1. (40 points) No need to explain how you got the answer on this question.

   (a) Suppose $D \subset \mathbb{R}^2$ has area 3 and the boundary is oriented positively. Compute $\int_{\partial D} x\,dx + 2x\,dy$ $= \iint_D (2-0)\,dx\,dy = 2 \cdot 3 = \boxed{6}$

   (b) Let $\Phi\colon \mathbb{R}^2 \to \mathbb{R}^2$ be such that the Jacobian determinant is the constant -4. What's the area of the image of the rectangle $[0,2] \times [0,1]$.

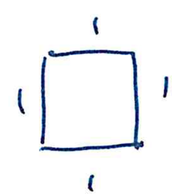
   $2 \cdot |-4| = \boxed{8}$

   (c) Suppose that M is an oriented surface in $\mathbb{R}^3$ of area 2, and suppose that $\mathbf{n}$ represents the unit normal. Compute $\iint_M 2\mathbf{n} \cdot d\mathbf{S}$ $= \iint_M 2\vec{n} \cdot \vec{n}\,dS$

   $= \iint_M 2\,dS = 2 \cdot 2 = \boxed{4}$

   (d) Suppose C is a curve and C^- is the same curve with the opposite orientation. Compute $\int_C \mathbf{F} \cdot d\mathbf{s} + \int_{C^-} \mathbf{F} \cdot d\mathbf{s}$ $= \boxed{0}$

   (e) Let C be the boundary of the unit square, that is the boundary of $[0,1] \times [0,1]$. Compute $\int_C ds$ $= \boxed{4}$



2. (40 points) Use cylindrical coordinates to compute

$$\iiint_R z e^{x^2+y^2}\, dV \quad =$$

where R is the region $0 \le z \le 1$, $x^2 + y^2 \le 1$.

$$= \int_0^{2\pi}\int_0^1\int_0^1 z e^{r^2} r \, dz\, dr\, d\theta$$

$$= 2\pi \int_0^1\int_0^1 zr\, e^{r^2} \, dz\, dr$$

$$= \pi \int_0^1 r\, e^{r^2}\, dr \qquad \begin{aligned} u &= r^2 \\ du &= 2r\,dr \end{aligned}$$

$$= \frac{\pi}{2} \int_0^1 e^u\, du$$

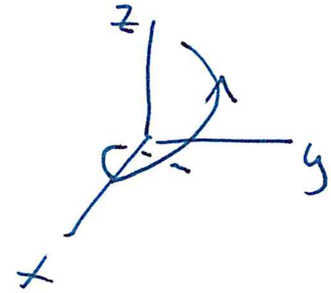
$$= \frac{\pi}{2}(e-1)$$

3. (40 points) Let C be the curve $\mathbb{R}^3$ defined by $x = \cos t$, $y = \sin t$, $z = t$, for $-\pi \leq t \leq \pi$ oriented in the direction of increasing t.

   Evaluate: $\int_C z\ dx + x\ dy + y\ dz$

$\vec{c}(t) = (\cos t, \sin t, t)$

$\vec{c}\,'(t) = (-\sin t, \cos t, 1)$



$$\int_C z\,dx + x\,dy + y\,dz$$

$$= \int_{-\pi}^{\pi} \left(-t \sin t + (\cos t)(\cos t) + (\sin t)\, 1 \right) dt$$

$$= -\int_{-\pi}^{\pi} t \sin t\, dt + \int_{-\pi}^{\pi} \cos^2 t\, dt + \int_{-\pi}^{\pi} \sin t\, dt$$

$$= -\left[-t \cos t + \sin t \right]_{-\pi}^{\pi} + \int_{-\pi}^{\pi} \frac{1}{2} + \frac{1}{2}\cos(2t)\, dt + 0$$

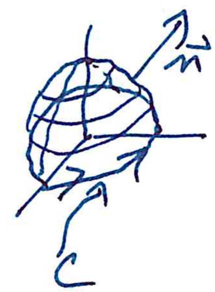
$$= -\left(-\pi(-1) - (-(-\pi(-1)))\right) + \frac{2\pi}{2} + 0 = -\pi$$

Useful formulas: $\frac{d}{dx}(-x\cos x + \sin x) = x \sin x$, $\frac{d}{dx}(x \sin x + \cos x) = x\cos x$, $\cos^2(x) = \frac{\cos(2x)+1}{2}$, $\sin^2(x) = \frac{1-\cos(2x)}{2}$, $\sin(x)\cos(x) = \frac{\sin(2x)}{2}$.

