Call a permutation *even* (resp. *odd*) if it takes an even (resp. odd) number of transpositions to get from σ to $(1,2,\ldots,n)$. It can be shown that this is well defined (exercise). In fact, define

$$\operatorname{sgn}(\sigma) := \operatorname{sgn}(\sigma_1,\ldots,\sigma_n) = \prod_{p<q} \operatorname{sgn}(\sigma_q - \sigma_p). \tag{8.3}$$

Then it can be shown that $\operatorname{sgn}(\sigma)$ is is 1 if σ is even and -1 if σ is odd. This fact can be proved by noting that applying a transposition changes the sign. Then note that the sign of $(1,2,\ldots,n)$ is 1.

Let S_n be the set of all permutations on n elements (the *symmetric group*). Let $A = [a_{i,j}]$ be a square $n \times n$ matrix. Define the *determinant* of A

$$\det(A) := \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{i,\sigma_i}.$$

Proposition 8.2.8.

(i) $\det(I) = 1$.

(ii) $\det([x_1 \;\; x_2 \;\; \cdots \;\; x_n])$ *as a function of column vectors* x_j *is linear in each variable* x_j *separately.*

(iii) If two columns of a matrix are interchanged, then the determinant changes sign.

(iv) If two columns of A *are equal, then* $\det(A) = 0$.

(v) If a column is zero, then $\det(A) = 0$.

(vi) $A \mapsto \det(A)$ *is a continuous function.*

(vii) $\det \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] = ad - bc$, *and* $\det[a] = a$.

In fact, the determinant is the unique function that satisfies (i), (ii), and (iii). But we digress. By (ii), we mean that if we fix all the vectors $x_1, \ldots, x_n$ except for x_j and think of the determinant as function of x_j, it is a linear function, that is, if $v, w \in \mathbb{R}^n$ are two vectors, and $a, b \in \mathbb{R}$ are scalars then

$$\det([x_1 \;\; \cdots \;\; x_{j-1} \;\; (av+bw) \;\; x_{j+1} \;\; \cdots \;\; x_n]) = \\ a\det([x_1 \;\; \cdots \;\; x_{j-1} \;\; v \;\; x_{j+1} \;\; \cdots \;\; x_n]) + b\det([x_1 \;\; \cdots \;\; x_{j-1} \;\; w \;\; x_{j+1} \;\; \cdots \;\; x_n]).$$

Proof. We go through the proof quickly, as you have likely seen this before.

(i) is trivial. For (ii), notice that each term in the definition of the determinant contains exactly one factor from each column.

Part (iii) follows by noting that switching two columns is like switching the two corresponding numbers in every element in S_n. Hence all the signs are changed. Part (iv) follows because if two columns are equal and we switch them we get the same matrix back and so part (iii) says the determinant must have been 0.

Part (v) follows because the product in each term in the definition includes one element from the zero column. Part (vi) follows as det is a polynomial in the entries of the matrix and hence continuous. We have seen that a function defined on matrices is continuous in the operator norm if it is continuous in the entries. Finally, part (vii) is a direct computation. □

The determinant tells us about areas and volumes, and how they change. For example, in the 1×1 case, a matrix is just a number, and the determinant is exactly this number. It says how the linear mapping "stretches" the space. Similarly for $\mathbb{R}^2$ (and in fact for $\mathbb{R}^n$). Suppose $A \in L(\mathbb{R}^2)$ is a linear transformation. It can be checked directly that the area of the image of the unit square $A([0,1]^2)$ is precisely $|\det(A)|$. The sign of the determinant tells us if the image is flipped or not. This works with arbitrary figures, not just the unit square. The determinant tells us the stretch in the area. In $\mathbb{R}^3$ it will tell us about the 3 dimensional volume, and in n-dimensions about the n-dimensional volume. We claim this without proof.

Proposition 8.2.9. *If* A *and* B *are* $n \times n$ *matrices, then* $\det(AB) = \det(A)\det(B)$. *In particular,* A *is invertible if and only if* $\det(A) \neq 0$ *and in this case,* $\det(A^{-1}) = \frac{1}{\det(A)}$.

Proof. Let $b_1, b_2, \ldots, b_n$ be the columns of B. Then

$$AB = [Ab_1 \quad Ab_2 \quad \cdots \quad Ab_n].$$

That is, the columns of AB are $Ab_1, Ab_2, \ldots, Ab_n$.

Let $b_{j,k}$ denote the elements of B and a_j the columns of A. Note that $Ae_j = a_j$. By linearity of the determinant as proved above we have

$$\begin{aligned}
\det(AB) &= \det([Ab_1 \quad Ab_2 \quad \cdots \quad Ab_n]) = \det\left(\left[\sum_{j=1}^{n} b_{j,1} a_j \quad Ab_2 \quad \cdots \quad Ab_n\right]\right) \\
&= \sum_{j=1}^{n} b_{j,1} \det([a_j \quad Ab_2 \quad \cdots \quad Ab_n]) \\
&= \sum_{1 \leq j_1, j_2, \ldots, j_n \leq n} b_{j_1,1} b_{j_2,2} \cdots b_{j_n,n} \det([a_{j_1} \quad a_{j_2} \quad \cdots \quad a_{j_n}]) \\
&= \left(\sum_{(j_1, j_2, \ldots, j_n) \in S_n} b_{j_1,1} b_{j_2,2} \cdots b_{j_n,n} \operatorname{sgn}(j_1, j_2, \ldots, j_n)\right) \det([a_1 \quad a_2 \quad \cdots \quad a_n]).
\end{aligned}$$

In the above, go from all integers between 1 and n, to just elements of S_n by noting that when two columns in the determinant are the same then the determinant is zero. We then reorder the columns to the original ordering and obtain the sgn.

The conclusion that $\det(AB) = \det(A)\det(B)$ follows by recognizing the determinant of B. We obtain this by plugging in $A = I$. The expression we got for the determinant of B has rows and columns swapped, so as a sidenote, we have also just proved that the determinant of a matrix and its transpose are equal.

To prove the second part of the theorem, suppose A is invertible. Then $A^{-1}A = I$ and consequently $\det(A^{-1})\det(A) = \det(A^{-1}A) = \det(I) = 1$. If A is not invertible, then the columns are linearly dependent. That is, suppose

$$\sum_{j=1}^{n} \gamma_j a_j = 0,$$

where not all γ_j are equal to 0. Without loss of generality suppose $\gamma_1 \neq 1$. Take

$$B := \begin{bmatrix} \gamma_1 & 0 & 0 & \cdots & 0 \\ \gamma_2 & 1 & 0 & \cdots & 0 \\ \gamma_3 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \gamma_n & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

Applying the definition of the determinant we see $\det(B) = \gamma_1 \neq 0$. Then $\det(AB) = \det(A)\det(B) = \gamma_1 \det(A)$. The first column of AB is zero, and hence $\det(AB) = 0$. Thus $\det(A) = 0$. □

Proposition 8.2.10. *Determinant is independent of the basis. In other words, if B is invertible then,*

$$\det(A) = \det(B^{-1}AB).$$

Proof follows by noting $\det(B^{-1}AB) = \frac{1}{\det(B)} \det(A)\det(B) = \det(A)$. If in one basis A is the matrix representing a linear operator, then for another basis we can find a matrix B such that the matrix $B^{-1}AB$ takes us to the first basis, applies A in the first basis, and takes us back to the basis we started with. Therefore, the determinant can be defined as a function on the space $L(X)$ for some finite dimensional metric space X, not just on matrices. We choose a basis on X, and we can represent a linear mapping using a matrix with respect to this basis. We obtain the same determinant as if we had used any other basis. It follows that

$$\det \colon L(X) \to \mathbb{R}$$

is a well-defined function.

There are three types of so-called *elementary matrices*. Recall again that e_j are the standard basis of $\mathbb{R}^n$. First for some $j = 1,2,\ldots,n$ and some $\lambda \in \mathbb{R}$, $\lambda \neq 0$, an $n \times n$ matrix E defined by

$$Ee_i = \begin{cases} e_i & \text{if } i \neq j, \\ \lambda e_i & \text{if } i = j. \end{cases}$$

Given any $n \times m$ matrix M the matrix EM is the same matrix as M except with the kth row multiplied by λ. It is an easy computation (exercise) that $\det(E) = \lambda$.

Second, for some j and k with $j \neq k$, and $\lambda \in \mathbb{R}$ an $n \times n$ matrix E defined by

$$Ee_i = \begin{cases} e_i & \text{if } i \neq j, \\ e_i + \lambda e_k & \text{if } i = j. \end{cases}$$

Given any $n \times m$ matrix M the matrix EM is the same matrix as M except with λ times the kth row added to the jth row. It is an easy computation (exercise) that $\det(E) = 1$.

Finally, for some j and k with $j \neq k$ an $n \times n$ matrix E defined by

$$Ee_i = \begin{cases} e_i & \text{if } i \neq j \text{ and } i \neq k, \\ e_k & \text{if } i = j, \\ e_j & \text{if } i = k. \end{cases}$$

Given any $n \times m$ matrix M the matrix EM is the same matrix with jth and kth rows swapped. It is an easy computation (exercise) that $\det(E) = -1$.

Elementary matrices are useful for computing the determinant. The proof of the following proposition is left as an exercise.

Proposition 8.2.11. *Let T be an $n \times n$ invertible matrix. Then there exists a finite sequence of elementary matrices $E_1, E_2, \dots, E_k$ such that*

$$T = E_1 E_2 \cdots E_k,$$

and

$$\det(T) = \det(E_1) \det(E_2) \cdots \det(E_k).$$

8.2.4 Exercises

Exercise* 8.2.1*: *If X is a vector space with a norm $\lVert\cdot\rVert$, then show that $d(x,y) := \lVert x-y \rVert$ makes X a metric space.*

Exercise* 8.2.2** (Easy):*** *Show that for square matrices A and B,* $\det(AB) = \det(BA)$.

Exercise* 8.2.3*: *For $\mathbb{R}^n$ define*

$$\lVert x \rVert_\infty := \max\{\lvert x_1 \rvert, \lvert x_2 \rvert, \dots, \lvert x_n \rvert\},$$

sometimes called the sup or the max norm.
a) Show that $\lVert\cdot\rVert_\infty$ is a norm on $\mathbb{R}^n$ (defining a different distance).
b) What is the unit ball $B(0,1)$ in this norm?

