Lewy extension for smooth hypersurfaces in $\mathbb{C}^n \times \mathbb{R}$

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joint work with Alan Noell and Sivaguru Ravisankar

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Hypersurface in $\mathbb{C}^n$

Any smooth hypersurface $M$ can be locally written as

$$\operatorname{Im} w = \sum_{j=1}^{n} \epsilon_j |z_j|^2 + E(z, \bar{z}, \operatorname{Re} w)$$

for $E \in O(3)$, and $\epsilon_j = -1, 0, 1$.

The form $\sum_{j=1}^{n} \epsilon_j |z_j|^2$ is the Levi-form at the origin.
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If $f \in \mathcal{O}(H_+ \setminus M) \cap C^\infty(H_+)$, then $f$ is CR on $M$, that is, $\nu f = 0$ for every $\nu \in \Gamma(T^{0,1}_p M)$

Here $T^{0,1}_p M = \text{span}_\mathbb{C} \left\{ \frac{\partial}{\partial \bar{z}_k}, \frac{\partial}{\partial w} \right\} \cap \mathbb{C} \otimes T_p M$
Lewy extension

\[ M : \ \text{Im} \ w = \sum_{j=1}^{n} \epsilon_j |z_j|^2 + E \quad H_+ : \ \text{Im} \ w \geq \sum_{j=1}^{n} \epsilon_j |z_j|^2 + E \]
Theorem (Lewy extension)

Let $M, H_+ \subset \mathbb{C}^{n+1}, n \geq 1$, be as above.

Then $\exists$ a neighborhood $U$ of 0, such that given $f \in CR(M) \cap C^\infty(M)$:

(i) If the Levi-form at 0 has a positive eigenvalue, then $\exists F \in C^\infty(U \cap H_+) \cap O(U \cap H_+ \setminus M)$ such that $F|_{M \cap U} = f|_{M \cap U}$

(ii) If the Levi-form at 0 has eigenvalues of both signs, then $\exists F \in O(U)$ such that $F|_{M \cap U} = f|_{M \cap U}$

\begin{align*}
M: \quad \text{Im} \ w &= \sum_{j=1}^{n} \epsilon_j |z_j|^2 + E \\
H_+: \quad \text{Im} \ w &\geq \sum_{j=1}^{n} \epsilon_j |z_j|^2 + E
\end{align*}
Theorem (Severi)

Suppose $M \subset \mathbb{C}^{n+1}$ is a real-analytic hypersurface and $f \in CR(M) \cap C^\omega(M)$

Then $\exists$ neighborhood $U$ of $M$ and $F \in O(U)$ s.t. $F|_{M \cap U} = f|_{M \cap U}$. 

Real-analytic CR functions
Write coordinates as \((z, s) \in \mathbb{C}^n \times \mathbb{R}\).
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Call the sets \(\mathbb{C}^n \times \{s\}\) the \textit{leaves} of \(\mathbb{C}^n \times \mathbb{R}\).

For \(X \subset \mathbb{C}^n \times \mathbb{R}\) define \(X_s = \{z \in \mathbb{C}^n \mid (z, s) \in X\}\).
Hypersurfaces in $\mathbb{C}^n \times \mathbb{R}$

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Let $M \subset \mathbb{C}^n \times \mathbb{R}$ be a smooth real hypersurface.

\[ T^{0,1}_p M = \text{span}_{\mathbb{C}} \left\{ \frac{\partial}{\partial \bar{z}_k} \right\} \cap \mathbb{C} \otimes T_p M = T^{0,1}_p M_s \]
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\(M\) is CR at \(p\) if \(\dim T^0,1_q M\) is constant on \(M\) near \(p\).

Let \(M_{CR}\) be the set of CR points of \(M\).

Otherwise \(M\) has a CR singularity at \(p\).
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$f \in C^\infty(M)$ is CR if $\nu f = 0$ for all $\nu \in \Gamma(T_{0,1}M_{CR})$. 
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\(f \in C^\infty(M)\) is CR if \(\nu f = 0\) for all \(\nu \in \Gamma( T_{\nu}^{0,1}M_{CR})\).

(Equivalently, \(\nu f = 0\) for all \(\nu \in \Gamma( \mathbb{C} \otimes TM)\) where \(\nu_p \in T_{\nu}^{0,1}M\) for all \(p\)).
(Severi strikes again)

Suppose $M \subset \mathbb{C}^n \times \mathbb{R}$ is a real-analytic CR hypersurface and $f \in CR(M) \cap C^\omega(M)$

Then $\exists$ a neighborhood $U \subset \mathbb{C}^{n+1}$ of $M \subset \mathbb{C}^n \times \mathbb{R} \subset \mathbb{C}^{n+1}$ and $F \in \mathcal{O}(U)$ s.t. $F|_M = f$. 
Theorem (Special case of Hill-Taiani ’84)

Let $M \subset \mathbb{C}^n \times \mathbb{R}$, $n \geq 2$, be a real smooth CR hypersurface of CR dimension $n - 1$ (not complex). Let $p = (z_0, s_0) \in M$. Let $(a, b)$ be the number of positive and negative eigenvalues of the Levi-form of $M_{(s_0)}$ at $z_0$.

