Tasty Bits of Several Complex Variables (1)

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$z = (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n$ and $\mathbb{C}^n \cong \mathbb{R}^n \times \mathbb{R}^n = \mathbb{R}^{2n}$ via

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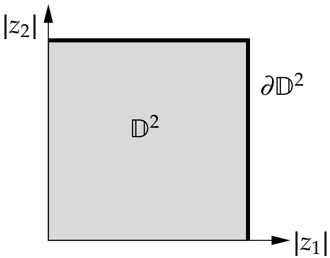

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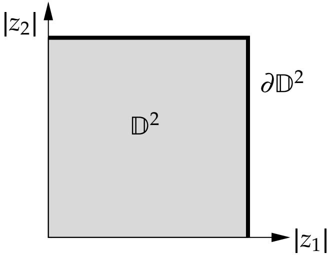

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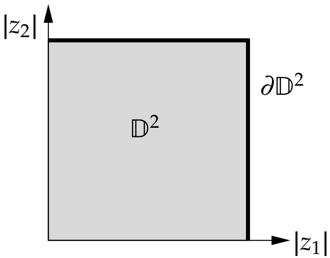

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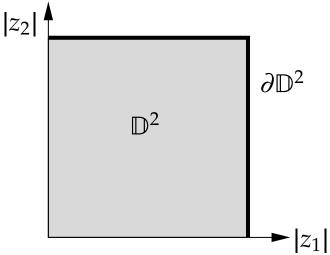

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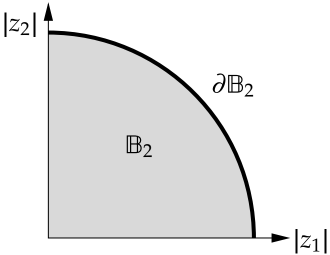
$B_\rho(a)$ is the ball in metric $\|\cdot\|$.

$\mathbb{B}_n = B_1(0)$ (*unit ball*)



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Alternatively, f is holomorphic if it satisfies

$$\frac{\partial f}{\partial \bar{z}_\ell} = 0 \quad \text{for } \ell = 1, 2, \ldots, n \quad \text{(the \textit{Cauchy–Riemann (CR) equations})}.$$

If f is holomorphic, then

$$\frac{\partial f}{\partial z_k}(z) = \lim_{\xi \in \mathbb{C} \to 0} \frac{f(z_1, \ldots, z_k + \xi, \ldots, z_n) - f(z)}{\xi}.$$

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Chain rule: Suppose $f\colon U \subset \mathbb{C}^n \to V \subset \mathbb{C}^m$ and $g\colon V \to \mathbb{C}$ variables are $z \in \mathbb{C}^n$ and $w \in \mathbb{C}^m$.

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If g and f are holomorphic.

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We cheat and write

$$\frac{1}{\zeta - z} \overset{\text{def}}{=} \frac{1}{(\zeta_1 - z_1)(\zeta_2 - z_2)\cdots(\zeta_n - z_n)}$$

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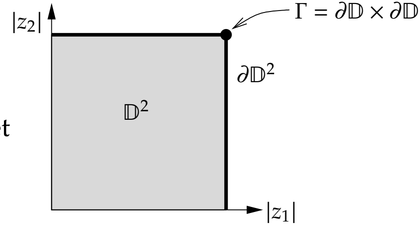

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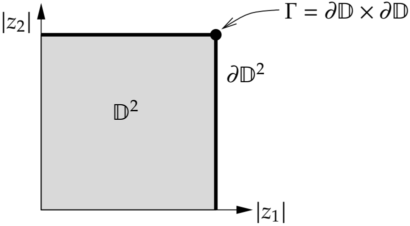

First big difference with 1D: Γ (a torus) is a small part of the boundary. Γ is called the *distinguished boundary*.

For $\alpha \in \mathbb{N}_0^n$, we cheat some more

$$z^\alpha \stackrel{\text{def}}{=} z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_n^{\alpha_n}, \qquad \frac{\partial^{|\alpha|}}{\partial z^\alpha} \stackrel{\text{def}}{=} \frac{\partial^{\alpha_1}}{\partial z_1^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial z_2^{\alpha_2}} \cdots \frac{\partial^{\alpha_n}}{\partial z_n^{\alpha_n}},$$

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Differentiate under the integral $\Rightarrow$ f is infinitely $\mathbb{C}$-differentiable and

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From this we get *the Cauchy estimates*:

$$\left| \frac{\partial^{|\alpha|} f}{\partial z^\alpha}(a) \right| \leq \frac{\alpha! \, \|f\|_\Gamma}{\rho^\alpha} = \frac{\alpha! \, \sup_{z \in \Gamma} |f(z)|}{\rho^\alpha}.$$

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$$\alpha! \overset{\text{def}}{=} \alpha_1! \alpha_2! \cdots \alpha_n!, \qquad \alpha + 1 \overset{\text{def}}{=} (\alpha_1 + 1, \alpha_2 + 1, \cdots \alpha_n + 1).$$

Let Δ be a polydisc with distinguished boundary Γ, centered at a, of polyradius ρ. Suppose f is continuous on $\overline{\Delta}$, holomorphic on Δ.