Exercise* 8.2.4*: *For $\mathbb{R}^n$ define*

$$\lVert x \rVert_1 := \sum_{j=1}^{n} \lvert x_j \rvert,$$

sometimes called the 1*-norm (or L^1 norm).*
a) Show that $\lVert\cdot\rVert_1$ is a norm on $\mathbb{R}^n$ (defining a different distance, sometimes called the taxicab distance).
b) What is the unit ball $B(0,1)$ in this norm?

Exercise* 8.2.5*: *Using the euclidean norm on $\mathbb{R}^2$. Compute the operator norm of the operators in $L(\mathbb{R}^2)$ given by the matrices:*
a) $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ *b)* $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ *c)* $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ *d)* $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

Exercise* 8.2.6*: *Using the standard euclidean norm $\mathbb{R}^n$. Show*
a) Suppose $A \in L(\mathbb{R},\mathbb{R}^n)$ is defined for $x \in \mathbb{R}$ by $Ax = xa$ for a vector $a \in \mathbb{R}^n$. Then the operator norm $\lVert A \rVert_{L(\mathbb{R},\mathbb{R}^n)} = \lVert a \rVert_{\mathbb{R}^n}$. (that is the operator norm of A is the euclidean norm of a).
b) Suppose $B \in L(\mathbb{R}^n,\mathbb{R})$ is defined for $x \in \mathbb{R}^n$ by $Bx = b \cdot x$ for a vector $b \in \mathbb{R}^n$. Then the operator norm $\lVert B \rVert_{L(\mathbb{R}^n,\mathbb{R})} = \lVert b \rVert_{\mathbb{R}^n}$

Exercise* 8.2.7*: *Suppose $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_n)$ is a permutation of $(1,2,\dots,n)$.*
a) Show that we can make a finite number of transpositions (switching of two elements) to get to $(1,2,\dots,n)$.
b) Using the definition (8.3) *show that σ is even if* $\operatorname{sgn}(\sigma) = 1$ *and σ is odd if* $\operatorname{sgn}(\sigma) = -1$. *In particular showing that being odd or even is well defined.*

Exercise* 8.2.8*: *Verify the computation of the determinant for the three types of elementary matrices.*

***Exercise* 8.2.9:** *Prove Proposition 8.2.11.*

***Exercise* 8.2.10:** *a) Suppose $D = [d_{i,j}]$ is an n-by-n* diagonal matrix*, that is, $d_{i,j} = 0$ whenever $i \neq j$. Show that* $\det(D) = d_{1,1}d_{2,2}\cdots d_{n,n}$.
b) Suppose A is a diagonalizable matrix. That is, there exists a matrix B such that $B^{-1}AB = D$ for a diagonal matrix $D = [d_{i,j}]$. Show that $\det(A) = d_{1,1}d_{2,2}\cdots d_{n,n}$.

***Exercise* 8.2.11:** *Take the vectorspace of polynomials $\mathbb{R}[t]$ and the linear operator $D \in L(\mathbb{R}[t])$ that is the differentiation (we proved in an earlier exercise that D is a linear operator). Define the norm on $P(t) = c_0 + c_1 t + \cdots + c_n t^n$ as $\|P\| := \sup\{\left|c_j\right| : j = 0, 1, \ldots, n\}$.*
a) Show that $\|P\|$ is a norm on $\mathbb{R}[t]$.
b) Show that D does not have bounded operator norm, that is $\|D\| = \infty$. Hint: consider the polynomials t^n as n tends to infinity.

***Exercise* 8.2.12:** *In this exercise we finish the proof of Proposition 8.2.4. Let X be any finite dimensional vector space with a norm. Let $\{x_1, x_2, \ldots, x_n\}$ be a basis for X.*
a) Show that the function $f : \mathbb{R}^n \to \mathbb{R}$

$$f(c_1, c_2, \ldots, c_n) = \|c_1 x_1 + c_2 x_2 + \cdots + c_n x_n\|$$

is continuous.
b) Show that there exists numbers m and M such that if $c = (c_1, c_2, \ldots, c_n) \in \mathbb{R}^n$ with $\|c\| = 1$ (standard euclidean norm), then $m \leq \|c_1 x_1 + c_2 x_2 + \cdots + c_n x_n\| \leq M$ (here the the norm is on X).
c) Show that there exists a number B such that if $\|c_1 x_1 + c_2 x_2 + \cdots + c_n x_n\| = 1$, then $\left|c_j\right| \leq B$.
d) Use part (c) to show that if X and Y are finite dimensional vector spaces and $A \in L(X,Y)$, then $\|A\| < \infty$.

***Exercise* 8.2.13:** *Let X be any finite dimensional vector space with a norm $\|\cdot\|$ and basis $\{x_1, x_2, \ldots, x_n\}$. Let $c = (c_1, \ldots, c_n) \in \mathbb{R}^n$ and $\|c\|$ be the standard euclidean norm on $\mathbb{R}^n$.*
a) Find that there exist positive numbers $m, M > 0$ such that for

$$m\|c\| \leq \|c_1 x_1 + c_2 x_2 + \cdots + c_n x_n\| \leq M\|c\|.$$

Hint: See previous exercise.
b) Use part (a) to show that of $\|\cdot\|_1$ and $\|\cdot\|_2$ are two norms on X, then there exist positive numbers $m, M > 0$ (perhaps different than above) such that for all $x \in X$ we have

$$m\|x\|_1 \leq \|x\|_2 \leq M\|x\|_1.$$

c) Now show that $U \subset X$ is open in the metric defined by $\|x - y\|_1$ if and only if it is open in the metric defined by $\|x - y\|_2$. In other words, convergence of sequences, continuity of functions is the same in either norm.

8.3 The derivative

Note: 2–3 lectures

8.3.1 The derivative

Recall that for a function $f\colon \mathbb{R} \to \mathbb{R}$, we defined the derivative at x as

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}.$$

In other words, there was a number a (the derivative of f at x) such that

$$\lim_{h\to 0} \left| \frac{f(x+h)-f(x)}{h} - a \right| = \lim_{h\to 0} \left| \frac{f(x+h)-f(x)-ah}{h} \right| = \lim_{h\to 0} \frac{\lvert f(x+h)-f(x)-ah \rvert}{\lvert h \rvert} = 0.$$

Multiplying by a is a linear map in one dimension. That is, we think of $a \in L(\mathbb{R}^1,\mathbb{R}^1)$ which is the best linear approximation of f near x. We use this definition to extend differentiation to more variables.

Definition 8.3.1. Let $U \subset \mathbb{R}^n$ be an open subset and $f\colon U \to \mathbb{R}^m$. We say f is *differentiable* at $x \in U$ if there exists an $A \in L(\mathbb{R}^n,\mathbb{R}^m)$ such that

$$\lim_{\substack{h\to 0\\ h\in\mathbb{R}^n}} \frac{\lVert f(x+h)-f(x)-Ah \rVert}{\lVert h \rVert} = 0.$$

We write $Df(x) := A$, or $f'(x) := A$, and we say A is the *derivative* of f at x. When f is *differentiable* at all $x \in U$, we say simply that f is differentiable.

For a differentiable function, the derivative of f is a function from U to $L(\mathbb{R}^n,\mathbb{R}^m)$. Compare to the one dimensional case, where the derivative is a function from U to $\mathbb{R}$, but we really want to think of $\mathbb{R}$ here as $L(\mathbb{R}^1,\mathbb{R}^1)$.

The norms above must be on the right spaces of course. The norm in the numerator is on $\mathbb{R}^m$, and the norm in the denominator is on $\mathbb{R}^n$ where h lives. Normally it is understood that $h \in \mathbb{R}^n$ from context. We will not explicitly say so from now on.

We have again cheated somewhat and said that A is *the* derivative. We have not shown yet that there is only one, let us do that now.

Proposition 8.3.2. *Let $U \subset \mathbb{R}^n$ be an open subset and $f\colon U \to \mathbb{R}^m$. Suppose $x \in U$ and there exist $A,B \in L(\mathbb{R}^n,\mathbb{R}^m)$ such that*

$$\lim_{h\to 0} \frac{\lVert f(x+h)-f(x)-Ah \rVert}{\lVert h \rVert} = 0 \qquad \textit{and} \qquad \lim_{h\to 0} \frac{\lVert f(x+h)-f(x)-Bh \rVert}{\lVert h \rVert} = 0.$$

Then $A = B$.

Proof.

$$\begin{aligned}\frac{\|(A-B)h\|}{\|h\|} &= \frac{\|f(x+h)-f(x)-Ah-(f(x+h)-f(x)-Bh)\|}{\|h\|}\\ &\leq \frac{\|f(x+h)-f(x)-Ah\|}{\|h\|}+\frac{\|f(x+h)-f(x)-Bh\|}{\|h\|}.\end{aligned}$$

So $\frac{\|(A-B)h\|}{\|h\|} \to 0$ as $h \to 0$. That is, given $\varepsilon > 0$, then for all h in some δ-ball around the origin

$$\varepsilon > \frac{\|(A-B)h\|}{\|h\|} = \left\|(A-B)\frac{h}{\|h\|}\right\|.$$

For any x with $\|x\| = 1$, let $h = (\delta/2)x$, then $\|h\| < \delta$ and $\frac{h}{\|h\|} = x$. So $\|(A-B)x\| < \varepsilon$. Taking the supremum over all x with $\|x\| = 1$ we get the operator norm $\|A-B\| \leq \varepsilon$. As $\varepsilon > 0$ was arbitrary $\|A-B\| = 0$ or in other words $A = B$. ☐

Example 8.3.3: If $f(x) = Ax$ for a linear mapping A, then $f'(x) = A$. This is easily seen:

$$\frac{\|f(x+h)-f(x)-Ah\|}{\|h\|} = \frac{\|A(x+h)-Ax-Ah\|}{\|h\|} = \frac{0}{\|h\|} = 0.$$

Example 8.3.4: Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $f(x,y) = \big(f_1(x,y), f_2(x,y)\big) := (1+x+2y+x^2, 2x+3y+xy)$. Let us show that f is differentiable at the origin and let us compute compute the derivative, directly using the definition. The derivative is in $L(\mathbb{R}^2,\mathbb{R}^2)$ so it can be represented by a 2×2 matrix $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$. Suppose $h = (h_1,h_2)$. We need the following expression to go to zero.

$$\begin{aligned}&\frac{\|f(h_1,h_2)-f(0,0)-(ah_1+bh_2, ch_1+dh_2)\|}{\|(h_1,h_2)\|} =\\ &\qquad\qquad \frac{\sqrt{\big((1-a)h_1+(2-b)h_2+h_1^2\big)^2+\big((2-c)h_1+(3-d)h_2+h_1h_2\big)^2}}{\sqrt{h_1^2+h_2^2}}.\end{aligned}$$

If we choose $a = 1$, $b = 2$, $c = 2$, $d = 3$, the expression becomes

$$\frac{\sqrt{h_1^4+h_1^2h_2^2}}{\sqrt{h_1^2+h_2^2}} = |h_1|\frac{\sqrt{h_1^2+h_2^2}}{\sqrt{h_1^2+h_2^2}} = |h_1|.$$

And this expression does indeed go to zero as $h \to 0$. Therefore the function is differentiable at the origin and the derivative can be represented by the matrix $\left[\begin{smallmatrix} 1 & 2 \\ 2 & 3 \end{smallmatrix}\right]$.

