Singular Levi-flat hypersurfaces (1)

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$z = (z_1, z_2, \dots, z_n) \in \mathbb{C}^n$ and $\mathbb{C}^n \cong \mathbb{R}^n \times \mathbb{R}^n = \mathbb{R}^{2n}$ via

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$f\colon U \subset \mathbb{C}^n \to \mathbb{C}$ is *holomorphic* if f satisfies

$$\frac{\partial f}{\partial \bar{z}_\ell} = 0 \quad \text{for } \ell = 1, 2, \dots, n \quad \text{(the } \textit{Cauchy–Riemann (CR) equations}\text{)}.$$

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Write $\mathcal{O}(U)$ for set of holomorphic functions on U.

We write a smooth (C^∞) function $f\colon U \subset \mathbb{C}^n \to \mathbb{C}$ as $f(z, \bar{z})$.

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$$2x_1 + 2y_1 + 4y_2^2 = (1-i)z_1 + (1+i)\bar{z}_1 - z_2^2 + 2z_2\bar{z}_2 - \bar{z}_2^2$$

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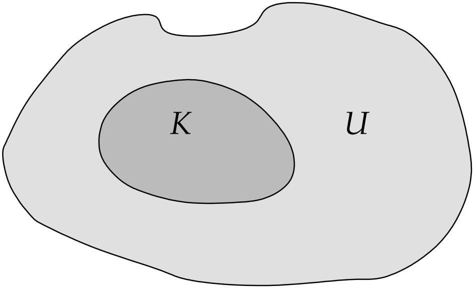
We must worry about convergence! More on all this later.

Theorem (Hartogs)

Let $U \subset \mathbb{C}^n$, $n \geq 2$, be a domain, and $K \subset\subset U$ be compact with $U \setminus K$ connected. If $f \in \mathcal{O}(U \setminus K)$, then there exists a unique $F \in \mathcal{O}(U)$ such that $F|_{U \setminus K} = f$.



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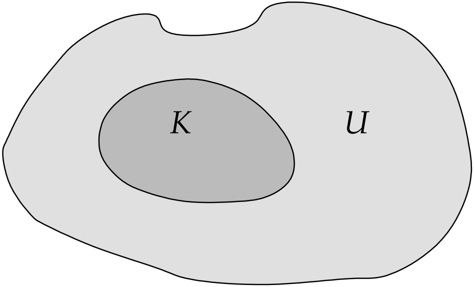

Note: *Not every domain is a natural domain of definition for a holomorphic function. Geometry of the boundary plays a role!*

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Example: $U = B(0,2) \setminus \overline{B(0,1)}$. The outer (convex) and the inner (concave) boundaries have very different properties. In fact it is a form of "convexity" that we need to study to understand boundaries.

Take the real tangent space $T_p\mathbb{C}^n = T_p\mathbb{R}^{2n}$. Write

$$\mathbb{C} \otimes T_p\mathbb{C}^n = \operatorname{span}_{\mathbb{C}} \left\{ \left.\frac{\partial}{\partial x_1}\right|_p, \left.\frac{\partial}{\partial y_1}\right|_p, \ldots, \left.\frac{\partial}{\partial x_n}\right|_p, \left.\frac{\partial}{\partial y_n}\right|_p \right\}.$$

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$B_p \cong \mathbb{C} \otimes T_pM \big/ T_p^{(1,0)}M \oplus T_p^{(0,1)}M$ is a one-dimensional space.

