Lewy extension for smooth hypersurfaces in $\mathbb{C}^n \times \mathbb{R}$

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joint work with Alan Noell and Sivaguru Ravisankar

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Hypersurface in $\mathbb{C}^n$

Any smooth hypersurface M can be locally written as

$$\operatorname{Im} w = \sum_{j=1}^{n} \epsilon_j \left|z_j\right|^2 + E(z, \bar{z}, \operatorname{Re} w)$$

for $E \in O(3)$, and $\epsilon_j = -1, 0, 1$.
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Define manifold with boundary H_+ by

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If $f \in \mathcal{O}(H_+ \setminus M) \cap C^\infty(H_+)$, then f has to be CR, that is,
let $T_p^{0,1} M = \operatorname{span}_{\mathbb{C}} \left\{ \frac{\partial}{\partial \bar{z}_k}, \frac{\partial}{\partial \bar{w}} \right\} \cap \mathbb{C} \otimes T_p M$

$f \colon M \to \mathbb{C}$ is CR whenever $\nu f = 0$ for every $\nu \in \Gamma(T^{0,1} M)$

Lewy extension

$$M: \quad \operatorname{Im} w = \sum_{j=1}^{n} \epsilon_j \left|z_j\right|^2 + E \qquad H_+: \quad \operatorname{Im} w \geq \sum_{j=1}^{n} \epsilon_j \left|z_j\right|^2 + E$$

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Theorem (Lewy extension)

Let $M, H_+ \subset \mathbb{C}^n$, $n \geq 2$, be as above. There exists a neighbourhood U of the origin such that given any $f \in CR(M) \cap C^\infty(M)$ we have:

(i) *If the Levi-form at the origin has a positive eigenvalue, there exists $F \in C^\infty(H_+ \cap U) \cap \mathcal{O}((H_+ \cap U) \setminus M)$ such that $F|_{M\cap U} = f|_{M\cap U}$*

(ii) *If the Levi-form at the origin has eigenvalues of both signs, there exists $F \in \mathcal{O}(U)$ such that $F|_{M\cap U} = f|_{M\cap U}$*

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Write coordinates as $(z, s) \in \mathbb{C}^n \times \mathbb{R}$.

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Call the sets $\mathbb{C}^n \times \{s\}$ the *leaves* of $\mathbb{C}^n \times \mathbb{R}$.
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Let $M \subset \mathbb{C}^n \times \mathbb{R}$ be a smooth real hypersurface.

$$T_p^{0,1} M = \operatorname{span}_{\mathbb{C}} \left\{ \frac{\partial}{\partial \bar{z}_k} \right\} \cap \mathbb{C} \otimes T_p M = T_p^{0,1} M_{(s)}$$

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M is CR at p if $\dim T_q^{0,1} M$ is constant on M near p.
Let M_{CR} be the set of CR points of M.

Otherwise M has a CR singularity at p.

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(Equivalently, $\nu f = 0$ for all $\nu \in \Gamma(\mathbb{C} \otimes TM)$ where $\nu_p \in T_p^{0,1} M$ for all p).

Theorem (Special case of Hill-Taiani '84)

Let $M \subset \mathbb{C}^n \times \mathbb{R}$, $n \geq 2$, be a real smooth CR hypersurface of CR dimension $n-1$ (not complex). Let $p = (z_0, s_0) \in M$. Then there exists a neighborhood $U \subset \mathbb{C}^n \times \mathbb{R}$ of p, such that given a smooth CR function $f \colon M \to \mathbb{C}$, we have:

(i) *If the Levi-form of $M_{(s_0)}$ at z_0 has at least one positive eigenvalue, and H_+ is the side of M in U corresponding to the positive eigenvalue, then there exists a smooth function $F \colon H_+ \to \mathbb{C}$, which is CR in $H_+ \setminus M$, and $f|_{M \cap U} = F|_{M \cap U}$.*

(ii) *If the Levi-form of $M_{(s_0)}$ at z_0 has eigenvalues of both signs, then there exists a smooth CR function $F \colon U \to \mathbb{C}$, such that $f|_{M \cap U} = F|_{M \cap U}$.*

CR singular submanifolds

Real dimension 2 CR singular manifolds in $\mathbb{C}^2$ first studied by Bishop.

