Severi’s theorem for codimension two CR singular submanifolds of $\mathbb{C}^3$
Let $\mathbb{R}^n \subset \mathbb{C}^n$ be the natural embedding (that is $\text{Im} \, z = 0$).

Suppose $M \subset \mathbb{R}^n$ is a domain and $f : M \to \mathbb{C}$ is real-analytic.

$\Rightarrow \exists$ a domain $V \subset \mathbb{C}^n$, $M \subset V$, and $F : V \to \mathbb{C}$ holomorphic such that $F|_M = f$. (We say $f$ extends holomorphically)
Complexification

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May not work if $M$ is another submanifold. Two examples:

(a) Consider $M = \{ z \in \mathbb{C}^2 \mid \text{Im} \, z_2 = 0 \}$, $f : M \to \mathbb{C}$ given by $f(z) = \text{Re} \, z_1$.

$\Rightarrow f$ does not extend holomorphically.

(b) Consider $M = \{ z \in \mathbb{C}^2 \mid z_2 = |z_1|^2 \}$, $f : M \to \mathbb{C}$ given by $f(z) = \bar{z}_1$.

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Note: all my submanifolds are embedded, all issues considered are local, and everything is real-analytic.
Let \( M \subset \mathbb{C}^n \) be a submanifold, write

\[
T^{0,1}_p M = \left( \mathbb{C} \otimes T_p M \right) \cap \text{span}_\mathbb{C} \left\{ \frac{\partial}{\partial \bar{z}_1} \bigg|_p, \ldots, \frac{\partial}{\partial \bar{z}_n} \bigg|_p \right\}
\]

**Def.:** \( M \) is CR if

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T^{0,1} M = \bigcup_{p \in M} T^{0,1}_p M \quad \text{is a vector bundle.}
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$F$ is holomorphic $\Rightarrow \frac{\partial F}{\partial \bar{z}_j} = 0$.

$\therefore$ if $M$ is CR, $\Rightarrow L(F|_M) = 0 \ \forall L \in \Gamma(T^{0,1} M)$. 
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Suppose $M \subset \mathbb{C}^n$ is a real-analytic CR submanifold and $f : M \to \mathbb{C}$ is a real-analytic CR function. 
$\Rightarrow$ $f$ extends holomorphically.
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Step 5) Profit!
Example:

Suppose $M \subset \mathbb{C}^2$ is a real-analytic real hypersurface.

Write $M$ as

$$\bar{w} = \Phi(z, \bar{z}, w),$$

and consider a real-analytic CR function $f(z, \bar{z}, w, \bar{w})$. Treat $\bar{z}$ and $\bar{w}$ as independent.
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Write

$$F'(z, \bar{z}, w) = f(z, \bar{z}, w, \Phi(z, \bar{z}, w))$$

Find CR vector field:

$$L = \frac{\partial}{\partial \bar{z}} + \frac{\partial \Phi}{\partial \bar{z}} \frac{\partial}{\partial \bar{w}}$$

$L F = 0 \quad \Rightarrow \quad \frac{\partial}{\partial \bar{z}} F = 0.$
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A function $f: M \to \mathbb{C}$ is CR if and only if it is CR on the CR submanifold $M_{CR} \subset M$ ($M_{CR} = \text{“CR points of } M\text{”}$).
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There are no CR vector fields, and you can’t easily solve for $\bar{z}$ and $\bar{w}$ in terms of $z$ and $w$:

$$\bar{z} = \frac{w}{z}, \quad \bar{w} = w$$

Extra conditions needed on $f$ here (codimension 2 in $\mathbb{C}^2$).
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We will switch to $\mathbb{C}^3$, where there is an actual CR vector field.
Harris ('78) provides a complete (but difficult to apply) criterion for $f$ on an arbitrary CR singular $M$ to be a restriction of a holomorphic function in $\mathcal{C}^\infty$ case.

In '11 we (L.–Minor–Shroff–Son–Zhang) proved that if a real-analytic CR singular manifold $M = \mathcal{N}'(\mathbb{N})$ for a real-analytic CR map $\mathcal{N}' : \mathbb{N} \rightarrow \mathbb{C}^n$ of a CR submanifold $\mathcal{N}$, and $\mathcal{N}'$ is a diffeomorphism onto $\mathcal{N}'(\mathbb{N}) = M$, then there exists a real-analytic CR function on $M$ that does not extend holomorphically.