4. (40 points) Let $\mathbf{u}(x,y,z) = ze^{y^2-x^2}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$. Let M be the upper hemisphere of the unit sphere, i.e. the set of points described by $z \geq 0$ and $x^2+y^2+z^2=1$. Let $\mathbf{n}$ be the upper unit normal. **Remember, show all your work!** Hint: Stokes.

Compute: $\iint_M (\nabla \times \mathbf{u}) \cdot \mathbf{n}\, dS$

let C be the boundary of M



$$\iint_M (\nabla \times \vec{u}) \cdot \vec{n}\, dS = \int_C \vec{u} \cdot d\vec{s}$$

$$= \int_C \vec{0} \cdot d\vec{s}$$

$$= 0$$

Along C: $x^2+y^2=1$, $z=0$

$\vec{u}(x,y,0) = \vec{0}$!!!

so on C $\vec{u} = \vec{0}$

5. (40 points) Suppose $\mathbf{F}(x, y, z) = (-xy, y, z)$. Let M be the graph $z = x^2 - y^2$ for $0 \leq x \leq 1$, $0 \leq y \leq 1$ and orient M in the standard way with the normal pointing upwards. Compute:

$$\iint_M \mathbf{F} \cdot d\mathbf{S}$$

$$\Phi(x,y) = (x, y, x^2 - y^2)$$

$$\frac{\partial \Phi}{\partial x} \times \frac{\partial \Phi}{\partial y} = \vec{T}_x \times \vec{T}_y = \det \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 2x \\ 0 & 1 & -2y \end{pmatrix}$$

$$= -2x\vec{i} + 2y\vec{j} + \vec{k}$$

$$\iint_M \vec{F} \cdot d\vec{S} = \int_0^1 \int_0^1 (-xy, y, x^2 - y^2) \cdot (-2x, 2y, 1)\, dx\, dy$$

$$= \int_0^1 \int_0^1 (2x^2 y + 2y^2 + x^2 - y^2)\, dx\, dy$$

$$= \int_0^1 \int_0^1 (2x^2 y + y^2 + x^2)\, dx\, dy$$

$$= \int_0^1 \left(\frac{2}{3} y + y^2 + \frac{1}{3}\right) dy$$

$$= \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = \underline{\underline{1}}$$

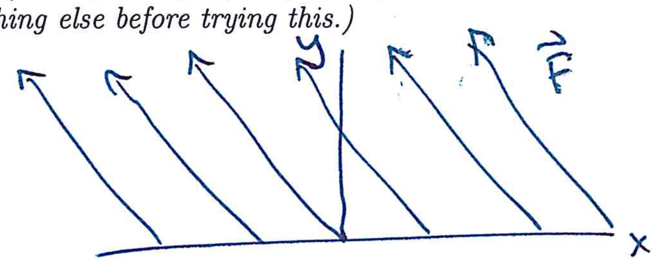
6. (10 points (bonus)) Solve the following differential equation for $y \geq 0$

$$\frac{\partial u}{\partial y} - \frac{\partial u}{\partial x} = xy, \qquad u(x,0) = \frac{1}{1+x^2}.$$

That is, write an expression $u(x,y) = \ldots$ that satisfies the above. I will accept the answer as a *definite* integral, though it is easily solvable.

Hint: Think of the equation as $\nabla u \cdot \mathbf{F} = xy$ for a certain vector field $\mathbf{F}$. Follow the vector field and integrate. It might be useful to draw.

(Very little partial credit available (it's a bonus). No points for just guessing the answer without work or explanation. Work on everything else before trying this.)



$\vec{F} = (-1, 1)$

$\vec{c}(t) = (x_0, 0) + t(-1, 1) = (x_0 - t, t)$

$\vec{c}\,'(t) = (-1, 1) = \vec{F}$

!!!

$$\frac{d}{dt}\left(u(\vec{c}(t))\right) = \nabla u(\vec{c}(t)) \cdot \vec{c}\,'(t)$$

$$= \nabla u \cdot \vec{F} = xy = (x_0 - t)t$$

$$u(x_0 - t, t) = \int_0^t (x_0 - \gamma)\gamma \, d\gamma + \frac{1}{1+x_0^2}$$

$x = x_0 - t \quad x_0 = x + t$

$y = t \quad\quad\;\; = x + y$

$$= \int_0^t (x_0\gamma - \gamma^2)\, d\gamma + \frac{1}{1+x_0^2} = \frac{x_0 t^2}{2} - \frac{t^3}{3} + \frac{1}{1+x_0^2}$$

$$u(x,y) = \frac{(x+y)y^2}{2} - \frac{y^3}{3} + \frac{1}{1+(x+y)^2}$$