Then $\exists$ a neighborhood $U \subset \mathbb{C}^n \times \mathbb{R}$ of $p$, such that given $f \in C^\infty(M) \cap CR(M)$:

(i) If $a \geq 1$, and $H_+$ is the corresponding side, then $\exists \ F \in C^\infty(U \cap H_+) \cap CR(U \cap H_+ \setminus M)$ such that $f|_{M \cap U} = F|_{M \cap U}$.

(ii) If $a \geq 1$ and $b \geq 1$, then $\exists \ F \in C^\infty(U) \cap CR(U)$ such that $f|_{M \cap U} = F|_{M \cap U}$. 
CR singular manifolds in $\mathbb{C}^2$ of real dim 2 first studied by Bishop ('65).

Later by Moser-Webster, Moser, Kenig-Webster, Gong, Huang-Krantz, Huang, Huang-Yin, etc...
Mostly interested in normal form.

In two dimensions we (at least formally) can generally realize such manifolds as real hypersurfaces in $\mathbb{C} \times \mathbb{R}$.

Higher dimensions far less understood. See e.g. Huang-Yin, Burcea, Gong-L., Dolbeault-Tomassini-Zaitsev, Coffman, Slapar, (and of course L.-Noell-Ravisankar), etc...
CR singular submanifolds and $\mathbb{C}^n \times \mathbb{R}$

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In $\mathbb{C}^m$ for $m > 2$ generally a codimension 2 submanifold is not realizable as a submanifold of $\mathbb{C}^{m-1} \times \mathbb{R}$.
A real codimension 2 CR singular submanifold $M \subset \C^m$ does not in general have the extension property in the analytic case. (Harris ’78, L.-Minor-Shroff-Son-Zhang).
A real codimension 2 CR singular submanifold \( M \subset \mathbb{C}^m \) does not in general have the extension property in the analytic case. (Harris ’78, L.-Minor-Shroff-Son-Zhang).

Simplest example: Let \( M \) in \((z, w) \in \mathbb{C}^n \times \mathbb{C}\) be given by

\[
w = z_1 \bar{z}_1
\]

Then \( \bar{z}_1 = \frac{w}{z_1} \) on \( M \) and so does not extend to a neighborhood.
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Simplest example: Let $M$ in $(z, w) \in \mathbb{C}^n \times \mathbb{C}$ be given by

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Then $\bar{z}_1 = \frac{w}{z_1}$ on $M$ and so does not extend to a neighborhood.

In early 20th century several authors considered extensions of holomorphic functions (e.g. Hartogs phenomenon) in $\mathbb{C}^n \times \mathbb{R}^k$ (e.g. Bochner, Brown, Severi, etc...)
Let $M \subset \mathbb{C}^n \times \mathbb{R}$ be a hypersurface with a CR singularity. Write $M$ as

$$s = Q(z, \bar{z}) + E(z, \bar{z})$$

where $Q$ is a real quadratic form, and $E \in O(3)$. If $Q$ is nondegenerate then the CR singularity is isolated.
Let $M \subset \mathbb{C}^n \times \mathbb{R}$ be a hypersurface with a CR singularity. Write $M$ as

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Diagonalize $A$

$$s = \sum_{j=1}^{a} |z_j|^2 - \sum_{j=a+1}^{a+b} |z_j|^2 + B(z, z) + \overline{B(z, z)} + E(z, \bar{z}),$$

We can’t generally also diagonalize $B$ (unless $a = n$).
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Define manifold with boundary $H_+$ by

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$$= A + B + \overline{B} + E = Q + E$$

**Theorem (L.-Noell-Ravisankar ’16)**

Suppose $M$ is real-analytic ($E$ is real-analytic), $A$ is nondegenerate ($a + b = n$), $n \geq 2$, and $f \in C^\omega(M) \cap CR(M_{CR})$.

Then $\exists$ neighborhood $U$ of 0 in $\mathbb{C}^n \times \mathbb{C}$ and $F \in \mathcal{O}(U)$ such that $F|_{M \cap U} = f|_{M \cap U}$. 
Smooth CR singular case

\[ H_+: s \geq \sum_{j=1}^{a} |z_j|^2 - \sum_{j=a+1}^{a+b} |z_j|^2 + B(z, z) + \overline{B(z, z)} + E(z, \overline{z}) \]

\[ M: s = \sum_{j=1}^{a} |z_j|^2 - \sum_{j=a+1}^{a+b} |z_j|^2 + B(z, z) + \overline{B(z, z)} + E(z, \overline{z}) \]

**Theorem (L.-Noell-Ravisankar ’17)**

Suppose \( Q \) is nondegenerate, and \( a \geq 2 \).

