Vector calculus background

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This class is really the vector calculus that you haven't really gotten to in Calc III. Let us start with a very quick review of the concepts from Calc III that we will need—a crash course if you will. We won't cover nearly everything needed in this quick overview, just the very basics. You should look back at your Calc III textbook.

1 Vectors

In calculus, one deals with $\mathbb{R}$, the real numbers, a one-dimensional space, or the *line*. In vector calculus, we consider the two dimensional cartesian space $\mathbb{R}^2$, the *plane*; three dimensional space $\mathbb{R}^3$; and in general the n-dimensional cartesian space $\mathbb{R}^n$. A *point* in $\mathbb{R}^2, \mathbb{R}^3$, or $\mathbb{R}^n$ is simply a tuple, a 3-tuple, or an n-tuple (respectively) of real numbers. For example, the following are points in $\mathbb{R}^2$

$$(1, -2), \qquad (0, 1), \qquad (-1, 10), \quad \text{etc} \ldots$$

The following are points in $\mathbb{R}^3$

$$(1, -2, 3), \qquad (0, 0, 1), \qquad (-1, -1, 10), \quad \text{etc} \ldots$$

Of course, $\mathbb{R}^n$ includes $\mathbb{R}^2$ and $\mathbb{R}^3$, and even $\mathbb{R} = \mathbb{R}^1$, as n can always be 1, 2, or 3. The coordinates used in calculus are x for $\mathbb{R}$, then (x, y) for $\mathbb{R}^2$, and (x, y, z) for $\mathbb{R}^3$. In general in $\mathbb{R}^n$, we run out of letters so generally use something like subscripts $(x_1, x_2, \ldots, x_n)$. But other letters are sometimes used. We mostly focus on $\mathbb{R}^3$ (and $\mathbb{R}^2$ to some extent) in this course.

Now that we have points, another object is a *vector*. When we talk about vectors, we wish to give them names. People use $\vec{v}$ or $\mathbf{v}$, although mathematicians often just write v and simply remember that v is a vector. On the board I write $\vec{v}$ although the book uses $\mathbf{v}$ (it is difficult to write bold on the board ☺). A vector is an object that describes a *direction* and a *magnitude*. It is simply an arrow in space, although it does not really care as to where the arrow starts, it only cares about its direction and its magnitude. The best way to think about it is thinking of a moving particle in space. A *point* describes the position of a particle, while a *vector* describes velocity, that is, the direction the object is traveling, and its speed. Forces and displacements are also described by vectors. That is a vector can say how to go from point A to point B (start in this direction and go this far). In fact such a vector is often written $\vec{AB}$.

A space $\mathbb{R}^n$ has one special point $O = (0, 0, \ldots, 0)$, the *origin*. We can describe a vector $\vec{v}$ via a point A in space if the vector describes the displacement from O to A, so $\vec{v} = \vec{OA}$. Then we can say that $\vec{v}$ is the *position vector* of A. Of course, this means that a vector can be described by 3 numbers just like a point. We don't necessarily want to use the same notation as for points, to distinguish a common notation for vectors is

$$\langle a, b, c \rangle,$$

which is the position vector of the point (a, b, c) in $\mathbb{R}^3$. Even though both point and vector are represented by 3 numbers in $\mathbb{R}^3$, we distinguish them. As far as computations are concerned, they are often just 3 numbers, but they are different things. Just like say temperature and speed are two very different things which are described by a single number, so we don't want to confuse speed and temperature.

The analogue of the origin is the zero vector $\vec{0}$, for example,

$$\vec{0} = \langle 0, 0, 0\rangle$$

in $\mathbb{R}^3$. It is the single vector which does not have a well-defined direction, and has a zero magnitude. If you go distance zero, then it doesn't matter in which direction you traveled.