Proposition 8.3.5. *Let $U \subset \mathbb{R}^n$ be open and $f\colon U \to \mathbb{R}^m$ be differentiable at $p \in U$. Then f is continuous at p.*

Proof. Another way to write the differentiability of f at p is to first write

$$r(h) := f(p+h) - f(p) - f'(p)h,$$

and $\frac{\|r(h)\|}{\|h\|}$ must go to zero as $h \to 0$. So $r(h)$ itself must go to zero. The mapping $h \mapsto f'(p)h$ is a linear mapping between finite dimensional spaces, it is therefore continuous and goes to zero as $h \to 0$. Therefore, $f(p+h)$ must go to $f(p)$ as $h \to 0$. That is, f is continuous at p. □

Theorem 8.3.6 (Chain rule)**.** *Let $U \subset \mathbb{R}^n$ be open and let $f \colon U \to \mathbb{R}^m$ be differentiable at $p \in U$. Let $V \subset \mathbb{R}^m$ be open, $f(U) \subset V$ and let $g \colon V \to \mathbb{R}^\ell$ be differentiable at $f(p)$. Then*

$$F(x) = g\bigl(f(x)\bigr)$$

is differentiable at p and

$$F'(p) = g'\bigl(f(p)\bigr) f'(p).$$

Without the points where things are evaluated, this is sometimes written as $F' = (f \circ g)' = g' f'$. The way to understand it is that the derivative of the composition $g \circ f$ is the composition of the derivatives of g and f. That is, if $f'(p) = A$ and $g'\bigl(f(p)\bigr) = B$, then $F'(p) = BA$.

Proof. Let $A := f'(p)$ and $B := g'\bigl(f(p)\bigr)$. Take $h \in \mathbb{R}^n$ and write $q = f(p)$, $k = f(p+h) - f(p)$. Let

$$r(h) := f(p+h) - f(p) - Ah.$$

Then $r(h) = k - Ah$ or $Ah = k - r(h)$. We look at the quantity we need to go to zero:

$$\begin{aligned}
\frac{\|F(p+h) - F(p) - BAh\|}{\|h\|} &= \frac{\|g\bigl(f(p+h)\bigr) - g\bigl(f(p)\bigr) - BAh\|}{\|h\|} \\
&= \frac{\|g(q+k) - g(q) - B\bigl(k - r(h)\bigr)\|}{\|h\|} \\
&\leq \frac{\|g(q+k) - g(q) - Bk\|}{\|h\|} + \|B\| \frac{\|r(h)\|}{\|h\|} \\
&= \frac{\|g(q+k) - g(q) - Bk\|}{\|k\|} \frac{\|f(p+h) - f(p)\|}{\|h\|} + \|B\| \frac{\|r(h)\|}{\|h\|}.
\end{aligned}$$

First, $\|B\|$ is constant and f is differentiable at p, so the term $\|B\| \frac{\|r(h)\|}{\|h\|}$ goes to 0. Next as f is continuous at p, we have that as h goes to 0, then k goes to 0. Therefore $\frac{\|g(q+k) - g(q) - Bk\|}{\|k\|}$ goes to 0 because g is differentiable at q. Finally

$$\frac{\|f(p+h) - f(p)\|}{\|h\|} \leq \frac{\|f(p+h) - f(p) - Ah\|}{\|h\|} + \frac{\|Ah\|}{\|h\|} \leq \frac{\|f(p+h) - f(p) - Ah\|}{\|h\|} + \|A\|.$$

As f is differentiable at p, for small enough h $\|f(p+h) - f(p) - Ah\| \|h\|$ is bounded. Therefore the term $\frac{\|f(p+h) - f(p)\|}{\|h\|}$ stays bounded as h goes to 0. Therefore, $\frac{\|F(p+h) - F(p) - BAh\|}{\|h\|}$ goes to zero, and $F'(p) = BA$, which is what was claimed. □

8.3.2 Partial derivatives

There is another way to generalize the derivative from one dimension. We can hold all but one variable constant and take the regular derivative.

Definition 8.3.7. Let $f: U \to \mathbb{R}$ be a function on an open set $U \subset \mathbb{R}^n$. If the following limit exists we write

$$\frac{\partial f}{\partial x_j}(x) := \lim_{h\to 0} \frac{f(x_1,\ldots,x_{j-1},x_j+h,x_{j+1},\ldots,x_n)-f(x)}{h} = \lim_{h\to 0} \frac{f(x+he_j)-f(x)}{h}.$$

We call $\frac{\partial f}{\partial x_j}(x)$ the *partial derivative* of f with respect to x_j. Sometimes we write $D_j f$ instead.

For a mapping $f: U \to \mathbb{R}^m$ we write $f = (f_1, f_2, \ldots, f_m)$, where f_k are real-valued functions. Then we define $\frac{\partial f_k}{\partial x_j}$ (or write it as $D_j f_k$).

Partial derivatives are easier to compute with all the machinery of calculus, and they provide a way to compute the derivative of a function.

Proposition 8.3.8. *Let $U \subset \mathbb{R}^n$ be open and let $f: U \to \mathbb{R}^m$ be differentiable at $p \in U$. Then all the partial derivatives at p exist and in terms of the standard basis of $\mathbb{R}^n$ and $\mathbb{R}^m$, $f'(p)$ is represented by the matrix*

$$\begin{bmatrix} \frac{\partial f_1}{\partial x_1}(p) & \frac{\partial f_1}{\partial x_2}(p) & \ldots & \frac{\partial f_1}{\partial x_n}(p) \\ \frac{\partial f_2}{\partial x_1}(p) & \frac{\partial f_2}{\partial x_2}(p) & \ldots & \frac{\partial f_2}{\partial x_n}(p) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(p) & \frac{\partial f_m}{\partial x_2}(p) & \ldots & \frac{\partial f_m}{\partial x_n}(p) \end{bmatrix}.$$

In other words

$$f'(p)\, e_j = \sum_{k=1}^{m} \frac{\partial f_k}{\partial x_j}(p)\, e_k.$$

If $v = \sum_{j=1}^{n} c_j e_j = (c_1, c_2, \ldots, c_n)$, then

$$f'(p)\, v = \sum_{j=1}^{n}\sum_{k=1}^{m} c_j \frac{\partial f_k}{\partial x_j}(p)\, e_k = \sum_{k=1}^{m} \left(\sum_{j=1}^{n} c_j \frac{\partial f_k}{\partial x_j}(p) \right) e_k.$$

Proof. Fix a j and note that

$$\begin{aligned} \left\| \frac{f(p+he_j)-f(p)}{h} - f'(p)e_j \right\| &= \left\| \frac{f(p+he_j)-f(p)-f'(p)he_j}{h} \right\| \\ &= \frac{\|f(p+he_j)-f(p)-f'(p)he_j\|}{\|he_j\|}. \end{aligned}$$

As h goes to 0, the right hand side goes to zero by differentiability of f, and hence

$$\lim_{h\to 0} \frac{f(p+he_j)-f(p)}{h} = f'(p)e_j.$$

Note that f is vector valued. So represent f by components $f = (f_1, f_2, \ldots, f_m)$, and note that taking a limit in $\mathbb{R}^m$ is the same as taking the limit in each component separately. Therefore for any k the partial derivative

$$\frac{\partial f_k}{\partial x_j}(p) = \lim_{h\to 0} \frac{f_k(p+he_j)-f_k(p)}{h}$$

exists and is equal to the kth component of $f'(p)e_j$, and we are done. □

The converse of the proposition is not true. Just because the partial derivatives exist, does not mean that the function is differentiable. See the exercises. However, when the partial derivatives are continuous, we will prove that the converse holds. One of the consequences of the proposition is that if f is differentiable on U, then $f' \colon U \to L(\mathbb{R}^n, \mathbb{R}^m)$ is a continuous function if and only if all the $\frac{\partial f_k}{\partial x_j}$ are continuous functions.

8.3.3 Gradient and directional derivatives

Let $U \subset \mathbb{R}^n$ be open and $f \colon U \to \mathbb{R}$ is a differentiable function. We define the *gradient* as

$$\nabla f(x) := \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(x)\, e_j.$$

Notice that the gradient gives us a way to represent the action of the derivative as a dot product: $f'(x)v = \nabla f(x) \cdot v$.

Suppose $\gamma \colon (a,b) \subset \mathbb{R} \to \mathbb{R}^n$ is a differentiable function and the image $\gamma\bigl((a,b)\bigr) \subset U$. Such a function and its image is sometimes called a *curve*, or a *differentiable curve*. Write $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n)$. Let

$$g(t) := f\bigl(\gamma(t)\bigr).$$

The function g is differentiable. For purposes of computation we identify $L(\mathbb{R}^1)$ with $\mathbb{R}$, and hence $g'(t)$ can be computed as a number:

$$g'(t) = f'\bigl(\gamma(t)\bigr)\gamma'(t) = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}\bigl(\gamma(t)\bigr)\frac{d\gamma_j}{dt}(t) = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}\frac{d\gamma_j}{dt}.$$

For convenience, we sometimes leave out the points where we are evaluating as on the right hand side above. Let us rewrite this with the notation of the gradient and the dot product:

$$g'(t) = (\nabla f)\bigl(\gamma(t)\bigr) \cdot \gamma'(t) = \nabla f \cdot \gamma'.$$

We use this idea to define derivatives in a specific direction. A direction is simply a vector pointing in that direction. So pick a vector $u \in \mathbb{R}^n$ such that $\|u\| = 1$. Fix $x \in U$. Then define a curve

$$\gamma(t) := x + tu.$$

It is easy to compute that $\gamma'(t) = u$ for all t. By chain rule

$$\frac{d}{dt}\Big|_{t=0} \big[f(x+tu)\big] = (\nabla f)(x) \cdot u,$$

where the notation $\frac{d}{dt}\big|_{t=0}$ represents the derivative evaluated at $t = 0$. We also compute directly

$$\frac{d}{dt}\Big|_{t=0} \big[f(x+tu)\big] = \lim_{h \to 0} \frac{f(x+hu) - f(x)}{h}.$$

We obtain the *directional derivative*, denoted by

$$D_u f(x) := \frac{d}{dt}\Big|_{t=0} \big[f(x+tu)\big],$$

which can be computed by one of the methods above.

