Levi-flat Plateau Problem

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In $\mathbb{R}^3$ a minimal surface is an isometric immersion of a Riemann surface using harmonic functions. (That sounds like complex analysis is involved !)
Problem: Given $M \subset \mathbb{C}^m = \mathbb{R}^{2m}$ of real dimension $2p - 1$, find a complex manifold (or variety) $H$ of complex dimension $p$ such that the boundary of $H$ is $M$ ... 

(Complex manifold of dimension $p$ is locally an immersion of a neighborhood of $\mathbb{C}^p$ via holomorphic functions)
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(Complex manifold of dimension $p$ is locally an immersion of a neighborhood of $\mathbb{C}^p$ via holomorphic functions)

Harvey–Lawson ’75: Not possible in general, but in the right sense (in the sense of currents) and under some natural condition on $M$, it is true.
Consider a smooth 
\[ f : S^1 \to \mathbb{C}^m \]

Is there an analytic disc with boundary \( f(S^1) \)? That is, is there 
\[ F : \overline{D} \to \mathbb{C}^m \]

holomorphic in \( \mathbb{D} \) and smooth up to the boundary such that 
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holomorphic in \( D \) and smooth up to the boundary such that \( F|_{S^1} = f \)?

We solve the Dirichlet problem, and for \( F \) to be holomorphic we need all the negative Fourier coefficients of \( f \) to be zero:
\[
\int_{S^1} f(z) z^k \, dz = 0
\]
for all \( k = 0, 1, 2, 3, \ldots \)
So $M = f(S^1)$ was given as an image of a subset of $\mathbb{C}$ and by extending the function to all of $\overline{\mathbb{D}}$ we found that $H = F(\overline{\mathbb{D}})$ is our solution.
Simple example: Analytic disc with smooth boundary

So \( M = f(S^1) \) was given as an image of a subset of \( \mathbb{C} \) and by extending the function to all of \( \overline{\mathbb{D}} \) we found that \( H = F(\overline{\mathbb{D}}) \) is our solution.

Singularities might crop up even if \( M \) is not singular:

\[
f(z) = F(z) = (z^2, z^3)
\]

Then \( M = f(S^1) \) is a nice smooth curve, but \( F(\mathbb{D}) \) is a cusp.
Levi-flat as a “minimal surface”

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Consider a hypersurface $H$ (dimension $2m - 1$) with as much structure of a complex manifold: foliated by complex hypersurfaces; locally a one parameter family of complex hypersurfaces. Such a hypersurface is Levi-flat.
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Complex hypersurfaces

A simple example: $\mathbb{C}^n \times \mathbb{R} \subset \mathbb{C}^{n+1}$. 
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In $n \geq 2$, Dolbeault–Tomassini–Zaitsev (’05 and ’11) found a possibly singular solution given some conditions on $M$ (elliptic CR singular points, nowhere minimal at CR points).

The nowhere-minimality is necessary, the ellipticity is not.
If $M \subset \mathbb{C}^{n+1}$ is a real submanifold, the CR vectors are

\[ T^{0,1}_p M = \text{span}_\mathbb{C} \left\{ \frac{\partial}{\partial \bar{z}_k}, \frac{\partial}{\partial \bar{w}} \right\} \cap \mathbb{C} \otimes T_p M \]

If $\dim T^{0,1}_p M$ (the CR dimension) is constant, then $M$ is a CR manifold.
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Real hypersurfaces in $\mathbb{C}^{n+1}$ are always CR submanifolds of CR dimension $n$.

Real codimension two submanifolds generically have isolated CR singularities.
Let $M$ be a CR submanifold and $p \in M$. If necessarily $M = N$, then $M$ is minimal at $p$. (and nowhere minimal just means... nowhere minimal) Let's stick to real-analytic submanifolds, and let $N$ be the smallest such submanifold. $N$ is then the CR orbit, the submanifold reachable by CR vector fields, their conjugates, and all the commutators. A hypersurface is Levi-flat iff it is nowhere minimal. The CR orbits then give a foliation by complex hypersurfaces.
Nowhere minimal

Let $M$ be a CR submanifold and $p \in M$.

Let $N \subset M$ be a CR submanifold, $p \in N$, such that $T_{p,1}^0 M = T_{p,1}^0 N$.