In '16 we (L.–Noell–Ravisankar) proved that a real-analytic codimension 2 real-analytic CR singular manifold in $\mathbb{C}^n (n \geq 3)$ that is flat (subset of $\mathbb{C}^n \cap \mathbb{R}$) and nondegenerate has the extension property.
Some previous work

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In ’11 we (L.–Minor–Shroff–Son–Zhang) proved that if a real-analytic CR singular manifold $M = \varphi(N)$ for a real-analytic CR map

$$\varphi: N \subset \mathbb{C}^n \rightarrow \mathbb{C}^n$$

of a CR submanifold $N$, and $\varphi$ is a diffeomorphism onto $\varphi(N) = M$, then there exists a real-analytic CR function on $M$ that does not extend holomorphically.
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A CR singular submanifold of codimension 2 in \( \mathbb{C}^3 \) is written as (after a rotation by a unitary)

\[
\begin{align*}
    w &= \rho(z, \bar{z}) \\
    &= Q(z, \bar{z}) + E(z, \bar{z}) \\
    &= z^* A z + \bar{z}^t B z + z^t C z + E(z, \bar{z}),
\end{align*}
\]

\((z, w) \in \mathbb{C}^2 \times \mathbb{C}, \quad \rho \text{ is } O(||z||^2), \quad E \text{ is } O(||z||^3).\)

A, B, C, 2 \times 2 complex matrices,

z column vector,

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$A, B, C$, $2 \times 2$ complex matrices,
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$M$ can be parametrized by $z$ (and $\bar{z}$)
Adam Coffman (’09) has a normal form of $Q$ up to local biholomorphisms (and it is a rather long list).

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This might be a good place to note that normal forms for codimension 2 CR singular manifolds has a long history:

$\mathbb{C}^2$: Bishop ’65, Moser–Webster ’83, Moser ’85, Kenig–Webster ’82, Gong ’94, Huang–Krantz ’95, Huang–Yin ’09, Slapar ’16, etc...

### Table 1. Normal forms for Theorem 7.1

<table>
<thead>
<tr>
<th>$N$</th>
<th>$P$</th>
<th>Conditions</th>
<th>Signatures</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\begin{pmatrix} 1 &amp; 0 \ 0 &amp; e^{i\theta} \end{pmatrix}$</td>
<td>$\begin{pmatrix} a &amp; b \ b &amp; d \end{pmatrix}$</td>
<td>$a &gt; 0, d &gt; 0, b \sim -b \in \mathbb{C}$</td>
<td>$+ - 0$</td>
</tr>
<tr>
<td>$0 &lt; \theta &lt; \pi$</td>
<td>$a = 0, b = 0$</td>
<td>$b \geq 0, d \geq 0$</td>
<td>$+ - 0$</td>
</tr>
<tr>
<td></td>
<td>$a = 0, b = 0$</td>
<td>$a &gt; 0, b \geq 0$</td>
<td>$+ - 0$</td>
</tr>
<tr>
<td>$\begin{pmatrix} 1 &amp; 0 \ 0 &amp; 1 \end{pmatrix}$</td>
<td>$\begin{pmatrix} a &amp; 0 \ 0 &amp; d \end{pmatrix}$</td>
<td>$0 \leq a \leq d$</td>
<td>$+ - 0$</td>
</tr>
<tr>
<td>$\begin{pmatrix} 1 &amp; 0 \ 0 &amp; -1 \end{pmatrix}$</td>
<td>$\begin{pmatrix} 0 &amp; b \ b &amp; 0 \end{pmatrix}$</td>
<td>$b &gt; 0$</td>
<td>$+$</td>
</tr>
<tr>
<td></td>
<td>$\begin{pmatrix} 0 &amp; 1 \ 1 &amp; 1 \end{pmatrix}$</td>
<td>$b \geq 0, d \geq 0 +$</td>
<td>$+$</td>
</tr>
<tr>
<td></td>
<td>$\begin{pmatrix} 0 &amp; 1 \ 1 &amp; 0 \end{pmatrix}$</td>
<td>$a \geq 0 \leq d$</td>
<td>$+$</td>
</tr>
<tr>
<td></td>
<td>$\begin{pmatrix} 0 &amp; 1 \ 0 &amp; 0 \end{pmatrix}$</td>
<td>$a \geq 0 \not\in \mathbb{C}$</td>
<td>$+$</td>
</tr>
<tr>
<td>$\begin{pmatrix} 0 &amp; 1 \ 1 &amp; 0 \end{pmatrix}$</td>
<td>$\begin{pmatrix} a &amp; b \ b &amp; d \end{pmatrix}$</td>
<td>$b &gt; 0,</td>
<td>a</td>
</tr>
<tr>
<td>$0 &lt; \tau &lt; 1$</td>
<td>$\begin{pmatrix} 0 &amp; 1 \ 1 &amp; 0 \end{pmatrix}$</td>
<td>$b &gt; 0,</td>
<td>d</td>
</tr>
<tr>
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<td>$\begin{pmatrix} 0 &amp; 1 \ 0 &amp; 0 \end{pmatrix}$</td>
<td>$b &gt; 0$</td>
<td>$+ - 0$</td>
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<tr>
<td></td>
<td>$\begin{pmatrix} 1 &amp; 1 \ 0 &amp; 0 \end{pmatrix}$</td>
<td>$d \in \mathbb{C}$</td>
<td>$+$</td>
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<td></td>
<td>$\begin{pmatrix} 1 &amp; 1 \ 0 &amp; 0 \end{pmatrix}$</td>
<td>$d \not\in \mathbb{C}$</td>
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\( M \subset \mathbb{C}^3, (z, w) \in \mathbb{C}^2 \times \mathbb{C} \)