Let us suppose $(\nabla f)(x) \neq 0$. By Cauchy-Schwarz inequality we have

$$|D_u f(x)| \leq \|(\nabla f)(x)\|.$$

Equality is achieved when u is a scalar multiple of $(\nabla f)(x)$. That is, when

$$u = \frac{(\nabla f)(x)}{\|(\nabla f)(x)\|},$$

we get $D_u f(x) = \|(\nabla f)(x)\|$. The gradient points in the direction in which the function grows fastest, in other words, in the direction in which $D_u f(x)$ is maximal.

8.3.4 The Jacobian

Definition 8.3.9. Let $U \subset \mathbb{R}^n$ and $f \colon U \to \mathbb{R}^n$ be a differentiable mapping. Then define the *Jacobian*, or *Jacobian determinant**, of f at x as

$$J_f(x) := \det\big(f'(x)\big).$$

Sometimes this is written as

$$\frac{\partial(f_1, f_2, \ldots, f_n)}{\partial(x_1, x_2, \ldots, x_n)}.$$

*The matrix from Proposition 8.3.8 representing $f'(x)$ is sometimes called the *Jacobian matrix*.

This last piece of notation may seem somewhat confusing, but it is useful when you need to specify the exact variables and function components used.

The Jacobian J_f is a real valued function, and when $n = 1$ it is simply the derivative. From the chain rule and the fact that $\det(AB) = \det(A)\det(B)$, it follows that:

$$J_{f\circ g}(x) = J_f\big(g(x)\big)J_g(x).$$

As we mentioned the determinant tells us what happens to area/volume. Similarly, the Jacobian measures how much a differentiable mapping stretches things locally, and if it flips orientation. In particular if the Jacobian is non-zero than we would assume that locally the mapping is invertible (and we would be correct as we will later see).

8.3.5 Exercises

Exercise* 8.3.1*: *Suppose* $\gamma\colon (-1,1) \to \mathbb{R}^n$ *and* $\alpha\colon (-1,1) \to \mathbb{R}^n$ *be two differentiable curves such that* $\gamma(0) = \alpha(0)$ *and* $\gamma'(0) = \alpha'(0)$. *Suppose* $F\colon \mathbb{R}^n \to \mathbb{R}$ *is a differentiable function. Show that*

$$\frac{d}{dt}\Big|_{t=0} F\big(\gamma(t)\big) = \frac{d}{dt}\Big|_{t=0} F\big(\alpha(t)\big).$$

Exercise* 8.3.2*: *Let* $f\colon \mathbb{R}^2 \to \mathbb{R}$ *be given by* $f(x,y) = \sqrt{x^2+y^2}$. *Show that* f *is not differentiable at the origin.*

Exercise* 8.3.3*: *Using only the definition of the derivative, show that the following* $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ *are differentiable at the origin and find their derivative.*
a) $f(x,y) := (1+x+xy, x)$,
b) $f(x,y) := (y-y^{10}, x)$,
c) $f(x,y) := \big((x+y+1)^2, (x-y+2)^2\big)$.

Exercise* 8.3.4*: *Suppose* $f\colon \mathbb{R} \to \mathbb{R}$ *and* $g\colon \mathbb{R} \to \mathbb{R}$ *are differentiable functions. Using only the definition of the derivative, show that* $h\colon \mathbb{R}^2 \to \mathbb{R}^2$ *defined by* $h(x,y) := \big(f(x), g(y)\big)$ *is a differentiable function and find the derivative at any point* (x,y).

Exercise* 8.3.5*: *Define a function* $f\colon \mathbb{R}^2 \to \mathbb{R}$ *by*

$$f(x,y) := \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

a) Show that partial derivatives $\frac{\partial f}{\partial x}$ *and* $\frac{\partial f}{\partial y}$ *exist at all points (including the origin).*
b) Show that f *is not continuous at the origin (and hence not differentiable).*

Exercise* 8.3.6*: *Define a function* $f: \mathbb{R}^2 \to \mathbb{R}$ *by*

$$f(x,y) := \begin{cases} \frac{x^2y}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

a) Show that partial derivatives $\frac{\partial f}{\partial x}$ *and* $\frac{\partial f}{\partial y}$ *exist at all points.*
b) Show that for all $u \in \mathbb{R}^2$ *with* $\|u\| = 1$*, the directional derivative* $D_u f$ *exists at all points.*
c) Show that f *is continuous at the origin.*
d) Show that f *is not differentiable at the origin.*

Exercise* 8.3.7*: *Suppose* $f: \mathbb{R}^n \to \mathbb{R}^n$ *is one-to-one, onto, differentiable at all points, and such that* f^{-1} *is also differentiable at all points.*
a) Show that $f'(p)$ *is invertible at all points* p *and compute* $(f^{-1})'\big(f(p)\big)$*. Hint: consider* $p = f^{-1}\big(f(p)\big)$*.*
b) Let $g: \mathbb{R}^n \to \mathbb{R}^n$ *be a function differentiable at* $q \in \mathbb{R}^n$ *and such that* $g(q) = q$*. Suppose* $f(p) = q$ *for some* $p \in \mathbb{R}^n$*. Show* $J_g(q) = J_{f^{-1}\circ g \circ f}(p)$ *where* J_g *is the Jacobian determinant.*

Exercise* 8.3.8*: *Suppose* $f: \mathbb{R}^2 \to \mathbb{R}$ *is differentiable and such that* $f(x,y) = 0$ *if and only if* $y = 0$ *and such that* $\nabla f(0,0) = (1,1)$*. Prove that* $f(x,y) > 0$ *whenever* $y > 0$ *and* $f(x,y) < 0$ *whenever* $y < 0$*.*

Exercise* 8.3.9*: *Suppose* $U \subset \mathbb{R}^n$ *is open and* $f: U \to \mathbb{R}$ *is differentiable. Suppose that* f *has a local maximum at* $p \in U$*. Show that* $f'(p) = 0$*, that is the zero mapping in* $L(\mathbb{R}^n, \mathbb{R})$*. That is* p *is a* critical point *of* f*.*

Exercise* 8.3.10*: *Suppose* $f: \mathbb{R}^2 \to \mathbb{R}$ *is differentiable and suppose that whenever* $x^2 + y^2 = 1$*, then* $f(x,y) = 0$*. Prove that there exists at least one point* (x_0, y_0) *such that* $\frac{\partial f}{\partial x}(x_0,y_0) = \frac{\partial f}{\partial y}(x_0,y_0) = 0$*.*

Exercise* 8.3.11** (The Peano surface):*** *Define* $f(x,y) := (x - y^2)(2y^2 - x)$*. Show*
a) $(0,0)$ *is a critical point, that is* $f'(0,0) = 0$*, that is the zero linear map in* $L(\mathbb{R}^2, \mathbb{R})$*.*
b) For every direction, that is (x,y) *such that* $x^2 + y^2 = 1$ *the "restriction of* f *to the line containing the points* $(0,0)$ *and* (x,y)*", that is a function* $g(t) := f(tx,ty)$ *has a local maximum at* $t = 0$*.*
c) f *does not have a local maximum at* $(0,0)$*.*

Exercise* 8.3.12*: *Suppose* $f: \mathbb{R} \to \mathbb{R}^n$ *is differentiable and* $\|f(t)\| = 1$ *for all* t *(that is, we have a curve in the unit sphere). Then show that for all* t*, treating* f' *as a vector we have,* $f'(t) \cdot f(t) = 0$*.*

Exercise* 8.3.13*: *Define* $f: \mathbb{R}^2 \to \mathbb{R}^2$ *by* $f(x,y) := \big(x, y + \varphi(x)\big)$ *for some differentiable function* φ *of one variable. Show that* f *is differentiable and and find* f'*.*

8.4 Continuity and the derivative

Note: 1–2 lectures

8.4.1 Bounding the derivative

Let us prove a "mean value theorem" for vector valued functions.

Lemma 8.4.1. *If $\varphi \colon [a,b] \to \mathbb{R}^n$ is differentiable on (a,b) and continuous on $[a,b]$, then there exists a $t_0 \in (a,b)$ such that*

$$\|\varphi(b) - \varphi(a)\| \leq (b-a)\|\varphi'(t_0)\|.$$

Proof. By mean value theorem on the function $\big(\varphi(b)-\varphi(a)\big) \cdot \varphi(t)$ (the dot is the scalar dot product again) we obtain there is a $t_0 \in (a,b)$ such that

$$\big(\varphi(b)-\varphi(a)\big) \cdot \varphi(b) - \big(\varphi(b)-\varphi(a)\big) \cdot \varphi(a) = \|\varphi(b)-\varphi(a)\|^2 = (b-a)\big(\varphi(b)-\varphi(a)\big) \cdot \varphi'(t_0)$$

where we treat φ' as a simply a column vector of numbers by abuse of notation. Note that in this case, if we think of $\varphi'(t)$ as simply a vector, then by Exercise 8.2.6, $\|\varphi'(t)\|_{L(\mathbb{R},\mathbb{R}^n)} = \|\varphi'(t)\|_{\mathbb{R}^n}$. That is, the euclidean norm of the vector is the same as the operator norm of $\varphi'(t)$.

By Cauchy-Schwarz inequality

$$\|\varphi(b)-\varphi(a)\|^2 = (b-a)\big(\varphi(b)-\varphi(a)\big) \cdot \varphi'(t_0) \leq (b-a)\|\varphi(b)-\varphi(a)\|\|\varphi'(t_0)\|. \qquad \square$$

Recall that a set U is convex if whenever $x,y \in U$, the line segment from x to y lies in U.

