Extending CR functions from codimension 2 CR singular manifolds in 3 dimensions

Jiří Lebl
joint work with Alan Noell and Sivaguru Ravisankar

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Complexification

Let $\mathbb{R}^n \subset \mathbb{C}^n$ be the natural embedding (that is $\operatorname{Im} z = 0$).

Suppose $M \subset \mathbb{R}^n$ is a domain and $f\colon M \to \mathbb{C}$ is real-analytic.

$\Rightarrow \exists$ a domain $V \subset \mathbb{C}^n$, $M \subset V$, and $F\colon V \to \mathbb{C}$ holomorphic such that $F|_M = f$. (We say f *extends holomorphically*)

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Does not work in general:

(a) $M = \{z \in \mathbb{C}^2 \mid \operatorname{Im} z_2 = 0\}$, $f = \operatorname{Re} z_1$.
$\Rightarrow f$ does not extend holomorphically.

(b) $M = \{z \in \mathbb{C}^2 \mid z_2 = |z_1|^2\}$, $f = \bar{z}_1$.
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Note: all my submanifolds are embedded, all issues considered are local, and everything is real-analytic.

CR vectors

Let $M \subset \mathbb{C}^n$ be a submanifold,

$$T_p^{0,1} M = \left(\mathbb{C} \otimes T_p M\right) \cap \operatorname{span}_{\mathbb{C}} \left\{ \frac{\partial}{\partial \bar{z}_1}\bigg|_p, \ldots, \frac{\partial}{\partial \bar{z}_n}\bigg|_p \right\}$$

Definition: M is CR if

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Severi's theorem

Theorem (Severi[1], '31)

Suppose $M \subset \mathbb{C}^n$ is a real-analytic CR submanifold and $f\colon M \to \mathbb{C}$ is a real-analytic CR function.
$\Rightarrow f$ extends holomorphically.

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Step 5) Profit!

CR singular submanifolds

Definition: M is *CR singular* if it is not CR.

Definition: $f\colon M \to \mathbb{C}$ is *CR* if
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E.g. $M = \{w = |z|^2\} \subset \mathbb{C}^2$. $f = \bar{z}$ is CR (trivially),
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You could even take $f = \bar{z}^2$ to make $\frac{\partial}{\partial \bar{z}}\big|_0 f = 0$,
but f still does not extend.

Some previous work

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In (L.–Minor–Shroff–Son–Zhang '11) we proved that if M is real-analytic CR singular submanifold and $T^{0,1} M_{CR}$ extends to a vector bundle on M (so M is an image of a CR manifold), then there exists a real-analytic CR function on M that does not extend holomorphically.

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In (L.–Noell–Ravisankar '11) we proved a Severi type theorem for a real-analytic codimension 2 real-analytic CR singular manifold in $\mathbb{C}^{n+1}$ ($n \geq 2$) that is flat (subset of $\mathbb{C}^n \times \mathbb{R}$) and nondegenerate.

Main theorem setup

Write a CR singular submanifold of codimension 2 in $\mathbb{C}^{n+1}$ as (after a rotation by a unitary)

$$\begin{aligned} w &= \rho(z, \bar{z}) \\ &= Q(z, \bar{z}) + E(z, \bar{z}) \\ &= z^* A z + \overline{z^t B z} + z^t C z + E(z, \bar{z}), \end{aligned}$$

$(z, w) \in \mathbb{C}^n \times \mathbb{C}$, ρ is $O(\|z\|)^2$, E is $O(\|z\|^3)$.

A, B, C, $n \times n$ complex matrices,
z column vector,
B, C symmetric.

Normal forms

Adam Coffman ('09) proved a normal form of Q in $\mathbb{C}^3$ up to local biholomorphisms (and it is a rather long list).

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This might be a good place to note that normal forms for codimension 2 CR singular manifolds has a long history:

$\mathbb{C}^2$: Bishop '65, Moser–Webster '83, Moser '85, Kenig–Webster '82, Gong '94, Huang–Krantz '95, Huang–Yin '09, Slapar '16, etc...

$\mathbb{C}^n$ $(n \geq 3)$ Dolbeault–Tomassini–Zaitsev '05, '11, Huang–Yin '09, '16, '17, Burcea '13, Gong–L. '15, Fang–Huang '18.

Coffman's table

Table 1. Normal forms for Theorem 7.1

N	P			
$\begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{pmatrix}$ $0<\theta<\pi$	$\begin{pmatrix} a & b \\ b & d \end{pmatrix}$	5	$a>0, d>0, b\sim -b \in \mathbb{C}$	$+-0$
	$\begin{pmatrix} 0 & b \\ b & d \end{pmatrix}$	3	$b\geq 0, d\geq 0$	$+-0$
	$\begin{pmatrix} a & b \\ b & 0 \end{pmatrix}$	3	$a>0, b\geq 0$	$+-0$
$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$	2	$0\leq a\leq d$	$+-0$
$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$	2	$0\leq a\leq d$	$+-0$
	$\begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix}$	1	$b>0$	$+$
	$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$	0		$+$
$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & b \\ b & 1 \end{pmatrix}$	1	$b>0$	$+0$
	$\begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}$	2	$\mathrm{Im}(d)>0$	$+$
$\begin{pmatrix} 0 & 1 \\ \tau & 0 \end{pmatrix}$ $0<\tau<1$	$\begin{pmatrix} a & b \\ b & d \end{pmatrix}$	5	$b>0, \lvert a\rvert=1, (a,d)\sim(-a,-d)$	$+-0$
	$\begin{pmatrix} 0 & b \\ b & d \end{pmatrix}$	3	$b>0, \lvert d\rvert=1, d\sim -d$	$+-0$
	$\begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix}$	2	$b>0$	$+-0$
	$\begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}$	3	$d\in\mathbb{C}$	$+0$
	$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$	1		$+$
	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	1		$+$
$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} a & b \\ b & 1 \end{pmatrix}$	3	$b>0, a\in\mathbb{C}$	$+-0$
	$\begin{pmatrix} 1 & b \\ b & 0 \end{pmatrix}$	1	$b>0$	$+-0$
	$\begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix}$	1	$b>0$	$+-0$
	$\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$	1	$a\geq 0$	$+0$
	$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$	0		0
	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	0		0
$\begin{pmatrix} 0 & 1 \\ 1 & i \end{pmatrix}$	$\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$	3	$a>0, d\in\mathbb{C}$	$+-0$
	$\begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix}$	1	$b>0$	$+0$
	$\begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix}$	1	$d\geq 0$	$+$
$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$	1	$a\geq 0$	$+-0$
	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	0		$+$
	$\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$	1	$a\geq 0$	0
$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	0		$+$
	$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$	0		0
	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	0		0

