Homotopy equivalence for proper holomorphic mappings

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Proper maps

Let

$$z = (z_1, z_2, \dots, z_n) \in \mathbb{C}^n$$

be the coordinates. Let

$$\mathbb{B}_n = \text{unit ball in } \mathbb{C}^n = \{z : \|z\| < 1\}.$$

Question

Classify proper holomorphic maps from $\mathbb{B}_n$ to $\mathbb{B}_N$ up to some natural equivalence.

Proper means that f^{-1} takes compacts to compacts. If f is proper and extends to the boundary, then f takes the boundary to the boundary (CR map of spheres).

Spherical equivalence

The first obvious choice: *spherical equivalence*
$F\colon \mathbb{B}_n \to \mathbb{B}_N$ and $G\colon \mathbb{B}_n \to \mathbb{B}_N$ are spherically equivalent if there exist $\chi \in \mathrm{Aut}(\mathbb{B}_n)$ and $\tau \in \mathrm{Aut}(\mathbb{B}_N)$ such that

$$F = \tau \circ G \circ \chi$$

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Lots known in low-codimension $(N - n)$ (Alexander, Pinchuk, Faran, Huang, Ji, D'Angelo, L., etc...). E.g.

Theorem (Alexander '77 (complicated history))

If $F\colon \mathbb{B}_n \to \mathbb{B}_n$ *is proper, holomorphic, and* $n \geq 2$, *then* $F \in \operatorname{Aut}(\mathbb{B}_n)$ *(spherically equivalent to identity).*

Spherical equivalence

Theorem (Faran '82)

If $F\colon \mathbb{B}_2 \to \mathbb{B}_3$ is proper, holomorphic, and smooth up to the boundary, then F is (spherically) equivalent to

- ☛ $(z, w) \mapsto (z, w, 0)$
- ☛ $(z, w) \mapsto (z, zw, w^2)$
- ☛ $(z, w) \mapsto (z^2, \sqrt{2}\, zw, w^2)$
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For higher codimension, there are ∞ many equivalence classes. An example found by D'Angelo:

$$(z, w) \mapsto (z, tw, \sqrt{1 - t^2}\, zw, \sqrt{1 - t^2}\, z^2)$$

for $t \in [0, 1]$ are all spherically inequivalent proper maps.

One dimension

Exercise

If $F\colon \mathbb{B}_1 \to \mathbb{B}_1$ *is proper then* F *is a finite Blaschke product:*

$$F(z) = e^{i\theta} \prod_{j=1}^{k} \frac{z - a_j}{1 - \bar{a}_j z}$$

$|a_j| < 1$ *for all* j.

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However, we can continuously deform F to z^k:
Let $a_j \to 0$, and $\theta \to 0$.

Definition

Let $F, G\colon \mathbb{B}_n \to \mathbb{B}_N$ be proper holomorphic maps.

Definition

F and G are *homotopic* if, for each $t \in [0,1]$ there is a proper holomorphic mapping $H_t\colon \mathbb{B}_n \to \mathbb{B}_N$ such that

- ☛ $H_0 = f$ and $H_1 = g$.
- ☛ H_t is a continuous family.

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Proposition

If $n = N = 1$ and $F\colon \mathbb{B}_1 \to \mathbb{B}_1$ is a proper holomorphic map, then there exists a unique k such that F is homotopic (through rational maps) to z^k.

Different dimensions

Adding a few zero components does not a new map make.

Let

$$F\colon \mathbb{B}_n \to \mathbb{B}_{N_1}$$

$$G\colon \mathbb{B}_n \to \mathbb{B}_{N_2}$$

be proper holomorphic maps.

Definition

F and G are *homotopic in target dimension* k if, for each $t \in [0,1]$ there is a proper holomorphic mapping $H_t\colon \mathbb{B}_n \to \mathbb{B}_k$ such that

- H_0 is spherically equivalent to $F \oplus 0$
- H_1 is spherically equivalent to $G \oplus 0$
- H_t is a continuous family.

Everything is homotopic

Proposition (D'Angelo)

Given any proper F and G, there exists a large enough k such that F and G are homotopic in dimension k.

Proof: $tF \oplus \sqrt{1-t^2}\,G$ □

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Theorem (L., D'Angelo)

- $(z, w) \mapsto (z, w, 0)$
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are mutually not homotopy equivalent (through rational maps) in dimension 3. They are all homotopy equivalent in dimension 5.

Theorem (Forstnerič '89)

Let $F\colon \mathbb{B}_n \to \mathbb{B}_N$ $(n \geq 2)$ be a proper holomorphic map, C^∞ up to the boundary.
Then F is rational of degree bounded by a constant $D(n, N)$.

Rational maps

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Then F *is rational of degree bounded by a constant* $D(n,N)$.

Theorem (L., D'Angelo)

Suppose $n \geq 2$. *Let* S *be the set of homotopy classes (of rational mappings and in target dimension* N*) of proper rational mappings* $F\colon \mathbb{B}_n \to \mathbb{B}_N$.
Then S *is a finite set.*

S is countable if $n = 1$.

Closedness of spherical equivalence

A general version of the D'Angelo example is the following:

Theorem (L., D'Angelo)

If $F, G\colon \mathbb{B}_n \to \mathbb{B}_N$ are spherically inequivalent, and H_t is a homotopy of F and G, then there exist uncountably many mutually spherically inequivalent maps in H_t.

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The theorem is a corollary of the following lemma and the classical theorem of Sierpinski:

Lemma (L., D'Angelo)

Assume $n \geq 2$. Let $H_t\colon \mathbb{B}_n \to \mathbb{B}_N$ be a homotopy of rational proper maps. Fix $t_0 \in [0,1]$. The set

$$\{t \in [0,1] : H_t \text{ is spherically equivalent to } H_{t_0}\}$$

is closed in $[0,1]$.

Invariants

Degree is not an invariant no matter what the target dimension.

Example:

$$H_\theta(z,w) = \Big(cz - sw^2, zw, (cz - sw^2)(sz + cw^2),$$
$$zw(sz + cw^2), (sz + cw^2)^2\Big).$$

where $c = \cos\theta$ and $s = \sin\theta$.

$$H_0 = (z, zw, zw^2, zw^3, w^4)$$

$$H_{\pi/2} = (-w^2, zw, -zw^2, z^2w, z^2)$$

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$$F(z) = \sum_{\alpha} \mathbf{c}_{\alpha} z^{\alpha}$$

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The question is: are orbits under spherical equivalence closed in the subset of ℓ^1 corresponding to proper maps. This is nontrivial: Reiter (PhD thesis '14) shows this is not true in the similar hyperqudric case.

Rational maps

But rational maps $F = \frac{p}{q}$ have a different natural topology. One has to look at the pairs (p, q) in the space of polynomials.

We prove that if H_t is a homotopy of rational maps of some bounded degree, then there exist another homotopy of polynomials $p_t \colon \mathbb{C}^n \to \mathbb{C}^N$ and $q_t \colon \mathbb{C}^n \to \mathbb{C}$ such that

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We then work in a finite dimensional space by Forstnerič's theorem. We prove that p and q are bounded (once normalized by $q(0) = 1$), so the set is also bounded.