Proposition 8.4.2. *Let $U \subset \mathbb{R}^n$ be a convex open set, $f \colon U \to \mathbb{R}^m$ a differentiable function, and an M such that*

$$\|f'(x)\| \leq M$$

for all $x \in U$. Then f is Lipschitz with constant M, that is

$$\|f(x) - f(y)\| \leq M\|x-y\|$$

for all $x,y \in U$.

Proof. Fix x and y in U and note that $(1-t)x+ty \in U$ for all $t \in [0,1]$ by convexity. Next

$$\frac{d}{dt}\Big[f\big((1-t)x+ty\big)\Big] = f'\big((1-t)x+ty\big)(y-x).$$

By the mean value theorem above we get for some $t_0 \in (0,1)$

$$\|f(x)-f(y)\| \leq \left\| \frac{d}{dt}\Big|_{t=t_0} \Big[f\big((1-t)x+ty\big)\Big] \right\| \leq \|f'\big((1-t_0)x+t_0y\big)\|\|y-x\| \leq M\|y-x\|. \qquad \square$$

Example 8.4.3: If U is not convex the proposition is not true. To see this fact, take the set

$$U = \{(x,y) : 0.9 < x^2 + y^2 < 1.1\} \setminus \{(x,0) : x < 0\}.$$

Let $f(x,y)$ be the angle that the line from the origin to (x,y) makes with the positive x axis. You can even write the formula for f:

$$f(x,y) = 2 \arctan \left(\frac{y}{x + \sqrt{x^2+y^2}} \right).$$

Think spiral staircase with room in the middle. See Figure 8.2.

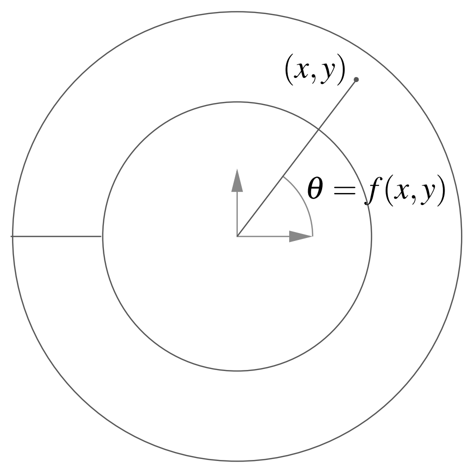

Figure 8.2: A non-Lipschitz function with uniformly bounded derivative.

The function is differentiable, and the derivative is bounded on U, which is not hard to see. Thinking of what happens near where the negative x-axis cuts the annulus in half, we see that the conclusion of the proposition cannot hold.

Let us solve the differential equation $f' = 0$.

Corollary 8.4.4. *If $U \subset \mathbb{R}^n$ is connected and $f \colon U \to \mathbb{R}^m$ is differentiable and $f'(x) = 0$, for all $x \in U$, then f is constant.*

Proof. For any $x \in U$, there is a ball $B(x,\delta) \subset U$. The ball $B(x,\delta)$ is convex. Since $\lVert f'(y) \rVert \leq 0$ for all $y \in B(x,\delta)$ then by the theorem, $\lVert f(x) - f(y) \rVert \leq 0 \lVert x - y \rVert = 0$, so $f(x) = f(y)$ for all $y \in B(x,\delta)$.

This means that $f^{-1}(c)$ is open for any $c \in \mathbb{R}^m$. Suppose $f^{-1}(c)$ is nonempty. The two sets

$$U' = f^{-1}(c), \qquad U'' = f^{-1}(\mathbb{R}^m \setminus \{c\}) = \bigcup_{\substack{a \in \mathbb{R}^m \\ a \neq c}} f^{-1}(a)$$

are open disjoint, and further $U = U' \cup U''$. So as U' is nonempty, and U is connected, we have that $U'' = \emptyset$. So $f(x) = c$ for all $x \in U$. $\square$

8.4.2 Continuously differentiable functions

Definition 8.4.5. We say $f: U \subset \mathbb{R}^n \to \mathbb{R}^m$ is *continuously differentiable*, or $C^1(U)$ if f is differentiable and $f': U \to L(\mathbb{R}^n, \mathbb{R}^m)$ is continuous.

Proposition 8.4.6. *Let $U \subset \mathbb{R}^n$ be open and $f: U \to \mathbb{R}^m$. The function f is continuously differentiable if and only if all the partial derivatives exist and are continuous.*

Without continuity the theorem does not hold. Just because partial derivatives exist does not mean that f is differentiable, in fact, f may not even be continuous. See the exercises for the last section and also for this section.

Proof. We have seen that if f is differentiable, then the partial derivatives exist. Furthermore, the partial derivatives are the entries of the matrix of $f'(x)$. So if $f': U \to L(\mathbb{R}^n, \mathbb{R}^m)$ is continuous, then the entries are continuous, hence the partial derivatives are continuous.

To prove the opposite direction, suppose the partial derivatives exist and are continuous. Fix $x \in U$. If we can show that $f'(x)$ exists we are done, because the entries of the matrix $f'(x)$ are then the partial derivatives and if the entries are continuous functions, the matrix valued function f' is continuous.

Let us do induction on dimension. First let us note that the conclusion is true when $n = 1$. In this case the derivative is just the regular derivative (exercise: you should check that the fact that the function is vector valued is not a problem).

Suppose the conclusion is true for $\mathbb{R}^{n-1}$, that is, if we restrict to the first $n-1$ variables, the conclusion is true. It is easy to see that the first $n-1$ partial derivatives of f restricted to the set where the last coordinate is fixed are the same as those for f. In the following we think of $\mathbb{R}^{n-1}$ as a subset of $\mathbb{R}^n$, that is the set in $\mathbb{R}^n$ where $x_n = 0$. Let

$$A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(x) & \dots & \frac{\partial f_1}{\partial x_n}(x) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(x) & \dots & \frac{\partial f_m}{\partial x_n}(x) \end{bmatrix}, \qquad A_1 = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(x) & \dots & \frac{\partial f_1}{\partial x_{n-1}}(x) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(x) & \dots & \frac{\partial f_m}{\partial x_{n-1}}(x) \end{bmatrix}, \qquad v = \begin{bmatrix} \frac{\partial f_1}{\partial x_n}(x) \\ \vdots \\ \frac{\partial f_m}{\partial x_n}(x) \end{bmatrix}.$$

Let $\varepsilon > 0$ be given. Let $\delta > 0$ be such that for any $k \in \mathbb{R}^{n-1}$ with $\|k\| < \delta$ we have

$$\frac{\|f(x+k) - f(x) - A_1 k\|}{\|k\|} < \varepsilon.$$

By continuity of the partial derivatives, suppose δ is small enough so that

$$\left| \frac{\partial f_j}{\partial x_n}(x+h) - \frac{\partial f_j}{\partial x_n}(x) \right| < \varepsilon,$$

for all j and all h with $\|h\| < \delta$.

Let $h = h_1 + te_n$ be a vector in $\mathbb{R}^n$ where $h_1 \in \mathbb{R}^{n-1}$ such that $\|h\| < \delta$. Then $\|h_1\| \leq \|h\| < \delta$. Note that $Ah = A_1 h_1 + tv$.

$$\begin{aligned}
\|f(x+h) - f(x) - Ah\| &= \|f(x+h_1+te_n) - f(x+h_1) - tv + f(x+h_1) - f(x) - A_1h_1\| \\
&\leq \|f(x+h_1+te_n) - f(x+h_1) - tv\| + \|f(x+h_1) - f(x) - A_1h_1\| \\
&\leq \|f(x+h_1+te_n) - f(x+h_1) - tv\| + \varepsilon\|h_1\|.
\end{aligned}$$

As all the partial derivatives exist then by the mean value theorem for each j there is some $\theta_j \in [0,t]$ (or $[t,0]$ if $t < 0$), such that

$$f_j(x+h_1+te_n) - f_j(x+h_1) = t\frac{\partial f_j}{\partial x_n}(x+h_1+\theta_j e_n).$$

Note that if $\|h\| < \delta$ then $\|h_1 + \theta_j e_n\| \leq \|h\| < \delta$. So to finish the estimate

$$\begin{aligned}
\|f(x+h) - f(x) - Ah\| &\leq \|f(x+h_1+te_n) - f(x+h_1) - tv\| + \varepsilon\|h_1\| \\
&\leq \sqrt{\sum_{j=1}^{m}\left(t\frac{\partial f_j}{\partial x_n}(x+h_1+\theta_j e_n) - t\frac{\partial f_j}{\partial x_n}(x)\right)^2} + \varepsilon\|h_1\| \\
&\leq \sqrt{m}\,\varepsilon\,|t| + \varepsilon\|h_1\| \\
&\leq (\sqrt{m}+1)\varepsilon\|h\|.
\end{aligned}$$

□

8.4.3 Exercises

Exercise* 8.4.1*: *Define $f\colon \mathbb{R}^2 \to \mathbb{R}$ as*

$$f(x,y) := \begin{cases} (x^2+y^2)\sin\big((x^2+y^2)^{-1}\big) & \textit{if } (x,y) \neq (0,0), \\ 0 & \textit{else.} \end{cases}$$

Show that f is differentiable at the origin, but that it is not continuously differentiable.

Exercise* 8.4.2*: *Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be the function from Exercise 8.3.5, that is,*

$$f(x,y) := \begin{cases} \frac{xy}{x^2+y^2} & \textit{if } (x,y) \neq (0,0), \\ 0 & \textit{if } (x,y) = (0,0). \end{cases}$$

Compute the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at all points and show that these are not continuous functions.

Exercise* 8.4.3*: *Let $B(0,1) \subset \mathbb{R}^2$ be the unit ball (disc), that is, the set given by $x^2 + y^2 < 1$. Suppose $f : B(0,1) \to \mathbb{R}$ is a differentiable function such that $|f(0,0)| \leq 1$, and $\left|\frac{\partial f}{\partial x}\right| \leq 1$ and $\left|\frac{\partial f}{\partial y}\right| \leq 1$ for all points in $B(0,1)$.*
a) Find an $M \in \mathbb{R}$ such that $\|f'(x,y)\| \leq M$ for all $(x,y) \in B(0,1)$.
b) Find a $B \in \mathbb{R}$ such that $|f(x,y)| \leq B$ for all $(x,y) \in B(0,1)$.