There are a certain number of special vectors called the *standard basis vectors*. In $\mathbb{R}^2$ and $\mathbb{R}^3$ they have special names. In $\mathbb{R}^2$:

$$\hat{\imath} = \langle 1, 0\rangle, \qquad \hat{\jmath} = \langle 0, 1\rangle.$$

In $\mathbb{R}^3$:

$$\hat{\imath} = \langle 1, 0, 0\rangle, \qquad \hat{\jmath} = \langle 0, 1, 0\rangle, \qquad \hat{k} = \langle 0, 0, 1\rangle$$

Why are hats used instead of arrows above the ijk? Because these vectors are unit vectors, that is vectors of magnitude 1, and it is common to write a hat instead of arrow for such vectors (see also below) Hats are more common than arrows if you can't have bold for these vectors.

The convenient way to write vectors is using the standard basis. That is, in $\mathbb{R}^2$ write,

$$\langle a, b\rangle = a\hat{\imath} + b\hat{\jmath}, \qquad \text{e.g.} \quad \langle 3, 4\rangle = 3\hat{\imath} + 4\hat{\jmath}.$$

In $\mathbb{R}^3$ write,

$$\langle a, b, c\rangle = a\hat{\imath} + b\hat{\jmath} + c\hat{k}, \qquad \text{e.g.} \quad \langle 3, 4, -2\rangle = 3\hat{\imath} + 4\hat{\jmath} - 2\hat{k}.$$

We also allow arithmetic with vectors. First, scalar multiplication. Real numbers are called *scalars* when vectors are around, because they are used to "scale" the vectors. If α is a scalar and $\vec{v}$ is a vector then the product $\alpha\vec{v}$ is the vector with the same direction as $\vec{v}$ (as long as $\alpha \geq 0$) and magnitude multiplied by α. if $\alpha < 0$, then the direction is reversed and the magnitude is multiplied by $|\alpha|$. It turns out that

$$\alpha(a\hat{\imath} + b\hat{\jmath} + c\hat{k}) = \alpha a\hat{\imath} + \alpha b\hat{\jmath} + \alpha c\hat{k} \qquad \text{e.g.} \quad 2(3\hat{\imath} + 4\hat{\jmath} - 2\hat{k}) = 6\hat{\imath} + 8\hat{\jmath} - 4\hat{k}.$$

We can also add vectors. Vector addition is defined by using the displacement interpretation of vectors. If $\vec{v}$ and $\vec{w}$ are vectors then $\vec{v} + \vec{w}$ is the vector where we travel along $\vec{v}$ first and then along $\vec{w}$. It turns out that

$$(a\hat{\imath} + b\hat{\jmath} + c\hat{k}) + (d\hat{\imath} + e\hat{\jmath} + f\hat{k}) = (a + d)\hat{\imath} + (b + e)\hat{\jmath} + (c + f)\hat{k}$$

e.g.

$$(\hat{\imath} + 2\hat{\jmath} + 3\hat{k}) + (5\hat{\imath} + \hat{\jmath} - 3\hat{k}) = 6\hat{\imath} + 2\hat{\jmath} + 0\hat{k} = 6\hat{\imath} + 2\hat{\jmath}.$$

We write the magnitude of $\vec{v}$ as $|\vec{v}|$. The following formulas compute the magnitude of a vector. In $\mathbb{R}^2$:

$$|a\hat{\imath} + b\hat{\jmath}| = \sqrt{a^2 + b^2}$$

and in $\mathbb{R}^3$:

$$|a\hat{\imath} + b\hat{\jmath} + c\hat{k}| = \sqrt{a^2 + b^2 + c^2}$$

Often when given a vector $\vec{r}$, its magnitude is written as simply r.

The direction of $\vec{v}$ written $\hat{v}$ is then the vector

$$\hat{v} = \frac{1}{|\vec{v}|}\vec{v} = \frac{\vec{v}}{|\vec{v}|}.$$

Although we'll try to state explicitly that $\hat{v}$ is the direction of $\vec{v}$. Notice again that we put a hat instead of an arrow on unit vectors. It is relatively common to use $\hat{v}$ for a unit vector skipping writing $\vec{v}$ altogether.