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The CR orbits then give a foliation by complex hypersurfaces.
Suppose $M = \partial H \subset \mathbb{C}^{n+1}$ for a Levi-flat hypersurface $H$. 
If $M$ is CR, it is of CR dimension $n$ ($M$ is complex) or $n - 1$. 
If $M$ is compact, it cannot be complex.
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$\Rightarrow$

$M$ is nowhere minimal.
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Near CR point? Yes if CR orbits are all of real codimension 1, possibly no otherwise (example in L. ’06). (Trivially yes if $n = 1$, but not unique!)
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Near a CR singular point? Yes ($n \geq 2$) if the CR singularity is nondegenerate (or an exceptional case), Fang–Huang ’17.

In $n = 1$, not always. Yes if the CR singularity is e.g. elliptic.
(e.g. Bishop ’65, Moser–Webster ’82, Moser ’85, Huang–Yin ’09 ...
... lots of others)
A codimension 2 CR singular submanifold $M$ is locally

$$ w = \rho(z, \bar{z}) = A(z, \bar{z}) + B(z, z) + \overline{B(z, z)} + O(\|z\|^3) $$

$(z, w) \in \mathbb{C}^n \times \mathbb{C}$, $A$ sesquilinear, $B$ bilinear.
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$M$ is $A$-nondegenerate (or just nondegenerate) if $A$ is nondegenerate. ($elliptic$ if $A$ is positive definite, and $B$ has small eigenvalues)
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To be locally boundary of a Levi-flat hypersurface, we need to have, after a change of variables, $A$ to be real-valued (Hermitian) and also the "$O(\|z\|^3)$" to be real valued.
A theorem

**Theorem (L.–Noell-Ravisankar ’17, ’18)**

Let \( \Omega \subset \mathbb{C}^n \times \mathbb{R} \), \( n \geq 2 \), be a bounded domain with connected real-analytic boundary such that \( \partial \Omega \) has only A-nondegenerate CR singularities. Let \( \Sigma \subset \partial \Omega \) be the set of CR singularities of \( \partial \Omega \). Let \( f: \partial \Omega \rightarrow \mathbb{C}^{n+1} \) be a real-analytic embedding that is CR at CR points of \( \partial \Omega \) and takes CR points of \( \partial \Omega \) to CR points of \( f(\partial \Omega) \).

Then, there exists a real-analytic CR map \( F: \overline{\Omega} \rightarrow \mathbb{C}^{n+1} \) such that \( F|_{\partial \Omega} = f \) and \( F|_{\overline{\Omega}\setminus \Sigma} \) is an immersion.

In other words, \( F(\overline{\Omega}) \) is the solution of the Levi-flat Plateau problem for \( f(\partial \Omega) \).
Proof? (simplified)

Work along “leaves”, extend \( f(\cdot, s) \) using Hartogs–Bochner (really Martinelli), or Severi and then Hartogs.
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The Jacobian of $F$ vanishes on too large of a set contradicting $f$ being a diffeomorphism.
A better result via Fang–Huang

We get a better result if $f(\partial \Omega)$ also has only nondegenerate singularities by applying Fang–Huang.

**Corollary**

Let $\Omega \subset \mathbb{C}^n \times \mathbb{R}$, $n \geq 2$, be a bounded domain with connected real-analytic boundary such that $\partial \Omega$ has only A-nondegenerate CR singularities, and let $f : \partial \Omega \to \mathbb{C}^{n+1}$ be a real-analytic embedding that is CR at CR points of $\partial \Omega$. Assume that $f(\partial \Omega)$ has only A-nondegenerate CR singularities. Further assume that either $n \geq 3$ or no CR singularity of $f(\partial \Omega)$ is the exceptional case (every CR singularity has an elliptic direction).

Then, there exists a real-analytic CR map $F : \overline{\Omega} \to \mathbb{C}^{n+1}$ such that $F|_{\partial \Omega} = f$ and $F$ is an immersion on $\overline{\Omega}$.