More explicitly,

$$X_p = \sum_{k=1}^{n} a_k \frac{\partial}{\partial z_k}\Big|_p + b_k \frac{\partial}{\partial \bar{z}_k}\Big|_p \;\in\; \mathbb{C} \otimes T_p M \quad \Leftrightarrow \quad \sum_{k=1}^{n} a_k \frac{\partial r}{\partial z_k}\Big|_p + b_k \frac{\partial r}{\partial \bar{z}_k}\Big|_p = 0.$$

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Example: $\operatorname{Im} z_n = \frac{z_n - \bar{z}_n}{2i} = 0$ defines $M = \mathbb{C}^{n-1}\times\mathbb{R} \subset \mathbb{C}^n$.

$$T_p^{(1,0)}M = \operatorname{span}_{\mathbb{C}}\left\{ \frac{\partial}{\partial z_1}\Big|_p, \dots, \frac{\partial}{\partial z_{n-1}}\Big|_p \right\} \quad T_p^{(0,1)}M = \operatorname{span}_{\mathbb{C}}\left\{ \frac{\partial}{\partial \bar{z}_1}\Big|_p, \dots, \frac{\partial}{\partial \bar{z}_{n-1}}\Big|_p \right\}$$

$$B_p = \operatorname{span}_{\mathbb{C}}\left\{ \frac{\partial}{\partial x_n}\Big|_p \right\} = \operatorname{span}_{\mathbb{C}}\left\{ \frac{\partial}{\partial (\operatorname{Re} z_n)}\Big|_p \right\} = \operatorname{span}_{\mathbb{C}}\left\{ \frac{\partial}{\partial z_n}\Big|_p + \frac{\partial}{\partial \bar{z}_n}\Big|_p \right\}$$

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If $T_p^{(1,0)}M$ and $T_p^{(0,1)}M$ have constant dimension as p ranges over M, then M is called a *CR submanifold*.

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Example 2: $M = \mathbb{C} \times \{0\} \subset \mathbb{C}^2$.

$$T_p^{(1,0)}M = \operatorname{span}_{\mathbb{C}} \left\{ \frac{\partial}{\partial z_1}\bigg|_p \right\}, \quad T_p^{(0,1)}M = \operatorname{span}_{\mathbb{C}} \left\{ \frac{\partial}{\partial \bar{z}_1}\bigg|_p \right\}, \quad B_p = \{0\}.$$

Suppose $M \subset \mathbb{C}^n$ is a smooth real hypersurface, $p \in M$. After a translation and rotation via a unitary matrix, $p = 0$ and near the origin M is written in variables $(z, w) \in \mathbb{C}^{n-1} \times \mathbb{C}$ $(w = z_n)$ as

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Consequently

$$T_0^{(1,0)}M = \operatorname{span}_{\mathbb{C}} \left\{ \frac{\partial}{\partial z_1}\bigg|_0, \ldots, \frac{\partial}{\partial z_{n-1}}\bigg|_0 \right\},$$

$$T_0^{(0,1)}M = \operatorname{span}_{\mathbb{C}} \left\{ \frac{\partial}{\partial \bar{z}_1}\bigg|_0, \ldots, \frac{\partial}{\partial \bar{z}_{n-1}}\bigg|_0 \right\},$$

$$B_0 = \operatorname{span}_{\mathbb{C}} \left\{ \frac{\partial}{\partial(\operatorname{Re} w)}\bigg|_0 \right\}.$$

In particular, $\dim_{\mathbb{C}} T_p^{(1,0)}M = \dim_{\mathbb{C}} T_p^{(0,1)}M = n - 1$ and $\dim_{\mathbb{C}} B_p = 1$.

Suppose $M = \{r = 0\}$ as before, and $p \in M$.

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A complex linear change of coordinates $A\colon \mathbb{C}^n \to \mathbb{C}^n$ acts like

$$\begin{bmatrix} A & 0 \\ 0 & \bar{A} \end{bmatrix}^* \begin{bmatrix} L & \bar{Z} \\ Z & L^t \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & \bar{A} \end{bmatrix} = \begin{bmatrix} A^* L A & \overline{A^t Z A} \\ A^t Z A & (A^* L A)^t \end{bmatrix}.$$

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Exercise: H_p and L_p depend on the defining function r, but their inertia on the tangent space does not change if we change the defining function r. (Assume the new r is negative on the same side of M).