Later by Moser-Webster, Moser, Kenig-Webster, Gong, Huang-Krantz, Huang, Huang-Yin, etc...

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Higher dimensions far less understood.
See e.g. Huang-Yin, Burcea, Gong-L., Dolbeault-Tomassini-Zaitsev, Coffman, Slapar, (and of course L.-Noell-Ravisankar), etc...

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In early 20th century several authors considered extensions of holomorphic functions (e.g. Hartogs phenomenon) in $\mathbb{C}^n \times \mathbb{R}^k$ (e.g. Bochner, Brown, Severi, etc...)

CR singular hypersurface in $\mathbb{C}^n \times \mathbb{R}$

Let $M \subset \mathbb{C}^n \times \mathbb{R}$ be a hypersurface with a CR singularity.
Write M as

$$s = Q(z, \bar{z}) + E(z, \bar{z})$$

where Q is a real quadratic form, and $E \in O(3)$.

If Q is nondegenerate then the CR singularity is isolated.

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Write $Q(z, \bar{z}) = A(z, \bar{z}) + B(z, z) + \overline{B(z, z)}$
for Hermitian form A and bilinear B.

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Suppose A is nondegenerate and diagonalize

$$s = \sum_{j=1}^{a} |z_j|^2 - \sum_{j=a+1}^{n} |z_j|^2 + B(z, z) + \overline{B(z, z)} + E(z, \bar{z}),$$

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Define manifold with boundary H_+ by

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CR singular extension

$$H_+ : \; s \geq \sum_{j=1}^{a} |z_j|^2 - \sum_{j=a+1}^{n} |z_j|^2 + B(z,z) + \overline{B(z,z)} + E(z,\bar{z})$$
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Theorem (L.-Noell-Ravisankar)

Suppose Q is nondegenerate.
Then there exists a neighborhood U of the origin, such that given a smooth CR function $f\colon M \to \mathbb{C}$:

(i) *If $a \geq 2$, then there exists a function $F \in C^\infty(H_+ \cap U)$ such that F is CR on $(H_+ \setminus M) \cap U$ and $F|_{M\cap U} = f|_{M\cap U}$.*

(ii) *If $a \geq 2$ and $n - a \geq 2$, then there exists a CR function $F \in C^\infty(U)$ such that $F|_{M\cap U} = f|_{M\cap U}$.*

In either case, F has a formal power series in z and w at 0.

Some sort of nondegeneracy is necessary

Define M by $s = \|z\|^4$.

The function $\sqrt{s}$ is smooth on M (it equals $\|z\|^2$ on M).

It is CR, and the unique extension to H_+ is $\sqrt{s}$, not smooth at the origin.

CR case: one nonzero eigenvalue is necessary

Define $M \subset \mathbb{C}^2 \times \mathbb{R}$ and f by

$$\operatorname{Im} z_1 = s \left| z_2 \right|^2 , \qquad f(z,s) = \begin{cases} \frac{e^{-1/s^2}}{z_1 + is} & \text{if } s \neq 0, \\ 0 & \text{if } s = 0. \end{cases}$$

The Levi-form is only zero when $s = 0$, and the extension of f to neither side is possible near the origin.

CR singular case: two eigenvalues of the same sign are necessary

Define $M \subset \mathbb{C}^2 \times \mathbb{R}$ and f by

$$M : \; s = |z_1|^2 - |z_2|^2, \qquad f(z,s) = \begin{cases} \frac{1}{z_1} e^{-1/s^2} & \text{if } s > 0, \\ 0 & \text{if } s = 0, \\ \frac{1}{z_2} e^{-1/s^2} & \text{if } s < 0. \end{cases}$$

f is smooth, CR, and cannot be extended to either side because of the poles.

