Signature pairs of positive polynomials

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Joint work with Jennifer Halfpap

Positivity in $\mathbb{R}^n$

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Question: How can we tell if $p(x) \geq 0$ for all $x \in \mathbb{R}^n$?

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Artin's 1927 solution to Hilbert 17th problem says that if $p \geq 0$, then there is a polynomial g such that pg^2 is a sum of squares.

In 1967 Pfister showed that you need at most 2^n squares!

Hermitian squares in $\mathbb{C}^n$

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for holomorphic polynomials p_j, then $p \geq 0$. In other words:

$$p(z,\bar{z}) = \|F(z)\|^2$$

for a holomorphic mapping $F\colon \mathbb{C}^n \to \mathbb{C}^k$.

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But e.g.

$$p(z,\bar{z}) = \left(|z_1|^2 - |z_2|^2\right)^2$$

is not a squared norm. It is not even a quotient of squared norms $\frac{\|F(z)\|^2}{\|G(z)\|^2}$. The zero set is too large!

Quillen's theorem

Quillen in 1968 proved that if $p(z, \bar{z})$ is bihomogeneous (that is, $p(tz, \bar{z}) = p(z, t\bar{z}) = t^d p(z, \bar{z})$), and positive on the sphere, then

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We can take the denominator G to be $z^{\otimes d}$, that is

$$\|G(z)\|^2 = \|z^{\otimes d}\|^2 = \|z\|^{2d} = \sum_{|\alpha|=d} \left| \sqrt{\binom{d}{\alpha}}\, z^\alpha \right|^2$$

Positivity classes Ψ_d

So we say that $p \in \Psi_d$ if $\|z\|^{2d} p(z, \bar{z})$ is a squared norm.

Ψ_d then interpolate between positive polynomials and squared norms

$$\Psi_0 \subsetneq \Psi_1 \subsetneq \Psi_2 \subsetneq \cdots \subset \Psi_\infty = \bigcup_d \Psi_d$$

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D'Angelo-Varolin showed that while

$$p(z, \bar{z}) = \left(|z_1|^2 + |z_2|^2 \right)^4 - \lambda \, |z_1 z_2|^4 .$$

is in Ψ_d for $\lambda < 16$, as $\lambda \to 16$, one requires larger and larger d.

Differences of squared norms

Any polynomial $p(z, \bar{z})$ can be written as

$$p(z, \bar{z}) = \|F(z)\|^2 - \|G(z)\|^2$$

for some mappings $F \colon \mathbb{C}^n \to \mathbb{C}^{N_+}$ and $G \colon \mathbb{C}^n \to \mathbb{C}^{N_-}$.

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Same for real-analytic functions if we allow ℓ^2-valued F and G.

(See D'Angelo's book for many applications of this idea to CR geometry)

Where the terminology comes from

If we let $\mathcal{Z} = (1, z_1, z_2, \ldots, z_n, z_1^2, z_1 z_2, \ldots, z^\alpha, \ldots)^t$, then we can write

$$p(z, \bar{z}) = \mathcal{Z}^* C \mathcal{Z}$$

where C is finite rank when p is a polynomial. In general, if p is real-analytic, and convergent on a neighbourhood of the closed unit polydisc, then C is a trace-class operator.

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$$p(z, \bar{z}) = \|F(z)\|^2 - \|G(z)\|^2$$

is obtained by diagonalizing C, and signature and rank have their usual meanings.

Class Ψ_1

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Question: If p is in Ψ_1, how many positive eigenvalues are needed to cancel each negative eigenvalue? That is, if

$$\begin{aligned} \|z\|^2 p(z,\bar{z}) &= \|z\|^2(\|F(z)\|^2 - \|G(z)\|^2) \\ &= \|z \otimes F(z)\|^2 - \|z \otimes G(z)\|^2 = \|H(z)\|^2, \end{aligned}$$

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By playing around, one might come to a conclusion that many positive eigenvalues are needed for every negative eigenvalue.

Theorem in Ψ_1

But!

Theorem

Let $r(z,\bar{z})$ be a real polynomial on $\mathbb{C}^n$, $n \geq 2$, and suppose that $r(z,\bar{z})\, \|z\|^2$ is a squared norm. Let (N_+, N_-) be the signature pair of r. Then

(i)
$$\frac{N_-}{N_+} < n - 1.$$

(ii) *The above inequality is sharp, i.e., for every $\varepsilon > 0$ there exists r with $\frac{N_-}{N_+} \geq n - 1 - \varepsilon$.*

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You can have (almost) $n - 1$ negatives for every positive! But to get close you need very large degree.

Theorem in Ψ_d

Theorem

Let $r(z,\bar{z})$ be a real polynomial on $\mathbb{C}^n$, $n \geq 2$, $d \geq 1$, and suppose that $r(z,\bar{z}) \left\lVert z \right\rVert^{2d}$ is a squared norm. Let (N_+, N_-) be the signature pair of r. Then

(i)
$$\frac{N_-}{N_+} \leq \binom{n-1+d}{d} - 1.$$

(ii) *For each fixed n, there exists a constant C_n such that for each d there is a polynomial $r \in \Psi_d$ with $\frac{N_-}{N_+} \geq C_n d^{n-1}$.*

Note $\binom{n-1+d}{d}$ is a polynomial in d of degree $n-1$. So (ii) says that the bound in (i) is of the correct order.

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It is possible to construct an example with just n positives, and an arbitrarily high number of negatives, if d is large enough.

Easier setting, similar question

Suppose $d = 1$ for simplicity. A similar question that is easier to play around with is the following:

If $p(x_1, \ldots, x_n)(x_1 + \cdots + x_n)$ has only positive coefficients, and p has N_+ positive coefficients and N_- negative coefficients, then we have the sharp bound

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The degrees required to get close to the bound are large. E.g. in degree 6 the largest ratio is for

$$\begin{aligned} p(x,y,z) = 2xyz^4 + 2x^3z^3 + 2y^3z^3 + 2x^2y^2z^2 + 2x^4yz + 2xy^4z + 2x^3y^3 \\ - x^2yz^3 - xy^2z^3 - x^3yz^2 - xy^3z^2 - x^3y^2z - x^2y^3z. \end{aligned}$$

$p(x,y,z)(x+y+z)$ has only, positive coefficients. Here $N_+ = 7$, $N_- = 6$, and $6/7$ is still much less than $n - 1 = 2$.

Thank you!