Exercise* 8.4.4*: *Define $\varphi : [0,2\pi] \to \mathbb{R}^2$ by $\varphi(t) = \big(\sin(t), \cos(t)\big)$. Compute $\varphi'(t)$ for all t. Compute $\|\varphi'(t)\|$ for all t. Notice that $\varphi'(t)$ is never zero, yet $\varphi(0) = \varphi(2\pi)$, therefore, Rolle's theorem is not true in more than one dimension.*

Exercise* 8.4.5*: *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function such that $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist at all points and there exists an $M \in \mathbb{R}$ such that $\left|\frac{\partial f}{\partial x}\right| \leq M$ and $\left|\frac{\partial f}{\partial y}\right| \leq M$ at all points. Show that f is continuous.*

Exercise* 8.4.6*: *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function and $M \in R$, such that for every $(x,y) \in \mathbb{R}^2$, the function $g(t) := f(xt, yt)$ is differentiable and $|g'(t)| \leq M$.*
a) Show that f is continuous at $(0,0)$.
b) Find an example of such an f which is not continuous at every other point of $\mathbb{R}^2$ (Hint: Think back to how did we construct a nowhere continuous function on $[0,1]$).

8.5 Inverse and implicit function theorem

Note: 2–3 lectures

To prove the inverse function theorem we use the contraction mapping principle we have seen in chapter 7 and that we have used to prove Picard's theorem. Recall that a mapping $f\colon X \to X'$ between two metric spaces (X,d) and (X',d') is called a contraction if there exists a $k < 1$ such that

$$d'\big(f(x), f(y)\big) \leq k d(x,y) \quad \text{for all } x,y \in X.$$

The contraction mapping principle says that if $f\colon X \to X$ is a contraction and X is a complete metric space, then there exists a unique fixed point, that is, there exists a unique $x \in X$ such that $f(x) = x$.

Intuitively if a function is differentiable, then it locally "behaves like" the derivative (which is a linear function). The idea of the inverse function theorem is that if a function is differentiable and the derivative is invertible, the function is (locally) invertible.

Theorem 8.5.1 (Inverse function theorem)**.** *Let $U \subset \mathbb{R}^n$ be a set and let $f\colon U \to \mathbb{R}^n$ be a continuously differentiable function. Also suppose $p \in U$, $f(p) = q$, and $f'(p)$ is invertible (that is, $J_f(p) \neq 0$). Then there exist open sets $V, W \subset \mathbb{R}^n$ such that $p \in V \subset U$, $f(V) = W$ and $f|_V$ is one-to-one and onto. Furthermore, the inverse $g(y) = (f|_V)^{-1}(y)$ is continuously differentiable and*

$$g'(y) = \big(f'(x)\big)^{-1}, \qquad \textit{for all } x \in V,\ y = f(x).$$

Proof. Write $A = f'(p)$. As f' is continuous, there exists an open ball V around p such that

$$\lVert A - f'(x) \rVert < \frac{1}{2\lVert A^{-1} \rVert} \qquad \text{for all } x \in V.$$

Note that $f'(x)$ is invertible for all $x \in V$.

Given $y \in \mathbb{R}^n$ we define $\varphi_y\colon C \to \mathbb{R}^n$

$$\varphi_y(x) = x + A^{-1}\big(y - f(x)\big).$$

As A^{-1} is one-to-one, then $\varphi_y(x) = x$ (x is a fixed point) if only if $y - f(x) = 0$, or in other words $f(x) = y$. Using chain rule we obtain

$$\varphi_y'(x) = I - A^{-1} f'(x) = A^{-1}\big(A - f'(x)\big).$$

So for $x \in V$ we have

$$\lVert \varphi_y'(x) \rVert \leq \lVert A^{-1} \rVert \lVert A - f'(x) \rVert < 1/2.$$

As V is a ball it is convex, and hence

$$\lVert \varphi_y(x_1) - \varphi_y(x_2) \rVert \leq \frac{1}{2} \lVert x_1 - x_2 \rVert \qquad \text{for all } x_1, x_2 \in V.$$

In other words φ_y is a contraction defined on V, though we so far do not know what is the range of φ_y. We cannot apply the fixed point theorem, but we can say that φ_y has at most one fixed point (note proof of uniqueness in the contraction mapping principle). That is, there exists at most one $x \in V$ such that $f(x) = y$, and so $f|_V$ is one-to-one.

Let $W = f(V)$. We need to show that W is open. Take a $y_1 \in W$, then there is a unique $x_1 \in V$ such that $f(x_1) = y_1$. Let $r > 0$ be small enough such that the closed ball $C(x_1,r) \subset V$ (such $r > 0$ exists as V is open).

Suppose y is such that

$$\|y - y_1\| < \frac{r}{2\|A^{-1}\|}.$$

If we can show that $y \in W$, then we have shown that W is open. Define $\varphi_y(x) = x + A^{-1}\bigl(y - f(x)\bigr)$ as before. If $x \in C(x_1,r)$, then

$$\begin{aligned}
\|\varphi_y(x) - x_1\| &\leq \|\varphi_y(x) - \varphi_y(x_1)\| + \|\varphi_y(x_1) - x_1\| \\
&\leq \frac{1}{2}\|x - x_1\| + \|A^{-1}(y - y_1)\| \\
&\leq \frac{1}{2}r + \|A^{-1}\|\|y - y_1\| \\
&< \frac{1}{2}r + \|A^{-1}\|\frac{r}{2\|A^{-1}\|} = r.
\end{aligned}$$

So φ_y takes $C(x_1,r)$ into $B(x_1,r) \subset C(x_1,r)$. It is a contraction on $C(x_1,r)$ and $C(x_1,r)$ is complete (closed subset of $\mathbb{R}^n$ is complete). Apply the contraction mapping principle to obtain a fixed point x, i.e. $\varphi_y(x) = x$. That is $f(x) = y$. So $y \in f\bigl(C(x_1,r)\bigr) \subset f(V) = W$. Therefore W is open.

Next we need to show that g is continuously differentiable and compute its derivative. First let us show that it is differentiable. Let $y \in W$ and $k \in \mathbb{R}^n$, $k \neq 0$, such that $y + k \in W$. Then there are unique $x \in V$ and $h \in \mathbb{R}^n$, $h \neq 0$ and $x + h \in V$, such that $f(x) = y$ and $f(x+h) = y+k$ as $f|_V$ is a one-to-one and onto mapping of V onto W. In other words, $g(y) = x$ and $g(y+k) = x+h$. We can still squeeze some information from the fact that φ_y is a contraction.

$$\varphi_y(x+h) - \varphi_y(x) = h + A^{-1}\bigl(f(x) - f(x+h)\bigr) = h - A^{-1}k.$$

So

$$\|h - A^{-1}k\| = \|\varphi_y(x+h) - \varphi_y(x)\| \leq \frac{1}{2}\|x + h - x\| = \frac{\|h\|}{2}.$$

By the inverse triangle inequality $\|h\| - \|A^{-1}k\| \leq \frac{1}{2}\|h\|$ so

$$\|h\| \leq 2\|A^{-1}k\| \leq 2\|A^{-1}\|\|k\|.$$

In particular as k goes to 0, so does h.

As $x \in V$, then $f'(x)$ is invertible. Let $B = \bigl(f'(x)\bigr)^{-1}$, which is what we think the derivative of g at y is. Then

$$\begin{aligned}
\frac{\|g(y+k) - g(y) - Bk\|}{\|k\|} &= \frac{\|h - Bk\|}{\|k\|} \\
&= \frac{\|h - B\bigl(f(x+h) - f(x)\bigr)\|}{\|k\|} \\
&= \frac{\|B\bigl(f(x+h) - f(x) - f'(x)h\bigr)\|}{\|k\|} \\
&\leq \|B\| \frac{\|h\|}{\|k\|} \frac{\|f(x+h) - f(x) - f'(x)h\|}{\|h\|} \\
&\leq 2\|B\| \|A^{-1}\| \frac{\|f(x+h) - f(x) - f'(x)h\|}{\|h\|}.
\end{aligned}$$

As k goes to 0, so does h. So the right hand side goes to 0 as f is differentiable, and hence the left hand side also goes to 0. And B is precisely what we wanted $g'(y)$ to be.

We have g is differentiable, let us show it is $C^1(W)$. Now, $g\colon W \to V$ is continuous (it is differentiable), f' is a continuous function from V to $L(\mathbb{R}^n)$, and $X \to X^{-1}$ is a continuous function. $g'(y) = \bigl(f'\bigl(g(y)\bigr)\bigr)^{-1}$ is the composition of these three continuous functions and hence is continuous. □

Corollary 8.5.2. *Suppose $U \subset \mathbb{R}^n$ is open and $f\colon U \to \mathbb{R}^n$ is a continuously differentiable mapping such that $f'(x)$ is invertible for all $x \in U$. Then given any open set $V \subset U$, $f(V)$ is open. (f is an open mapping).*

Proof. Without loss of generality, suppose $U = V$. For each point $y \in f(V)$, we pick $x \in f^{-1}(y)$ (there could be more than one such point), then by the inverse function theorem there is a neighbourhood of x in V that maps onto an neighbourhood of y. Hence $f(V)$ is open. □

Example 8.5.3: The theorem, and the corollary, is not true if $f'(x)$ is not invertible for some x. For example, the map $f(x,y) = (x,xy)$, maps $\mathbb{R}^2$ onto the set $\mathbb{R}^2 \setminus \{(0,y) : y \neq 0\}$, which is neither open nor closed. In fact $f^{-1}(0,0) = \{(0,y) : y \in \mathbb{R}\}$. This bad behaviour only occurs on the y-axis, everywhere else the function is locally invertible. If we avoid the y-axis, f is even one-to-one.

Example 8.5.4: Also note that just because $f'(x)$ is invertible everywhere does not mean that f is one-to-one globally. It is "locally" one-to-one but perhaps not "globally." For an example, take the map $f\colon \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^2$ defined by $f(x,y) = (x^2 - y^2, 2xy)$. It is left to student to show that f is differentiable and the derivative is invertible

On the other hand, the mapping is 2-to-1 globally. For every (a,b) that is not the origin, there are exactly two solutions to $x^2 - y^2 = a$ and $2xy = b$. We leave it to the student to show that there is at least one solution, and then notice that replacing x and y with $-x$ and $-y$ we obtain another solution.