Then there exists a neighborhood \( U \) of 0,

such that given \( f \in C^\infty(M) \cap CR(M) \):

(i) If \( a \geq 2 \),

then there exists \( F \in C^\infty(U \cap H_+) \cap CR(U \cap H_+ \setminus M) \)

such that \( F|_{M \cap U} = f|_{M \cap U} \).

(ii) If \( a \geq 2 \) and \( b \geq 2 \),

then there exists \( F \in C^\infty(U) \cap CR(U) \)

such that \( F|_{M \cap U} = f|_{M \cap U} \).

In either case, \( F \) has a formal power series in \( z \) and \( s \) at 0.
There are two problems for the extension:

1) existence of the extension
2) regularity up to the boundary
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2) regularity up to the boundary

For the first problem, the two eigenvalues are needed.

\[ M : s = A(z, \bar{z}) + B(z, z) + \overline{B(z, z)} + E(z, \bar{z}) \]

If \( A \) has two positive eigenvalues, then the Levi-form of \( M(s) \) has at least one positive eigenvalue.
Example 1: Define $M$ by $s = ||z||^4$ (isolated CR singularity).
The function $\sqrt{s}$ is $C^\omega(M)$ (it equals $||z||^2$ on $M$).
It is CR, and the unique extension to $H_+$ is $\sqrt{s}$, not smooth at the origin.
Some sort of nondegeneracy is necessary

Example 1: Define $M$ by $s = \| z \|^4$ (isolated CR singularity).
The function $\sqrt{s}$ is $C^\omega(M)$ (it equals $\| z \|^2$ on $M$).
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Example 2: Write $z = (z', z'')$. Define $M$ by
$s = (\| z' \|^2 - \| z'' \|^2)^3$.
The function $\sqrt[3]{s}$ is $C^\omega(M)$ (equals $\| z' \|^2 - \| z'' \|^2$ on $M$).
It is CR, and the unique extension to $H_+$ is $\sqrt[3]{s}$, not smooth at points of $H_+$ where $s = 0$ (including interior).
CR singularity is large. All points where $s = 0$ are CR singular.
Define $M \subset \mathbb{C}^2 \times \mathbb{R}$ and $f$ by

$$\text{Im}\, z_1 = s \, |z_2|^2, \quad f(z, s) = \begin{cases} \frac{e^{-1/s^2}}{z_1 + is} & \text{if } s \neq 0, \\ 0 & \text{if } s = 0. \end{cases}$$

The Levi-form is zero when $s = 0$. Extension of $f$ to neither side is possible near 0.
CR singular case: two eigenvalues of the same sign are necessary

Define $M \subset \mathbb{C}^2 \times \mathbb{R}$ and $f$ by

$$M : s = |z_1|^2 - |z_2|^2, \quad f(z, s) = \begin{cases} \frac{1}{z_1} e^{-1/s^2} & \text{if } s > 0, \\ 0 & \text{if } s = 0, \\ \frac{1}{z_2} e^{-1/s^2} & \text{if } s < 0. \end{cases}$$

$f \in C^\infty(M) \cap CR(M)$ but no extension exists due to the poles.
Analogue of Baouendi-Treves is not true

An idea for extension is to generalize Baouendi-Treves (B-T) (approximation by polynomials in $z$ and $s$).
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But \(M\):

\[ s = |z_1|^2 - |z_2|^2 \]

cannot have B-T.

There is a disc through every point attached to \(M\), so B-T would imply extension to a neighborhood.
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cannot have B-T.

There is a disc through every point attached to $M$, so B-T would imply extension to a neighborhood.

**Question:** What extra hypotheses to add to B-T to make it work.
Two eigenvalues of both signs needed for extension to a neighborhood

Define $M \subset \mathbb{C}^3 \times \mathbb{R}$ and $f$ by

$$M : s = |z_1|^2 + |z_2|^2 - |z_3|^2,$$

$$f(z, s) = \begin{cases} 0 & \text{if } s \geq 0, \\ \frac{1}{z_3} e^{-1/s^2} & \text{if } s < 0. \end{cases}$$

Again, $f \in C^\infty(M) \cap CR(M)$

And $f$ extends above $M$, but not below $M$. 
There exists an example that extends only to one side at every point.