All of these notions are generalized to $\mathbb{R}^n$ in the obvious manner. Higher number of dimensions do occur naturally. For example, if t is time, then time-space can have the coordinates (x, y, z, t), that is $\mathbb{R}^4$. Similarly, the space of all configurations of two particles in 3-space is really $\mathbb{R}^6$, that is $(x_1, y_1, z_1, x_2, y_2, z_2)$, where (x_1, y_1, z_1), is the position of the first particle and (x_2, y_2, z_2) is the position of the second.

2 Products of vectors

We saw one product, that is product of a scalar and a vector

$$\alpha\vec{v}$$

Another type of product is the so called *dot product*

$$(a\hat{\imath} + b\hat{\jmath} + c\hat{k}) \cdot (d\hat{\imath} + e\hat{\jmath} + f\hat{k}) = ad + be + cf.$$

e.g.

$$(3\hat{\imath} + \hat{\jmath} - 2\hat{k}) \cdot (-2\hat{\imath} + 5\hat{\jmath} + \hat{k}) = -6 + 5 - 2 = -3$$

This product is easy to generalize to any number of dimensions in the obvious way. Notice that the result of this product is a scalar and not a vector. For this reason it is sometimes called the scalar product. Notice also that

$$|\vec{v}|^2 = \vec{v} \cdot \vec{v}.$$

Geometrically in $\mathbb{R}^2$ or $\mathbb{R}^3$, this product is

$$\vec{v} \cdot \vec{w} = |\vec{v}||\vec{w}| \cos\theta$$

where θ is the angle between $\vec{v}$ and $\vec{w}$. So the dot product can be used to compute the angle. We saw that it also computes the magnitude. Note that it doesn't matter if you think of the angle between $\vec{v}$ and $\vec{w}$ or vice-versa, as we are taking the cosine here. Also there are two ways you could define the angle depending which direction you start in, but because of the cosine you get the same dot product. Two vectors are orthogonal (at right angle, perpendicular) if their dot product is zero.

The dot product is bilinear (if something is called a product, usually people want it to be bilinear):

$$(\alpha\vec{v} + \beta\vec{w}) \cdot \vec{u}\alpha(\vec{v} \cdot \vec{u}) + \beta(\vec{w} \cdot \vec{u})$$

and

$$\vec{u} \cdot (\alpha\vec{v} + \beta\vec{w}) = \alpha(\vec{u} \cdot \vec{v}) + \beta(\vec{u} \cdot \vec{w})$$

It is also commutative (not all products are commutative):

$$\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$$

Another type of product which really only exists in $\mathbb{R}^3$ is the so called *cross product*, or sometimes the *vector product*. This product results in a vector. Geometrically

$$\vec{v} \times \vec{w} = |\vec{v}||\vec{w}|(\sin\theta)\hat{n}$$

Where θ is the angle going from $\vec{v}$ to $\vec{w}$ in the plane spanned by them (now the order matters), and $\hat{n}$ is the normal vector to that plane oriented according to the right hand rule. The orientation can be figured out from the formula

$$\hat{\imath} \times \hat{\jmath} = \hat{k}$$

That is $\hat{k}$ is the normal vector to the xy-plane using the right hand rule.

There are a bunch of ways to compute the cross product, though perhaps the easiest to remember is using algebra. First, the cross product is bilinear:

$$(\alpha\vec{v} + \beta\vec{w}) \times \vec{u}\alpha(\vec{v} \times \vec{u}) + \beta(\vec{w} \times \vec{u})$$

and

$$\vec{u} \times (\alpha\vec{v} + \beta\vec{w}) = \alpha(\vec{u} \cdot \vec{v}) + \beta(\vec{u} \times \vec{w})$$

And also anti-commutative:

$$\vec{v} \times \vec{w} = -\vec{w} \times \vec{v}.$$

From this we find

$$\vec{v} \times \vec{v} = \vec{0}$$

And finally you use the identities:

$$\hat{\imath} \times \hat{\jmath} = \hat{k}, \qquad \hat{\jmath} \times \hat{k} = \hat{\imath}, \qquad \hat{k} \times \hat{\imath} = \hat{\jmath}$$

All three identities list $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$ in the same order, if you go in the opposite order you get a minus sign.