The invertibility of the derivative is not a necessary condition, just sufficient, for having a continuous inverse and being an open mapping. For example the function $f(x) = x^3$ is an open mapping from $\mathbb{R}$ to $\mathbb{R}$ and is globally one-to-one with a continuous inverse, although the inverse is not differentiable at $x = 0$.

8.5.1 Implicit function theorem

The inverse function theorem is really a special case of the implicit function theorem which we prove next. Although somewhat ironically we prove the implicit function theorem using the inverse function theorem. What we were showing in the inverse function theorem was that the equation $x - f(y) = 0$ was solvable for y in terms of x if the derivative in terms of y was invertible, that is if $f'(y)$ was invertible. That is there was locally a function g such that $x - f\big(g(x)\big) = 0$.

OK, so how about we look at the equation $f(x,y) = 0$. Obviously this is not solvable for y in terms of x in every case. For example, when $f(x,y)$ does not actually depend on y. For a slightly more complicated example, notice that $x^2 + y^2 - 1 = 0$ defines the unit circle, and we can locally solve for y in terms of x when 1) we are near a point which lies on the unit circle and 2) when we are not at a point where the circle has a vertical tangency, or in other words where $\frac{\partial f}{\partial y} = 0$.

To make things simple we fix some notation. We let $(x,y) \in \mathbb{R}^{n+m}$ denote the coordinates $(x_1,\ldots,x_n,y_1,\ldots,y_m)$. A linear transformation $A \in L(\mathbb{R}^{n+m},\mathbb{R}^m)$ can then always be written as $A = [A_x\, A_y]$ so that $A(x,y) = A_x x + A_y y$, where $A_x \in L(\mathbb{R}^n,\mathbb{R}^m)$ and $A_y \in L(\mathbb{R}^m)$.

Proposition 8.5.5. *Let $A = [A_x\, A_y] \in L(\mathbb{R}^{n+m},\mathbb{R}^m)$ and suppose A_y is invertible. If $B = -(A_y)^{-1}A_x$, then*

$$0 = A(x,Bx) = A_x x + A_y Bx.$$

The proof is obvious. We simply solve and obtain $y = Bx$. Let us therefore show that the same can be done for C^1 functions.

Theorem 8.5.6 (Implicit function theorem)**.** *Let $U \subset \mathbb{R}^{n+m}$ be an open set and let $f \colon U \to \mathbb{R}^m$ be a $C^1(U)$ mapping. Let $(p,q) \in U$ be a point such that $f(p,q) = 0$ and such that*

$$\frac{\partial(f_1,\ldots,f_m)}{\partial(y_1,\ldots,y_m)}(p,q) \neq 0.$$

Then there exists an open set $W \subset \mathbb{R}^n$ with $p \in W$, an open set $W' \subset \mathbb{R}^m$ with $q \in W'$, with $W \times W' \subset U$, and a $C^1(W)$ mapping $g \colon W \to W'$, with $g(p) = q$, and for all $x \in W$, the point $g(x)$ is the unique point in W' such that

$$f\big(x, g(x)\big) = 0.$$

Furthermore, if $[A_x\, A_y] = f'(p,q)$, then

$$g'(p) = -(A_y)^{-1}A_x.$$

The condition $\frac{\partial(f_1,\ldots,f_m)}{\partial(y_1,\ldots,y_m)}(p,q) = \det(A_y) \neq 0$ simply means that A_y is invertible.

Proof. Define $F \colon U \to \mathbb{R}^{n+m}$ by $F(x,y) := \big(x, f(x,y)\big)$. It is clear that F is C^1, and we want to show that the derivative at (p,q) is invertible.

Let us compute the derivative. We know that

$$\frac{\|f(p+h,q+k) - f(p,q) - A_x h - A_y k\|}{\|(h,k)\|}$$

goes to zero as $\|(h,k)\| = \sqrt{\|h\|^2 + \|k\|^2}$ goes to zero. But then so does

$$\frac{\big\|\big(h, f(p+h,q+k) - f(p,q)\big) - (h, A_x h + A_y k)\big\|}{\|(h,k)\|} = \frac{\|f(p+h,q+k) - f(p,q) - A_x h - A_y k\|}{\|(h,k)\|}.$$

So the derivative of F at (p,q) takes (h,k) to $(h, A_x h + A_y k)$. If $(h, A_x h + A_y k) = (0,0)$, then $h = 0$, and so $A_y k = 0$. As A_y is one-to-one, then $k = 0$. Therefore $F'(p,q)$ is one-to-one or in other words invertible and we can apply the inverse function theorem.

That is, there exists some open set $V \subset \mathbb{R}^{n+m}$ with $(p,0) \in V$, and an inverse mapping $G \colon V \to \mathbb{R}^{n+m}$, that is $F\big(G(x,s)\big) = (x,s)$ for all $(x,s) \in V$ (where $x \in \mathbb{R}^n$ and $s \in \mathbb{R}^m$). Write $G = (G_1, G_2)$ (the first n and the second m components of G). Then

$$F\big(G_1(x,s), G_2(x,s)\big) = \big(G_1(x,s), f(G_1(x,s), G_2(x,s))\big) = (x,s).$$

So $x = G_1(x,s)$ and $f\big(G_1(x,s), G_2(x,s)\big) = f\big(x, G_2(x,s)\big) = s$. Plugging in $s = 0$ we obtain

$$f\big(x, G_2(x,0)\big) = 0.$$

The set $G(V)$ contains a whole neighbourhood of the point (p,q) and therefore there are some open The set V is open and hence there exist some open sets $\widetilde{W}$ and W' such that $\widetilde{W} \times W' \subset G(V)$ with $p \in \widetilde{W}$ and $q \in W'$. Then take $W = \{x \in \widetilde{W} : G_2(x,0) \in W'\}$. The function that takes x to $G_2(x,0)$ is continuous and therefore W is open. We define $g \colon W \to \mathbb{R}^m$ by $g(x) := G_2(x,0)$ which is the g in the theorem. The fact that $g(x)$ is the unique point in W' follows because $W \times W' \subset G(V)$ and G is one-to-one and onto $G(V)$.

Next differentiate

$$x \mapsto f\big(x, g(x)\big),$$

at p, which should be the zero map. The derivative is done in the same way as above. We get that for all $h \in \mathbb{R}^n$

$$0 = A\big(h, g'(p)h\big) = A_x h + A_y g'(p) h,$$

and we obtain the desired derivative for g as well. $\square$

In other words, in the context of the theorem we have m equations in $n+m$ unknowns.

$$f_1(x_1,\ldots,x_n,y_1,\ldots,y_m) = 0$$
$$\vdots$$
$$f_m(x_1,\ldots,x_n,y_1,\ldots,y_m) = 0$$

And the condition guaranteeing a solution is that this is a C^1 mapping (that all the components are C^1, or in other words all the partial derivatives exist and are continuous), and the matrix

$$\begin{bmatrix} \frac{\partial f_1}{\partial y_1} & \cdots & \frac{\partial f_1}{\partial y_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial y_1} & \cdots & \frac{\partial f_m}{\partial y_m} \end{bmatrix}$$

is invertible at (p,q).

Example 8.5.7: Consider the set $x^2+y^2-(z+1)^3 = -1$, $e^x+e^y+e^z = 3$ near the point $(0,0,0)$. The function we are looking at is

$$f(x,y,z) = (x^2+y^2-(z+1)^3+1, e^x+e^y+e^z-3).$$

We find that

$$f' = \begin{bmatrix} 2x & 2y & -3(z+1)^2 \\ e^x & e^y & e^z \end{bmatrix}.$$

The matrix

$$\begin{bmatrix} 2(0) & -3(0+1)^2 \\ e^0 & e^0 \end{bmatrix} = \begin{bmatrix} 0 & -3 \\ 1 & 1 \end{bmatrix}$$

is invertible. Hence near $(0,0,0)$ we can find y and z as C^1 functions of x such that for x near 0 we have

$$x^2+y(x)^2-\big(z(x)+1\big)^3 = -1, \qquad e^x+e^{y(x)}+e^{z(x)} = 3.$$

The theorem does not tell us how to find $y(x)$ and $z(x)$ explicitly, it just tells us they exist. In other words, near the origin the set of solutions is a smooth curve in $\mathbb{R}^3$ that goes through the origin.

Note that there are versions of the theorem for arbitrarily many derivatives. If f has k continuous derivatives, then the solution also has k derivatives.

8.5.2 Exercises

Exercise* 8.5.1*: *Let $C = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 = 1\}$.*
a) Solve for y in terms of x near $(0,1)$.
b) Solve for y in terms of x near $(0,-1)$.
c) Solve for x in terms of y near $(-1,0)$.

Exercise* 8.5.2*: *Define $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ by $f(x,y) := \bigl(x, y+h(x)\bigr)$ for some continuously differentiable function h of one variable.*
a) Show that f is one-to-one and onto.
b) Compute f'.
c) Show that f' is invertible at all points, and compute its inverse.

Exercise* 8.5.3*: *Define $f\colon \mathbb{R}^2 \to \mathbb{R}^2 \setminus \{(0,0)\}$ by $f(x,y) := \bigl(e^x\cos(y), e^x\sin(y)\bigr)$.*
a) Show that f is onto.
b) Show that f' is invertible at all points.
c) Show that f is not one-to-one, in fact for every $(a,b) \in \mathbb{R}^2 \setminus \{(0,0)\}$, there exist infinitely many different points $(x,y) \in \mathbb{R}^2$ such that $f(x,y) = (a,b)$.
Therefore, invertible derivative at every point does not mean that f is invertible globally.

