Normal forms for proper maps of balls and associated groups

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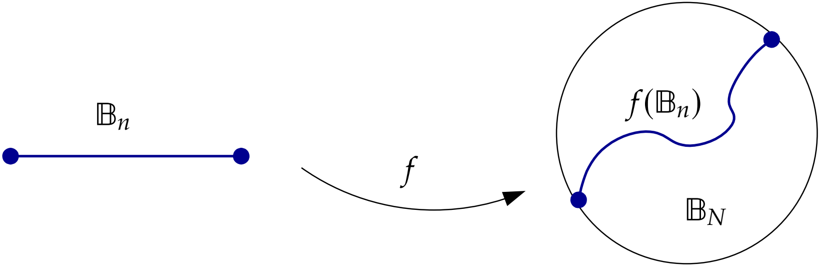
Question: Classify all rational proper maps $f\colon \mathbb{B}_n \to \mathbb{B}_N$



$f(z) = \frac{p(z)}{g(z)} = \frac{\left(p_1(z),\ldots,p_N(z)\right)}{g(z)}$, where $p_1, \ldots, p_N, g$ are polynomials.

$\mathbb{B}_n = \left\{z \in \mathbb{C}^n : \|z\|^2 = |z_1|^2 + \cdots + |z_n|^2 < 1\right\}$.

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Every proper holomorphic map $f : \mathbb{D} \to \mathbb{D}$ *is a finite Blaschke product.*

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Theorem (Forstnerič ’89)

Suppose $2 \leq n \leq N$. *If a proper holomorphic* $f\colon \mathbb{B}_n \to \mathbb{B}_N$ *extends smoothly up to the boundary, then* f *is rational, and its degree is bounded in terms of* n *and* N.

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Theorem (Cima–Suffridge '90)

If $f = \frac{p}{g}\colon \mathbb{B}_n \to \mathbb{B}_N$ *is rational proper map written in lowest terms, then* $g \neq 0$ *on* $\overline{\mathbb{B}_n}$.

f & F are *spherically equivalent* if $F = \tau \circ f \circ \psi$, where $\psi \in \mathrm{Aut}(\mathbb{B}_n)$, $\tau \in \mathrm{Aut}(\mathbb{B}_N)$.

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Faran ('82) showed that the $\mathbb{B}_2 \to \mathbb{B}_3$ case has 4 equivalence classes:

$(z_1, z_2) \mapsto (z_1,\ z_2,\ 0)$	(linear embedding)
$(z_1, z_2) \mapsto (z_1,\ z_1 z_2,\ z_2^2)$	(Whitney map)
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We want for something akin to:

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If $P\colon \mathbb{B}_n \to \mathbb{B}_N$ and $Q\colon \mathbb{B}_n \to \mathbb{B}_N$ are spherically equivalent polynomial proper maps and $P(0) = G(0) = 0$, then $P(z) = UQ(Vz)$ where U and V are unitary.

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We'll see in a bit that normal form up to the $V \in U(n)$ is then linear algebra.

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$$G(z) = 1 + G_2(z) + G_3(z) + \cdots + G_{d-1}(z), \qquad \textit{where} \quad G_2(z) = \sum_{k=1}^{n} \sigma_k z_k^2.$$

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So if F and Φ are spherically equivalent and in the form above, $\sigma_1, \ldots, \sigma_n$ are the same and $\Phi = U \circ F \circ V$, where U and V are unitaries and $G_2 \circ V = G_2$.

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If $0 < \sigma_1 < \cdots < \sigma_n$, then the only V that satisfy $G_2 \circ V = G_2$ are diagonal matrices with ± 1 on the diagonal.

$f = \frac{p}{g} : \mathbb{B}_n \to \mathbb{B}_N$ is proper if $f(S^{2n-1}) \subset S^{2N-1}$, or in other words, if

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Lemma (L. '11)

Suppose $\frac{p}{g} : \mathbb{B}_n \to \mathbb{B}_N$ *and* $\frac{P}{G} : \mathbb{B}_n \to \mathbb{B}_N$ *are proper rational maps written in lowest terms such that* $|g(0)|^2 - \|p(0)\|^2 = 1$ *and* $|G(0)|^2 - \|P(0)\|^2 = 1$. *Then there exists a* $\tau \in \operatorname{Aut}(\mathbb{B}_N)$ *such that*

$$\tau \circ \frac{p}{g} = \frac{P}{G} \qquad \text{if and only if} \qquad |g(z)|^2 - \|p(z)\|^2 = |G(z)|^2 - \|P(z)\|^2.$$

In other words, classification up to the target automorphism is classification of $|g(z)|^2 - \|p(z)\|^2$.

Define $\Lambda \colon \mathbb{B}_n \to \mathbb{R}$,

$$\Lambda(z, \bar{z}) = \Lambda_f(z, \bar{z}) = \frac{|g(z)|^2 - \|p(z)\|^2}{\left(1 - \|z\|^2\right)^d}.$$

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(iii) *Λ is a strongly plurisubharmonic exhaustion function for $\mathbb{B}_n$: Λ is strongly plurisubharmonic and $\Lambda(z)$ goes to $+\infty$ as $z \to \partial\mathbb{B}_n$. In fact, Λ is strongly convex near $\partial\mathbb{B}_n$.*

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(iii) Λ *is a strongly plurisubharmonic exhaustion function for* $\mathbb{B}_n$: Λ *is strongly plurisubharmonic and* $\Lambda(z)$ *goes to* $+\infty$ *as* $z \to \partial\mathbb{B}_n$. *In fact,* Λ *is strongly convex near* $\partial\mathbb{B}_n$.

(iv) Λ *has a unique critical point (a minimum) in* $\mathbb{B}_n$.

Definition (D'Angelo–Xiao)**:** Suppose $f : \mathbb{B}_n \to \mathbb{B}_N$ is a proper map.

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If compact, all groups can be conjugated to a subgroup of the unitary.

Suppose $f = \frac{p}{g} \colon \mathbb{B}_n \to \mathbb{B}_N$ is a rational proper map in normal form, then

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Remark: $\Gamma_f = D_f^{(*,*)}$, $D_f = D_f^{(*,0)}$, and $\Sigma_f = D_f^{(2,0)}$.

Theorem (L., Grundmeier)

Suppose $f : \mathbb{B}_n \longrightarrow \mathbb{B}_N$ *is a rational proper map in normal form and* f *is not linear. Then*

(i) $A_f \leq U(n) \oplus U(N)$ *is a closed subgroup.*

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In general, Γ_f can be computed by considering monomials that appear in $|g(z)|^2 - \|p(z)\|^2$

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Γ_f is a group that is given by a real invariant polynomial:
$\Gamma_f = \{U : p(Uz, \overline{Uz}) = p(z, \bar{z}) \text{ for all } z\}$.
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If we put constraints on the degree or target dimension, then Γ_f is not arbitrary:
1) E.g., degree-2 map is equivalent to a monomial map, so Γ_f contains a torus.
2) E.g., $\mathbb{B}_2 \to \mathbb{B}_3$ maps are known and there are exactly 4 possibilities for Γ_f.