Example:

$$(3\hat{\imath} + \hat{k}) \times (\hat{\jmath} + 2\hat{k}) = 3\hat{\imath} \times \hat{\jmath} + (3 \cdot 2)\hat{\imath} \times \hat{k} + \hat{k} \times \hat{\jmath} + 2\hat{k} \times \hat{k} = 3\hat{k} + 6(-\hat{\jmath}) + (-\hat{\imath}) + 2\vec{0} = -\hat{\imath} - 6\hat{\jmath} + 3\hat{k}.$$

A very important property of the cross product is that it is orthogonal to both of the vectors. In terms of the dot product:

$$\vec{v} \cdot (\vec{v} \times \vec{w}) = 0, \qquad \vec{w} \cdot (\vec{v} \times \vec{w}) = 0.$$

It is a useful easy to compute way to find the orthogonal vector to a plane.

3 Functions and partial derivatives

A function is simply an assignment of inputs to some output. A function of 3 real variables can be written as

$$w = f(x, y, z)$$

That is, given numbers x, y, and z, the function f returns the number $f(x, y, z)$. For example, the temperature in a room where x, y, z are some coordinates on the room, then the temperature $T(x, y, z)$ is a function. Sometimes this is called a “scalar function” or a “scalar field”.

If we keep y and z fixed, then the assignment that takes x to $f(x,y,z)$ is a function of one variable. Its derivative is the so-called *partial derivative* with respect to x, written $\frac{\partial f}{\partial x}$. That is

$$\frac{\partial f}{\partial x}(x,y,z) = \lim_{h\to 0} \frac{f(x+h,y,z) - f(x,y,z)}{h}$$

We will not use the notation f_x in this course for the partial derivative as it may get confusing; we reserve the subscript for another concept. Similarly $\frac{\partial f}{\partial y}$, $\frac{\partial f}{\partial z}$ and so on. These partial derivatives are again functions of $\mathbb{R}^3$.

To compute partials we simply consider all other variables constant. For example, if $f(x,y,z) = x^2yz + xy + z$ then

$$\frac{\partial f}{\partial x}(x,y,z) = 2xyz + y, \qquad \frac{\partial f}{\partial y}(x,y,z) = x^2z + x, \qquad \text{and} \qquad \frac{\partial f}{\partial z}(x,y,z) = x^2y + 1.$$

As these are functions we may take the derivative again. Let us show a couple of examples,

$$\frac{\partial^2 f}{\partial x \partial y}(x,y,z) = 2xz + 1, \qquad \text{and} \qquad \frac{\partial^2 f}{\partial x^2}(x,y,z) = 2yz.$$

The generalization to n variables is similar. E.g., if $f(x_1,x_2,x_3,x_4) = x_1^2x_2x_3 + x_2x_4$ then

$$\frac{\partial f}{\partial x_1}(x_1,x_2,x_3,x_4) = 2x_1x_2x_3, \qquad \text{or} \qquad \frac{\partial f}{\partial x_2}(x_1,x_2,x_3,x_4) = x_1^2x_3 + x_4$$

4 Multiple integrals

In the plane the area of a small rectangle with sides Δx and Δy is of area $\Delta A = \Delta x \Delta y$. Then if we build a box of height c above this small rectangle, the volume of the box is $c\Delta x \Delta y = c\Delta A$.