Differentiate under the integral $\Rightarrow$ f is infinitely $\mathbb{C}$-differentiable and

$$\frac{\partial^{|\alpha|} f}{\partial z^\alpha}(z) = \frac{1}{(2\pi i)^n} \int_\Gamma \frac{\alpha! f(\zeta)}{(\zeta - z)^{\alpha+1}} \, d\zeta.$$

From this we get *the Cauchy estimates*:

$$\left| \frac{\partial^{|\alpha|} f}{\partial z^\alpha}(a) \right| \leq \frac{\alpha! \, \|f\|_\Gamma}{\rho^\alpha} = \frac{\alpha! \, \sup_{z \in \Gamma} |f(z)|}{\rho^\alpha}.$$

Corollary: *The "correct" topology on $\mathcal{O}(U)$ is uniform convergence on compacts (normal convergence). If $f_n \in \mathcal{O}(U)$ and $f_n \to f$ uniformly on compacts, then $f \in \mathcal{O}(U)$ and $f_n^{(\ell)} \to f^{(\ell)}$ uniformly on compacts.*

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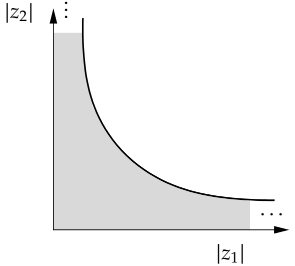

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Conversely, if f is defined by a power series, then it is holomorphic.

Theorem (Identity)**:** *Let* $U \subset \mathbb{C}^n$ *be a domain (connected open set) and let* $f : U \to \mathbb{C}$ *be holomorphic. If* $f|_N \equiv 0$ *for a nonempty open subset* $N \subset U$*, then* $f \equiv 0$*.*

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Here, even the argument goes back to 1D: just use the maximum principle on every 1D subspace.

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WLOG suppose $g(0) = (1, 0, \cdots, 0) \quad \Rightarrow \quad g_1$ attains a max at 0
$\Rightarrow \quad g_1$ is constant $\quad \Rightarrow \quad g$ is constant. $\square$

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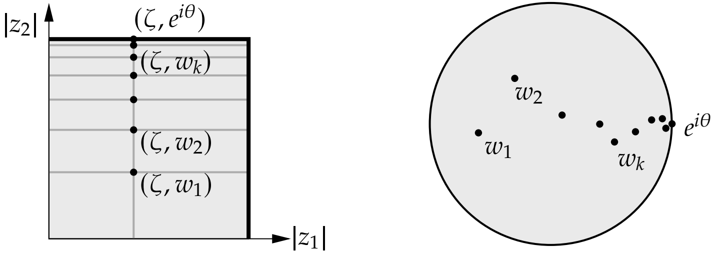

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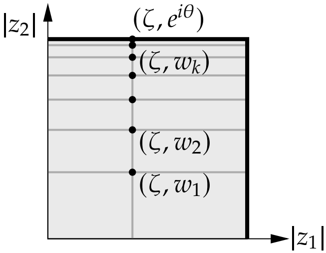

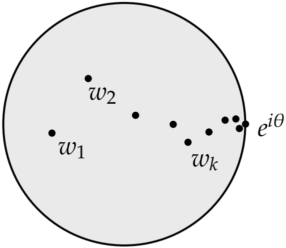

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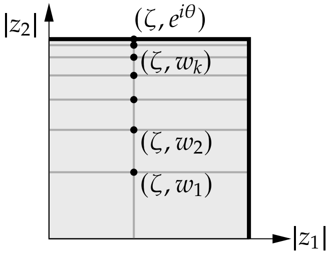

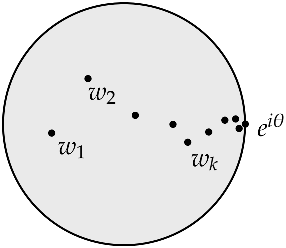

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Derivative of $\zeta \mapsto f(\zeta, w_k)$ goes to zero for every $e^{i\theta}$ and every $\{w_k\}$.
$\Rightarrow \quad \frac{\partial f}{\partial z_1} \equiv 0$ (and by symmetry $\frac{\partial f}{\partial z_2} \equiv 0$). □

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$a \in \mathbb{B}_n$, U a unitary, $s_a = \sqrt{1 - \|a\|^2}$ and $P_a z = \frac{\langle z,a\rangle}{\langle a,a\rangle}a$.

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$$z_n = g(z_1, \dots, z_{n-1})$$

for a holomorphic function g.

"Proof:" Consider all possible derivatives of f, one of them must be nonzero somewhere on N (analyticity). Then apply implicit function theorem.

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Example: The theorem does not hold in different dimensions. $f\colon \mathbb{C} \to \mathbb{C}^2$ given by $z \mapsto (z^2, z^3)$ is one-to-one and onto the cusp $(z_1^3 - z_2^2 = 0)$, but $f'(0) = 0$.