\[ M : w = \rho(z, \bar{z}) = Q(z, \bar{z}) + E(z, \bar{z}) = z^*Az + \bar{z}^tBz + z^tCz + E(z, \bar{z}) \]

**Theorem (L.–Noell–Ravisankar)**

*Suppose*

\[ \text{null } A^* \cap \text{null } B = \{0\} \]

*If \( f(z, \bar{z}) \) is real-analytic CR function defined near the origin, then \( f \) extends holomorphically near the origin. That is, \( \exists \ F(z, w) \) such that*

\[ f(z, \bar{z}) = F(z, \rho(z, \bar{z})). \]
Theorem (L.–Noell–Ravisankar)

Suppose \( M \subset \mathbb{C}^3 \) is a quadric given by

\[ w = Q(z, \bar{z}) = z^* A z + \bar{z}^t B z + z^t C z \]

Assume \( \bar{\partial} Q \neq 0 \). TFAE:

(a) \( \text{null } A^* \cap \text{null } B = \{0\} \)

(b) For every CR polynomial \( f(z, \bar{z}) \),

\[ \exists! \text{ holomorphic polynomial } F(z, w) \text{ such that } f(z, \bar{z}) = F(z, Q(z, \bar{z})). \]

If \( f \) is homogeneous, then \( F \) is weighted homogeneous.

(c) Every CR real-linear \( h(z, \bar{z}) \) is holomorphic

(\text{does not depend on } \bar{z}).
The difficulty

Consider $f(z, \bar{z})$ on

$$w = z_1^2 + z_2^2 + \bar{z}_1^2 + \bar{z}_2^2 \quad (B = I, A = 0, E = 0)$$

Solve for say $\bar{z}_1$:

$$\bar{z}_1 = \pm \sqrt{w - z_1^2 + z_2^2 + \bar{z}_2^2}$$

We can get rid of all but the first power of $\bar{z}_1$:

$$f = \alpha(z_1, z_2, w, \bar{z}_2) + \bar{z}_1 \beta(z_1, z_2, w, \bar{z}_2)$$
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Consider \( f(z, \bar{z}) \) on

\[
w = z_1^2 + z_2^2 + \bar{z}_1^2 + \bar{z}_2^2 \quad (B = I, A = 0, E = 0)
\]

Solve for say \( \bar{z}_1 \):

\[
\bar{z}_1 = \pm \sqrt{w - z_1^2 + z_2^2 + \bar{z}_2^2}
\]

We can get rid of all but the first power of \( \bar{z}_1 \):

\[
f = \alpha(z_1, z_2, w, \bar{z}_2) + \bar{z}_1 \beta(z_1, z_2, w, \bar{z}_2)
\]

\( Lf = 0 \) (CR vector field) must get rid of not only the dependence on \( \bar{z}_2 \) in \( \alpha \), but also force \( \beta \equiv 0 \).
Fix $Q$, $\partial Q \neq 0$, and suppose $\text{null } A^* \cap \text{null } B \neq \{0\}$. Let

$$f = \bar{v}_2 \bar{z}_1 - \bar{v}_1 \bar{z}_2,$$

where $(v_1, v_2)$ is a nonzero vector in $\text{null } A^* \cap \text{null } B$.