Let $R = [a,b] \times [c,d]$ be a rectangle in $\mathbb{R}^2$. That is, all the points (x,y) such that $a \leq x \leq b$ and $c \leq y \leq d$. We divide R into a bunch of rectangles of area $\Delta A = \Delta x \Delta y$. In each of these rectangles we pick a point (x_j, y_j) and then

$$f(x_j, y_j)\Delta A$$

is a reasonable approximation for the volume under the graph of f above the little rectangle. We sum all these approximations

$$\sum_j f(x_j, y_j)\Delta A,$$

which is a reasonable approximation for the volume under the graph of f above the rectangle R. Taking limit as Δx and Δy go to zero and therefore ΔA goes to zero we find the limit

$$\iint_R f(x,y)\, dA$$

which we call the double integral of f over R. For a reasonable function (such as continuous), this limit exists. The sum can be done column-wise or row-wise, in which case we have a double sum, and taking the limits we find that

$$\iint_R f(x,y)\, dA = \int_c^d \int_a^b f(x,y)\, dx\, dy = \int_a^b \int_c^d f(x,y)\, dy\, dx.$$

Notice the units of this quantity. It is the units of f times the units of x times the units of y. So for example if all three are meters, then the units of dA is m^2 and the units of $f(x,y)\,dA$ and therefore $\iint_R f(x,y)\,dA$ is m^3 or volume.

For example,

$$\iint_R dA = A(R) = (b-a)(d-c)$$

the area of of R. Another example: Let $R = [0,1] \times [0,1]$

$$\iint_R xy\,dA = \int_0^1 \int_0^1 xy\,dx\,dy = \int_0^1 \frac{1}{2} y\,dy = \frac{1}{4}.$$

Similarly in $\mathbb{R}^3$, a little box of sides Δx, Δy, Δz is of volume $\Delta V = \Delta x \Delta y \Delta z$. Let $B = [a,b] \times [c,d] \times [e,f]$. We follow the same procedure above to find for a function $f(x,y,z)$

$$\iiint_B f(x,y,z)\,dV = \int_e^f \int_c^d \int_a^b f(x,y,z)\,dx\,dy\,dz.$$

All the other orderings of dx, dy, and dz also work. Notice the units. If all dimensions are in meters, then $\iiint_B f(x,y)\,dV$ is in units m^4, or 4-dimensional volume.

Sometimes we might just write

$$\iiint_B f\,dV$$

for simplicity.

We could generalize this further, given $f(x_1,\ldots,x_n)$ on $\mathbb{R}^n$, we start with the n dimensional volume element dV_n or just dV, we integrate f, and obtain $n+1$ dimensional volume "under the graph". Mathematicians sometimes do not make distinction for $n = 1$ and $n = 2$, and simply call everything "volume", so 1 dimensional volume is simply length, 2 dimensional volume is area, etc…Mathematicians also often do not use $\iint$ and $\iiint$, and so on and simply use $\int$, as we all know in what dimension we are integrating based on the nature of the element dV or dA or whatnot.

The integral is additive, that is if B is the disjoint union of B_1 and B_2, then

$$\iiint_B f\,dV = \iiint_{B_1} f\,dV + \iiint_{B_2} f\,dV.$$

Also the integral is linear, that is if α, β are real numbers and f and g are functions then

$$\iiint_B \alpha f + \beta g\,dV = \alpha \iiint_B f\,dV + \beta \iiint_B g\,dV.$$

Understanding the above development of the integral is important in applications for recognizing quantities computed by integration. In fact, because calculus is so powerful, sometimes the approximation goes the other way. Instead of the sum being an approximation for the integral, the integral can be computed to approximate a sum. In fact, in some sense this is always the case, as our world is really composed on tiny bits rather than continuous unbroken things. But adding up a finite but large number of bits tends to be far harder to do than "adding up infinitely many", that is, integrating. A recurring theme in mathematics is that by making a problem more complicated (in just the right way), we make an impossible-to-solve problem into a tractable, solvable problem.