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(To see that f is smooth suppose $s > 0$. Write $f(z, |z_1|^2 - |z_2|^2) = \frac{1}{z_1} e^{-1/(|z_1|^2 - |z_2|^2)^2}$. Derivatives are of the form $\frac{P(z,\bar{z})}{z_1^d (|z_1|^2 - |z_2|^2)^k} e^{-1/(|z_1|^2 - |z_2|^2)^2}$, and as $s > 0$, then $\left|\frac{1}{z_1}\right| \leq \frac{1}{|z_1|^2 - |z_2|^2}$.)

Two eigenvalues of both signs needed for extension to a neighbourhood

Define $M \subset \mathbb{C}^3 \times \mathbb{R}$ and f by

$$M : \; s = |z_1|^2 + |z_2|^2 - |z_3|^2, \qquad f(z, s) = \begin{cases} 0 & \text{if } s \geq 0, \\ \frac{1}{z_3} e^{-1/s^2} & \text{if } s < 0. \end{cases}$$

Again, f is smooth and CR.

And f extends above M, but not below M.

There exists an example that extends only to one side at every point.

Let $M \subset \mathbb{C}^2 \times \mathbb{R}$ be

$$s = |z_1|^2 + |z_2|^2 = \|z\|^2,$$

$g\colon S^3 \subset \mathbb{C}^2 \to \mathbb{C}$ a smooth CR function not extending to the outside of S^3 through any point (e.g. Catlin or Hakim-Sibony).

$$f(z,s) = \begin{cases} e^{-1/s^2} g\left(\frac{z}{\sqrt{s}}\right) & \text{if } s < 0, \\ 0 & \text{if } s = 0, \end{cases}$$

is smooth, CR, extends above M (to H_+), but not below through any point.

Extension fails in $n = 1$.

Let $M \subset \mathbb{C} \times \mathbb{R}$ be a nonparabolic Bishop surface

$$s = |z|^2 + \lambda z^2 + \lambda \bar{z}^2, \qquad \text{(where } 0 \leq \lambda < \infty \text{ and } \lambda \neq \tfrac{1}{2}\text{)}.$$

Define a smooth $f : \mathbb{C} \to \mathbb{R}$ that is zero on the first quadrant of $\mathbb{C}$ and positive elsewhere.

Parametrize M by z, then $f(z, \bar{z})$ is smooth on M.
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For every $s \neq 0$, the leaf

$$(H_+)_{(s)} = \{z \in \mathbb{C} \mid s \geq |z|^2 + \lambda z^2 + \lambda \bar{z}^2\}$$

is either empty, or has part of its boundary in the first quadrant. So f cannot extend.

Levi-form on the leaves

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1) existence of the extension

2) regularity up to the boundary

For the first problem, the two eigenvalues are needed.

$$M: \; s = A(z, \bar{z}) + B(z, z) + \overline{B(z, z)} + E(z, \bar{z})$$

If A has two positive eigenvalues, then the Levi-form of $M_{(s)}$ has at least one positive eigenvalues.

Proof of the theorem

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4) Apply Kontinuitätssatz to find an extension F. (technicality: proving single valuedness, $M_{(s)}$ and $(H_+)_{(s)}$ need not be connected, and $(H_+)_{(s)}$ may not be simply connected.)

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5) Prove regularity at the CR points using Hill-Taiani.

Proof of the theorem, cont.

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7) Suppose M given by $s = \rho(z, \bar{z})$. Parametrize M by z and differentiate f outside the origin.

$$f_{\bar{z}_j} = (F_s|_M)\rho_{\bar{z}_j}$$

Division works formally at the origin by the formal solution. By Malgrange $F_s|_M$ is smooth. Similarly $F_{z_j}|_M$ is smooth.

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8) $F_s|_M$ and $F_{z_j}|_M$ are smooth CR functions, therefore their extensions are continuous up to the boundry. Now iterate.

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Is nondegeneracy of Q needed?
(It is not in the real-analytic/formal case).

Question: Is the nondegeneracy of A needed?
(we needed this in the real-analytic/formal case, though it does not seem totally necessary)

Thank you