Main theorem

$M \subset \mathbb{C}^{n+1}$, $(z, w) \in \mathbb{C}^n \times \mathbb{C}$

$M : w = \rho(z, \bar{z}) = Q(z, \bar{z}) + E(z, \bar{z}) = z^* Az + \overline{z^t Bz} + z^t Cz + E(z, \bar{z})$

Theorem (L.–Noell–Ravisankar)

Suppose

$$\operatorname{rank} \begin{bmatrix} A^* \\ B \end{bmatrix} \geq 2.$$

If $f(z, \bar{z})$ is real-analytic CR function defined near the origin, then f extends holomorphically near the origin. That is, $\exists$ $F(z, w)$ such that

$$f(z, \bar{z}) = F(z, \rho(z, \bar{z})).$$

The quadric

Theorem (L.–Noell–Ravisankar)

Consider $M \subset \mathbb{C}^{n+1}$ given by $w = Q(z,\bar{z}) = z^* A z + \overline{z^t B z} + z^t C z$. Assume $\bar{\partial} Q \not\equiv 0$. TFAE:

(a) $\operatorname{rank} \begin{bmatrix} A^* \\ B \end{bmatrix} \geq 2$

(b) *For every CR polynomial $f(z,\bar{z})$, $\exists!$ holo. poly. $F(z,w)$ such that $f(z,\bar{z}) = F\bigl(z, Q(z,\bar{z})\bigr)$.*

f homogeneous $\Rightarrow$ F weighted homogeneous.

(c) *Every CR real-linear $h(z,\bar{z})$ is holomorphic.*

(d) *M is **not** biholomorphically equivalent to one of the following (mutually inequivalent) exceptional cases:*

(1) $w = \bar{z}_1 z_2 + \bar{z}_1^2$,
(2) $w = \bar{z}_1 z_2$,
(3) $w = |z_1|^2 + a\bar{z}_1^2$, $a \geq 0$,
(4) $w = \bar{z}_1^2$.

$\bar\partial Q \equiv 0$, that is, rank $\left[\begin{smallmatrix} A^* \\ B \end{smallmatrix}\right] = 0$

$$M : w = z^t Cz + E(z, \bar z)$$

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$E \equiv 0$) M is complex and every "CR function" extends holomorphically.
So for some E we may have extension.

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So for some E we may have extension.

$E = \|z\|^4$) M is given by

$$w = \|z\|^4$$

and

$$f(z, \bar{z}) = \|z\|^2$$

is CR but equal to $\sqrt{w}$ on M, so does not extend.
So for some E we do not have extension.

rank $\begin{bmatrix} A^* \\ B \end{bmatrix} = 1$

$$M : w = Q(z, \bar{z}) + E(z, \bar{z})$$

$E \equiv 0)$ Extension does not hold. E.g. consider $M \subset \mathbb{C}^3$:

$$w = \bar{z}_1 z_2$$

then $\bar{z}_1$ is CR as the CR vector field is $L = -z_2 \frac{\partial}{\partial \bar{z}_2}$

Note: The theorem is an if-and-only-if when $E \equiv 0$.

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$E \not\equiv 0)$ Extension may or may not hold depending on E. E.g.

$$w = \bar{z}_1 z_2 + \bar{z}_2^3$$

then extension holds (explicit computation), but if

$$w = \bar{z}_1 z_2 + \bar{z}_1^3$$

then extension does not hold ($\bar{z}_1$ again).

Application: classification of CR images

Suppose $M \subset \mathbb{C}^{n+1}$ is a real-analytic CR singular submanifold of codimension 2, and there exists a real-analytic vector bundle $\mathcal{V}$ on M such that $\mathcal{V}_p = T^{0,1}_p M$ for all $p \in M_{CR}$.

Equivalently (locally), $\exists$ a generic submanifold $N \subset \mathbb{C}^{n+1}$ and a real-analytic CR map $\varphi \colon N \to \mathbb{C}^{n+1}$ such that $\varphi(N) = M$.

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Corollary

M is equivalent to exactly one of

(1) $w = \bar{z}_1 z_2 + \bar{z}_1^2 + O(\|z\|^3)$,

(2) $w = \bar{z}_1 z_2 + O(\|z\|^3)$,

(3) $w = |z_1|^2 + a\bar{z}_1^2 + O(\|z\|^3),\ a \geq 0$,

(4) $w = \bar{z}_1^2 + O(\|z\|^3)$.

(5) $w = O(\|z\|^3)$.

Thanks for listening!