Exercise* 8.5.4*: *Find a map $f\colon \mathbb{R}^n \to \mathbb{R}^n$ that is one-to-one, onto, continuously differentiable, but $f'(0) = 0$. Hint: Generalize $f(x) = x^3$ from one to n dimensions.*

Exercise* 8.5.5*: *Consider $z^2 + xz + y = 0$ in $\mathbb{R}^3$. Find an equation $D(x,y) = 0$, such that if $D(x_0,y_0) \neq 0$ and $z^2 + x_0 z + y_0 = 0$ for some $z \in \mathbb{R}$, then for points near (x_0,y_0) there exist exactly two distinct continuously differentiable functions $r_1(x,y)$ and $r_2(x,y)$ such that $z = r_1(x,y)$ and $z = r_2(x,y)$ solve $z^2 + xz + y = 0$. Do you recognize the expression D from algebra?*

Exercise* 8.5.6*: *Suppose $f\colon (a,b) \to \mathbb{R}^2$ is continuously differentiable and $\frac{\partial f}{\partial x}(t) \neq 0$ for all $t \in (a,b)$. Prove that there exists an interval (c,d) and a continuously differentiable function $g\colon (c,d) \to \mathbb{R}$ such that $(x,y) \in f\bigl((a,b)\bigr)$ if and only if $x \in (c,d)$ and $y = g(x)$. In other words, the set $f\bigl((a,b)\bigr)$ is a graph of g.*

Exercise* 8.5.7*: *Define $f\colon \mathbb{R}^2 \to \mathbb{R}^2$*

$$f(x,y) := \begin{cases} (x^2 \sin\bigl(1/x\bigr) + \frac{x}{2}, y) & \text{if } x \neq 0, \\ (0,y) & \text{if } x = 0. \end{cases}$$

a) Show that f is differentiable everywhere.
b) Show that $f'(0,0)$ is invertible.
c) Show that f is not one-to-one in any neighbourhood of the origin (it is not locally invertible, that is, the inverse theorem does not work).
d) Show that f is not continuously differentiable.

Exercise* 8.5.8** (Polar coordinates):*** *Define a mapping $F(r,\theta) := \bigl(r\cos(\theta), r\sin(\theta)\bigr)$.*
a) Show that F is continuously differentiable (for all $(r,\theta) \in \mathbb{R}^2$).
b) Compute $F'(0,\theta)$ for any θ.
c) Show that if $r \neq 0$, then $F'(r,\theta)$ is invertible, therefore an inverse of F exists locally as long as $r \neq 0$.
d) Show that $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ is onto, and for each point $(x,y) \in \mathbb{R}^2$, the set $F^{-1}(x,y)$ is infinite.
e) Show that $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ is an open map, despite not satisfying the condition of the inverse function theorem.
f) Show that $F|_{(0,\infty)\times[0,2\pi)}$ is one to one and onto $\mathbb{R}^2 \setminus \{(0,0)\}$.

8.6 Higher order derivatives

Note: less than 1 lecture

Let $U \colon \mathbb{R}^n$ be an open set and $f \colon U \to \mathbb{R}$ a function. Denote by $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ our coordinates. Suppose that $\frac{\partial f}{\partial x_j}$ exists everywhere in U, then we note that it is also a function $\frac{\partial f}{\partial x_j} \colon U \to \mathbb{R}$. Therefore it makes sense to talk about its partial derivatives. We denote the partial derivative of $\frac{\partial f}{\partial x_j}$ with respect to x_k by

$$\frac{\partial^2 f}{\partial x_k \partial x_j} := \frac{\partial \left(\frac{\partial f}{\partial x_j} \right)}{\partial x_k}.$$

If $k = j$ then we write $\frac{\partial^2 f}{\partial x_j^2}$ for simplicity.

In general we can define higher order derivatives inductively. Suppose $j_1, j_2, \ldots, j_\ell$ are integers between 1 and n, and suppose that

$$\frac{\partial^{\ell-1} f}{\partial x_{j_{\ell-1}} \partial x_{j_{\ell-2}} \cdots \partial x_{j_1}}$$

exists and is differentiable in the variable x_{j_ℓ}, then the partial derivative with respect to that variable is denoted by

$$\frac{\partial^{\ell} f}{\partial x_{j_\ell} \partial x_{j_{\ell-1}} \cdots \partial x_{j_1}} := \frac{\partial \left(\frac{\partial^{\ell-1} f}{\partial x_{j_{\ell-1}} \partial x_{j_{\ell-2}} \cdots \partial x_{j_1}} \right)}{\partial x_{j_\ell}}.$$

Such a derivative is called a *partial derivative of order ℓ*.

Remark that sometimes the notation $f_{x_j x_k}$ is used for $\frac{\partial^2 f}{\partial x_k \partial x_j}$. Notice that this notation swaps the order of derivatives, which may be important. Also note that since the subscript is often used for other things, it may be confusing.

Definition 8.6.1. If $U \subset \mathbb{R}^n$ is an open set and $f \colon U \to \mathbb{R}$ a function. We say f is *k-times continuously differentiable function*, or a C^k function, if all partial derivatives of all orders up to and including order k exist and are continuous.

So a continuously differentiable, or C^1, function is one where all partial derivatives exist and are continuous which agrees with our previous definition due to Proposition 8.4.6. Note also that we could have only required that the kth order partial derivatives exist and are continuous as the existence of lower order derivatives is clearly necessary to even define kth order partial derivatives, and these lower order derivatives will be continuous as they will be differentiable functions.

When the partial derivatives are continuous, we can swap their order.

Proposition 8.6.2. *Suppose $U \subset \mathbb{R}^n$ is open and $f \colon U \to \mathbb{R}$ is a C^2 function, and j and k are two integers between 1 and n. Then*

$$\frac{\partial^2 f}{\partial x_k \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_k}$$

Proof. Fix a point $p \in U$, and let e_j and e_k be the standard basis vectors and let s and t be two small nonzero real numbers. We pick s and t small enough so that $p + s_0 e_j + t_0 e_k \in U$ for all s_0 and t_0 with $|s_0| \leq |s|$ and $|t_0| \leq |t|$. This is possible since U is open and so contains a small ball (or a box if you wish).

Using the mean value theorem on the partial derivative in x_k on the function $f(p+se_j) - f(p)$ we find a t_0 between 0 and t such that

$$\frac{f(p+se_j+te_k) - f(p+te_k) - f(p+se_j) + f(p)}{t} = \frac{\partial f}{\partial x_k}(p+se_j+t_0e_k) - \frac{\partial f}{\partial x_k}(p+t_0e_k).$$

Next there exists a number s_0 between 0 and s such that

$$\frac{\frac{\partial f}{\partial x_k}(p+se_j+t_0e_k) - \frac{\partial f}{\partial x_k}(p+t_0e_k)}{s} = \frac{\partial^2 f}{\partial x_j \partial x_k}(p+s_0e_j+t_0e_k).$$

In other words

$$g(s,t) := \frac{f(p+se_j+te_k) - f(p+te_k) - f(p+se_j) + f(p)}{st} = \frac{\partial^2 f}{\partial x_j \partial x_k}(p+s_0e_j+t_0e_k).$$

Taking a limit as $(s,t) \in \mathbb{R}^2$ goes to zero we find that (s_0,t_0) also goes to zero and by continuity of the second partial derivatives we find that

$$\lim_{(s,t)\to 0} g(s,t) = \frac{\partial^2 f}{\partial x_j \partial x_k}(p).$$

We now reverse the ordering, starting with the function $f(p+te_k) - f(p)$ we find an s_1 between 0 and s such that

$$\frac{f(p+te_k+se_j) - f(p+se_j) - f(p+te_k) + f(p)}{s} = \frac{\partial f}{\partial x_j}(p+te_k+s_1e_j) - \frac{\partial f}{\partial x_j}(p+s_1e_j).$$

And we find a t_1 between 0 and t

$$\frac{\frac{\partial f}{\partial x_j}(p+te_k+s_1e_j) - \frac{\partial f}{\partial x_j}(p+s_1e_j)}{t} = \frac{\partial^2 f}{\partial x_k \partial x_j}(p+t_1e_k+s_1e_j).$$

Again we find that $g(s,t) = \frac{\partial^2 f}{\partial x_k \partial x_j}(p+t_1e_k+s_1e_j)$ and therefore

$$\lim_{(s,t)\to 0} g(s,t) = \frac{\partial^2 f}{\partial x_k \partial x_j}(p).$$

And therefore the two partial derivatives are equal. □

The proposition does not hold if the derivatives are not continuous. See the exercises. Notice also that we did not really need a C^2 function we only needed the two second order partial derivatives involved to be continuous functions.

8.6.1 Exercises

Exercise* 8.6.1*: *Suppose $f\colon U \to \mathbb{R}$ is a C^2 function for some open $U \subset \mathbb{R}^n$ and $p \in U$. Use the proof of Proposition 8.6.2 to find an expression in terms of just the values of f (analogue of the difference quotient for the first derivative), whose limit is $\frac{\partial^2 f}{\partial x_j \partial x_k}(p)$.*

Exercise* 8.6.2*: *Define*

$$f(x,y) := \begin{cases} \frac{xy(x^2-y^2)}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Show that
a) The first order partial derivatives exist and are continuous.
b) The partial derivatives $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$ exist, but are not continuous at the origin.
c) Show $\frac{\partial^2 f}{\partial x \partial y}(0,0) \neq \frac{\partial^2 f}{\partial y \partial x}(0,0)$.

Exercise* 8.6.3*: *Suppose $f\colon U \to \mathbb{R}$ is a C^k function for some open $U \subset \mathbb{R}^n$ and $p \in U$. Suppose $j_1, j_2, \ldots, j_k$ are integers between 1 and n, and suppose $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_k)$ is a permutation of $(1,2,\ldots,k)$. Prove*

$$\frac{\partial^\ell f}{\partial x_{j_\ell} \partial x_{j_{\ell-1}} \cdots \partial x_{j_1}}(p) = \frac{\partial^\ell f}{\partial x_{j_{\sigma_\ell}} \partial x_{j_{\sigma_{\ell-1}}} \cdots \partial x_{j_{\sigma_1}}}(p).$$

Exercise* 8.6.4*: *Suppose $\varphi\colon \mathbb{R}^2 \to \mathbb{R}$ be a C^k function such that $\varphi(r,\theta) = \varphi(s,\theta)$ for all $s,r,\theta \in \mathbb{R}$ and $\varphi(r,\theta) = \varphi(r,\theta+2\pi)$ for all $r,\theta \in \mathbb{R}$. Let $F(r,\theta) = \big(r\cos(\theta), r\sin(\theta)\big)$ from Exercise 8.5.8. Show that a function $g\colon \mathbb{R}^2 \to \mathbb{R}$, given $g(x,y) := f\big(F^{-1}(x,y)\big)$ is well defined (notice that $F^{-1}(x,y)$ can only be defined locally), and when restricted to $\mathbb{R}^2 \setminus \{0\}$ it is a C^k function.*