(exceptional case:

$w = |z_1|^2 - |z_2|^2 + \lambda(z_1^2 + \bar{z}_1^2) + \lambda(z_2^2 + \bar{z}_2^2) + O(\|z\|^3), \lambda \geq \frac{1}{2}$)
Examples... \((n = 1)\)

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\(N \subset \mathbb{C} \times \mathbb{R}: \quad s = |z|^2\)

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Elliptic singularity, \(F|_N\) a diffeomorphism, but \(F\) is a finite map, not an immersion (on either side of \(N\))

\[F(z, s) = (z, zs)\] is even worse

\((F(N)\) is degenerate in both cases)
Examples... \((n = 1)\)

\[ N \subset \mathbb{C} \times \mathbb{R}: \quad \text{Im } z = 0 \]

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This is impossible when \(n \geq 2,\) and \(F(N)\) is \(A\)-nondegenerate.
Examples...

\[ N \subset \mathbb{C}^2 \times \mathbb{R}: \quad s = z_1 + \bar{z}_1 + |z_2|^2 \]

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$F(z, s) = (z, s^2 + is^3)$

In $(\xi, \sigma + i\tau) \in \mathbb{C}^2 \times \mathbb{C}$

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\( F(N) \) is CR singular, \( F|_N \) is a diffeomorphism, \( F|_N \) is a CR diffeomorphism outside the CR singularity,
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The singular(!) Levi-flat hypersurface \( \{\sigma^3 = \tau^2\} \) is the unique Levi-flat hypersurface that contains \( F(N) \).
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The singularity of \( F(N) \) is degenerate!
Examples...

\[ N \subset \mathbb{C}^n \times \mathbb{R}: \quad s = ||z||^2 \]

\[ F(z, s) = (z, s^2) \]

In \((\xi, \sigma) \in \mathbb{C}^n \times \mathbb{C}, \)

\[ F(N): \quad \sigma = ||\xi||^4 \]

\(F(N)\) is CR singular and degenerate in every sense.
$F(z, s) = (z, s^2 + is^2)$

$F(N): \sigma = ||\xi||^4$ and $\tau = ||\xi||^6$

$N \subset \mathbb{C}^n \times \mathbb{R}: s = ||z||^2$

$F(N)$ is degenerate, and the singular $F$ is the unique Levi-flat that contains $F(N)$. $F(N)$ is an example of the necessity of nondegeneracy in Fang–Huang.
$N \subset \mathbb{C}^n \times \mathbb{R}: \quad s = \|z\|^2$

$F(z, s) = (z, s^2 + is^2)$

$F(N): \sigma = \|\xi\|^4 \quad \text{and} \quad \tau = \|\xi\|^6$

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$F(N)$ is an example of the necessity of nondegeneracy in Fang–Huang.
Examples... (now think globally)

\[ \Omega \subset \mathbb{C}^n \times \mathbb{R} : \quad ||z||^2 + (s + \epsilon)^2 < 1 \]

\[ F(z, s) = (z, s^2) \]

In \((\xi, \sigma) \in \mathbb{C}^n \times \mathbb{R},\)

\[ F(\partial \Omega) \text{ is } 4\epsilon^2 \sigma = (1 - \epsilon^2 - ||\xi||^2 - \sigma)^2 \]

\[ F|_{\partial \Omega} \text{ is a diffeomorphism,} \]

\[ F(\partial \Omega) \text{ has CR singularities at} \]

\[ \xi = 0, \quad 4\epsilon^2 \sigma = (1 - \epsilon^2 - \sigma)^2 \text{ (isolated)} \]

\[ \sigma = 0 \text{ and } ||\xi||^2 = 1 - \epsilon^2 \text{ (not isolated)} \]

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Examples... (now think globally)

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but ... \(F\) is not 1-1 on \(\Omega\)!
\[\Omega \subset \mathbb{C}^n \times \mathbb{R}: \quad ||z||^2 + (s + \epsilon)^2 < 1\]

\[F(z, s) = (z, 1 - 4s^2 + i(8s^3 - 2s))\]

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Let $H$ in $(\xi, \eta) \in \mathbb{C}^2$ be defined by ($\epsilon > 0$ small) 
\[
\text{Im}(\xi^2 + \eta^2) = 0, \quad |\xi|^2 + |\eta + \epsilon|^2 \leq 1
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$H$ is singular (as a variety) at the origin.
A final example ...

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Consider $M = \partial H$

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$H$ is not an image of a domain in $\mathbb{C} \times \mathbb{R}$!

(There is noting special about $\mathbb{C}^2$ here).
Thank you