Then $f$ is not a restriction to $M$ of a holomorphic function (in any neighborhood of the origin)
As $M$ is a graph of $w$ over $z$:

$$w = \rho(z, \bar{z}),$$

write everything on $M$ in terms of $z$.

A function on $M$ is a function $f(z, \bar{z})$.

The CR vector field in terms of $z$ as a parameter on $M$ is

$$L = \rho_{\bar{z}_2} \frac{\partial}{\partial \bar{z}_1} - \rho_{\bar{z}_1} \frac{\partial}{\partial \bar{z}_2}$$
Everything in \( z \)

As \( M \) is a graph of \( w \) over \( z \):

\[
w = \rho(z, \bar{z}),
\]

write everything on \( M \) in terms of \( z \).

A function on \( M \) is a function \( f(z, \bar{z}) \).

The CR vector field in terms of \( z \) as a parameter on \( M \) is

\[
L = \rho_{\bar{z}_2} \frac{\partial}{\partial \bar{z}_1} - \rho_{\bar{z}_1} \frac{\partial}{\partial \bar{z}_2}
\]

Normally to complexify in \( \mathbb{C}^3 \): we consider \((z, \bar{z}, w, \bar{w})\) in \( \mathbb{C}^6 \).

But we only need to complexify into \( \mathbb{C}^5 \) and consider \((z, \bar{z}, w)\).
If $\bar{\partial}Q \equiv 0$, things can be complicated.
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$E \equiv 0$) $M$ is complex and every “CR function” extends holomorphically.
So for some $E$ we may have extension.
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$E \equiv 0$) $M$ is complex and every “CR function” extends holomorphically. 
So for some $E$ we may have extension.

$E = ||z||^4$) $M$ is given by

$$w = ||z||^4 = (|z_1|^2 + |z_2|^2)^2$$

and

$$f(z, \bar{z}) = ||z||^2 = |z_1|^2 + |z_2|^2$$

is CR but equal to $\sqrt{w}$ on $M$, so does not extend. So for some $E$ we do not have extension.
null $A^* \cap \text{null } B \neq \{0\}$

Suppose $\text{null } A^* \cap \text{null } B \neq \{0\}$.

$E \equiv 0$) Extension does not hold. E.g. if

$$w = \bar{z}_1 z_2$$

then $\bar{z}_1$ is CR as the CR vector field is $L = -z_2 \frac{\partial}{\partial \bar{z}_2}$

Note: The theorem is an if-and-only-if when $E \equiv 0$. 

null $A^* \cap \text{null } B \neq \{0\}$

Suppose $null A^* \cap \text{null } B \neq \{0\}$.

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$$w = \bar{z}_1 z_2$$

then $\bar{z}_1$ is CR as the CR vector field is $L = -z_2 \frac{\partial}{\partial \bar{z}_2}$

Note: The theorem is an if-and-only-if when $E \equiv 0$.

$E \not\equiv 0$) Extension may or may not hold depending on $E$. E.g. if

$$w = \bar{z}_1 z_2 + \bar{z}_2^3$$

then extension holds (explicit computation), but if

$$w = \bar{z}_1 z_2 + \bar{z}_1^3$$

then extension does not hold ($\bar{z}_1$ again).
The proof has the following outline:

Step 1) Prove theorem for homogeneous polys. and quadrics.
Step 2) Prove a formal extension theorem.
Step 3) Prove that in $\mathbb{C}^2$ a formal solution is convergent.
Step 4) Use this to prove convergence of $F$ in $\mathbb{C}^3$. 
Proof sketch for the quadrics I

Suppose $\bar{\partial} Q \neq 0$ and $M$ is a quadric:

$$w = Q(z, \bar{z}) = z^* A z + z^t B z + z^t C z.$$

Suppose $f(z, \bar{z})$ is a homogeneous polynomial that is CR.
Suppose $\bar{\partial} \mathcal{Q} \neq 0$ and $M$ is a quadric:

$$w = Q(z, \bar{z}) = z^* A z + z^t B z + z^t C z.$$

Suppose $f(z, \bar{z})$ is a homogeneous polynomial that is CR.

There are two cases:

$B \neq 0)$ Let’s tackle that one first.