Let $M \subset \mathbb{C}^2 \times \mathbb{R}$ be

$$s = |z_1|^2 + |z_2|^2 = ||z||^2,$$

$g: S^3 \subset \mathbb{C}^2 \to \mathbb{C}$ a smooth CR function not extending to the outside of $S^3$ through any point (e.g. Catlin or Hakim-Sibony).

$$f(z, s) = \begin{cases} e^{-1/s^2} g\left(\frac{z}{\sqrt{s}}\right) & \text{if } s < 0, \\ 0 & \text{if } s = 0, \end{cases}$$

is smooth, CR, extends above $M$ (to $H_+$), but not below through any point.
Extension fails in $n = 1$.

Let $M \subset \mathbb{C} \times \mathbb{R}$ be a nonparabolic Bishop surface

$$s = |z|^2 + \lambda z^2 + \lambda \bar{z}^2,$$

(where $0 \leq \lambda < \infty$ and $\lambda \neq \frac{1}{2}$).

Define a smooth $f : \mathbb{C} \rightarrow \mathbb{R}$ that is zero on the first quadrant of $\mathbb{C}$ and positive elsewhere.

Parametrize $M$ by $z$, then $f(z, \bar{z})$ is smooth on $M$.

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The CR condition is vacuous.

For every $s \neq 0$, the leaf

$$(H_+)_s = \{z \in \mathbb{C} \mid s \geq |z|^2 + \lambda z^2 + \lambda \bar{z}^2\}$$

is either empty, or has part of its boundary in the first quadrant. So $f$ cannot extend.
Define $\mathcal{M}$ by

$$s = |z_1|^2 - |z_2|^2 + \lambda (z_1^2 + \bar{z}_1^2)$$

for $\lambda > \frac{1}{2}$.

For $s > 0$, the manifold with boundary $(H_+)_s$ has disconnected boundary.

Construct a function that is a different constant on each boundary component for each $(H_+)_s$. 
Example: Topology for degenerate $M$ can be evil

Take $\phi(x) = \sin^2(1/x) e^{-1/x^2}$, and let $M$ be given by

$$s = \phi(||z||^2)$$

$(H_+)(s)$ has multiple components with disconnected boundary.

The function $||z||^2$ is $C^\infty(M) \cap CR(M)$ but has no extension.

The CR singularity is large (an infinite set of concentric circles).
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2) Iterate the above to obtain a formal power series solution.
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3) Extend near the CR points using Hill-Taiani.
Proof of the theorem

1) Solve the problem for homogeneous polynomial CR functions on the model manifold $s = Q(z, \bar{z})$.

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4) Construct families of analytic discs inside a single leaf attached to CR points of $M$ shrinking to a CR point of $M$. 
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3) Extend near the CR points using Hill-Taiani.

4) Construct families of analytic discs inside a single leaf attached to CR points of $M$ shrinking to a CR point of $M$.

5) Apply Kontinuitätssatz to find an extension $F$. (technicality: proving single valuedness, $M_s$ and $(H_+)_s$ need not be connected, and $(H_+)_s$ may not be simply connected.)
6) Prove that $F$ is continuous up to the CR singularity.
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7) Suppose $M$ given by $s = \rho(z, \bar{z})$. Parametrize $M$ by $z$ and differentiate $f(z, \bar{z})$ outside the origin.

$$f_{\bar{z}_j} = (F_s|_M)\rho_{\bar{z}_j}$$

Division works formally at the origin by the formal solution. By Malgrange $F_s|_M$ is smooth. Similarly $F_{z_j}|_M$ is smooth.
Proof of the theorem, cont.

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7) Suppose $M$ given by $s = \rho(z, \bar{z})$. Parametrize $M$ by $z$ and differentiate $f(z, \bar{z})$ outside the origin.

$$f_{\bar{z}_j} = (F_s\big|_M)\rho_{\bar{z}_j}$$

Division works formally at the origin by the formal solution. By Malgrange $F_s|_M$ is smooth. Similarly $F_{\bar{z}_j}|_M$ is smooth.

8) $F_s|_M$ and $F_{\bar{z}_j}|_M$ are smooth CR functions, therefore their extensions are continuous up to the boundary. Now iterate.
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Not sure how much nondegeneracy is necessary.

Question: Is isolated singularity needed?  
Is nondegeneracy of $Q$ needed?  
(It is not in the real-analytic/formal case).
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Not sure how much nondegeneracy is necessary.

Question: Is isolated singularity needed?
   Is nondegeneracy of $Q$ needed?
   (It is not in the real-analytic/formal case).

Question: What nondegeneracy is needed in the $C^\omega$ case?
   (e.g., we can prove $C^\omega$ extension for
   $s = z_1^2 + \cdots + z_n^2 + \bar{z}_1^2 + \cdots + \bar{z}_n^2$.)
Thank you