Tasty Bits of Several Complex Variables (3)

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Complexification (traditional):

If $U \subset \mathbb{C}^n$ is a domain, $U \cap \mathbb{R}^n \neq \emptyset$, $f, g \in \mathcal{O}(U)$, and $f = g$ on $U \cap \mathbb{R}^n$.
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Goes the other way too: If $V \subset \mathbb{R}^n$, $f \colon V \to \mathbb{R}$ is real-analytic,
$\Rightarrow \quad \exists\, U \subset \mathbb{C}^n$ open, $V \subset U$, $F \in \mathcal{O}(U)$, $F|_V = f$.

Proof: Given real power series $\sum_\alpha c_n (x-p)^n$, plug in complex numbers: $\sum_\alpha c_n (z-p)^n$.

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We identify $\mathbb{C}^n$ and $D \subset \mathbb{C}^n \times \mathbb{C}^n$ with $\iota(z) = (z,\bar{z})$.

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Remark: There is no good control of the neighborhood to which f extends. Even in 1D: Given any interval (a,b) and any neighborhood U of (a,b), there is an $F \in \mathcal{O}(U)$ that does not extend past any boundary point of U. So $f = F|_{(a,b)}$ also cannot extend further.

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$$\bar{w} = \Phi(z, \bar{z}, w),$$

Φ, $\frac{\partial \Phi}{\partial z_k}$, $\frac{\partial \Phi}{\partial \zeta_k}$ vanish at 0 and $w = \bar{\Phi}\big(\zeta, z, \Phi(z, \zeta, w)\big)$.

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Φ, $\frac{\partial \Phi}{\partial z_k}$, $\frac{\partial \Phi}{\partial \zeta_k}$ vanish at 0 and $w = \bar{\Phi}\big(\zeta, z, \Phi(z, \zeta, w)\big)$. A basis for $T^{(0,1)}M$:

$$\frac{\partial}{\partial \bar{z}_k} + \frac{\partial \Phi}{\partial \bar{z}_k} \frac{\partial}{\partial \bar{w}} \quad \left(= \frac{\partial}{\partial \bar{z}_k} + \frac{\partial \Phi}{\partial \zeta_k} \frac{\partial}{\partial \bar{w}} \right), \qquad k = 1, \ldots, n-1.$$

So: M is $\bar{w} = \Phi(z, \bar{z}, w)$, $T^{(0,1)}M$ is given by $\frac{\partial}{\partial \bar{z}_k} + \frac{\partial \Phi}{\partial \bar{z}_k} \frac{\partial}{\partial \bar{w}}$.

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Example: Consider $M \subset \mathbb{C}^2$ given by $\operatorname{Im} w = |z|^2$, that is, $\frac{w - \bar{w}}{2i} = z\bar{z}$, or in other words, $\mathscr{M}$ is given by $\omega = -2iz\zeta + w$, and the CR vector field by $\frac{\partial}{\partial \bar{z}} - 2iz \frac{\partial}{\partial \bar{w}}$.

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If $f(z, w, \bar{z}, \bar{w})$ is a CR function, the holomorphic extension is $f(z, w, \bar{z}, -2iz\bar{z} + w)$, the $\bar{z}$ will cancel.

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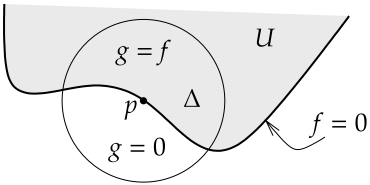
Proof: Use Radó's theorem to extend as 0 outside (g in the picture), then use identity. □



Theorem (Radó)**:** *If $U \subset \mathbb{C}^n$ is open and $g \colon U \to \mathbb{C}$ continuous and holomorphic on*

$$U' = \{ z \in U : g(z) \neq 0 \}.$$

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Example: Suppose $M = \mathbb{R} \subset \mathbb{C}$. Define $f : M \to \mathbb{C}$:

$$f(x) = \begin{cases} e^{-x^{-2}} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Then f is CR (trivially), but is not a restriction nor boundary value (from either side) of a holomorphic function continuous up to 0.

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Example: Define the function $f \in \overline{\mathbb{B}_2} \to \mathbb{C}$ by

$$f(z_1, z_2) = \begin{cases} e^{-1/\sqrt{z_1+1}} & \text{if } z_1 \neq -1, \\ 0 & \text{if } z_1 = -1. \end{cases}$$

Then f is smooth on $\overline{\mathbb{B}_2}$, holomorphic on $\mathbb{B}_2$, but near $(-1, 0)$ is not a restriction of a holomorphic function (only one sided extension).

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Baouendi–Trèves uses the same idea on a totally real subset of M and slightly modified version of the above.

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Remark: So if the Levi-form has eigenvalues of both signs, then every CR function is a restriction of a holomorphic function.

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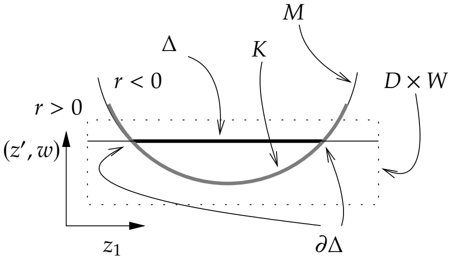
we find an analytic disc Δ "attached" to $K \subset M$ (i.e., $\partial\Delta \subset K$).

One can fill a one-sided neighborhood by such discs.



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Remark: These ideas led Lewy to find the example of the unsolvable PDE.

Another application is a special case of the following theorem:

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Example: Similarly, not true in general if U is unbounded. If $U = \mathbb{D} \times \mathbb{C} \subset \mathbb{C}^2$, then $\bar{z}_1$ is a CR function, but does not extend inside for the same reason.