$B = 0)$ Special case, needs to be handled differently.
Proof sketch for the quadrics II ($B \neq 0$)

If $B \neq 0$, then it can be diagonalized by a transformation in $z$:

$$Q(z, \bar{z}) = z^* A z + \bar{z}_1^2 + \epsilon \bar{z}_2^2 + z^t C z$$

where $\epsilon = 0, 1$. 

Weierstrass division algorithm (using $z_1$) says 

$$f(z, z^2) = h(z, z^2, w) Q(z, z) w + (z, z^2, w) z_1:$$

The remainder in Weierstrass is unique: Any equality on $M$, as long $z_1$ appears up to first power holds everywhere. $f$ equals a holomorphic polynomial $g(z, w)$ if and only if $+ z_1 g 0$, or in other words if $+ 0; \text{ and } 0:.$
If $B \neq 0$, then it can be diagonalized by a transformation in $z$:

$$Q(z, \bar{z}) = z^* Az + \bar{z}_1^2 + \epsilon \bar{z}_2^2 + z^t Cz$$

where $\epsilon = 0, 1$.

Weierstrass division algorithm (using $\bar{z}_1$) says

$$f(z, \bar{z}) = h(z, \bar{z}, w)(Q(z, \bar{z}) - w) + \alpha(z, \bar{z}_2, w) + \beta(z, \bar{z}_2, w)\bar{z}_1.$$  

The remainder in Weierstrass is unique: Any equality on $M$, as long $\bar{z}_1$ appears up to first power holds everywhere.
Proof sketch for the quadrics II ($B \neq 0$)

If $B \neq 0$, then it can be diagonalized by a transformation in $z$:

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Weierstrass division algorithm (using $\bar{z}_1$) says

$$f(z, \bar{z}) = h(z, \bar{z}, w)(Q(z, \bar{z}) - w) + \alpha(z, \bar{z}_2, w) + \beta(z, \bar{z}_2, w)\bar{z}_1.$$

The remainder in Weierstrass is unique: Any equality on $M$, as long $\bar{z}_1$ appears up to first power holds everywhere.

$f$ equals a holomorphic polynomial $g(z, w)$ if and only if $\alpha + \beta\bar{z}_1 - g \equiv 0$, or in other words if

$$\alpha\bar{z}_2 \equiv 0, \quad \text{and} \quad \beta \equiv 0.$$
Proof sketch for the quadrics III ($B \neq 0$)

Solving $Lf = 0$ we get differential equations for $\alpha$ and $\beta$, in fact a single equation for $\beta$.

Then it is an almost-undergraduate-first-order-DE computation to find for which coefficients in $A$ (and $B$) do we get a polynomial solution.
Solving $Lf = 0$ we get differential equations for $\alpha$ and $\beta$, in fact a single equation for $\beta$.

Then it is an almost-undergraduate-first-order-DE computation to find for which coefficients in $A$ (and $B$) do we get a polynomial solution.

QED!
Proof sketch for the quadrics IV \((B = 0)\)

Let \(L\) be the CR vector field.

Proof outline.

**Step 1)** For each degree \(d\), compute \(L\) as a matrix taking homogeneous polynomials of fixed degree to themselves.

**Step 2)** Compute the dimension of the kernel of \(L\) for each \(d\).

**Step 3)** Compute the dimension of weighted homogeneous polynomials \(F(z, w)\) of degree \(d\).

**Step 4)** ... the two dimensions match! (if the nullspace condition is met)

**Step 5)** QED!
Matrix for $L$ in degree 3

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### One subblock, degree 9

|       | z_1 | z_2 | z_3 | z_4 | z_5 | z_6 | z_7 | z_8 | z_9 | z_10 | z_11 | z_12 | z_13 | z_14 | z_15 | z_16 | z_17 | z_18 | z_19 | z_20 | z_21 | z_22 | z_23 | z_24 | z_25 |
|-------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| z_1  |     |   # |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |
| z_2  |     |   # |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |
| z_3  |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |
| z_4  |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |
| z_5  |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |
| z_6  |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |
| z_7  |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |
| z_8  |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |
| z_9  |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |
| z_10 |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |
| z_11 |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |
| z_12 |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |
| z_13 |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |
| z_14 |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |
| z_15 |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |
| z_16 |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |
| z_17 |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |
| z_18 |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |
| z_19 |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |
| z_20 |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |
| z_21 |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |
| z_22 |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |
| z_23 |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |
| z_24 |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |
| z_25 |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |
Thanks for listening!