Extensions of CR functions from CR singular submanifolds of codimension 2

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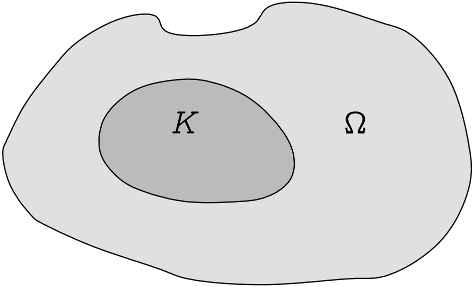
Hartogs phenomenon

Theorem (Hartogs)

Let $\Omega \subset \mathbb{C}^n$, $n \geq 2$, be a domain, and $K \subset\subset \Omega$ be compact with $\Omega \setminus K$ connected. If $f \in \mathcal{O}(\Omega \setminus K)$, then there exists a unique $F \in \mathcal{O}(\Omega)$ such that $F|_{\Omega \setminus K} = f$.



There are no hypotheses on the geometry of Ω, only a mild clearly required topological requirement on $\Omega \setminus K$.
Furthermore, K can be "as large as we want."

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M is *CR singular* at $q \in M$ if $\dim T_p^{0,1}M$ is not constant in any neighbourhood of q.

Write $M_{CR} = M \setminus \{$ CR singularities of M $\}$

Generically a codimension 2 submanifold of $\mathbb{C}^n$ will have isolated CR singularities.

CR submanifolds, singularities, and CR functions

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A smooth function $f \colon M \to \mathbb{C}$ on a CR submanifold is a *CR function* if $vf = 0$ for all $v \in T^{0,1} M$.
We will write $f \in CR(M)$.

Bochner-Hartogs

Theorem (Bochner-Hartogs)

Let $\Omega \subset \mathbb{C}^n$, $n \geq 2$, be a bounded domain with smooth connected boundary. If $f \in C^\infty(\partial\Omega) \cap CR(\partial\Omega)$, then there exists a unique $F \in C^\infty(\overline{\Omega}) \cap \mathcal{O}(\Omega)$ such that $F|_{\partial\Omega} = f$.

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Theorem (Severi)

Let $\Omega \subset \mathbb{C}^n$, $n \geq 2$, be a bounded domain with real-analytic connected boundary. If $f \in C^\omega(\partial\Omega) \cap CR(\partial\Omega)$, then there exists a unique $F \in \mathcal{O}(\overline{\Omega})$ such that $F|_{\partial\Omega} = f$.

Local extension CR case

A smooth CR function f on a strictly pseudoconvex smooth hypersurface $M \subset \mathbb{C}^{n+1}$ extends to one side.

If Levi-form has eigenvalues of both signs, then to both sides, so to a neighbourhood.

If f and M is real-analytic, then no need to check the Levi-form, f always extends to a neighbourhood.

In coordinates $(z, w) \in \mathbb{C}^n \times \mathbb{C}$, consider the hypersurface X given by

$$\operatorname{Im} w = 0.$$

Let $w = s + it$. Parametrize X using $(z, s) \in \mathbb{C}^n \times \mathbb{R}$.

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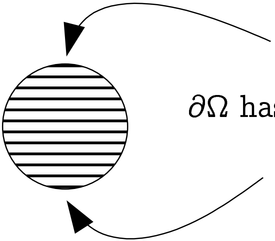
The CR vectors on X are $\frac{\partial}{\partial \bar{z}_j}$.

A function $f(z, s)$ is CR if it is holomorphic for fixed s.

$$\Omega = \{(z, s) \in \mathbb{C}^n \times \mathbb{R} : \|z\|^2 + s^2 < 1\}$$

Have $f \in C^\infty(\partial\Omega) \cap CR(\partial\Omega_{CR})$, want $F \in C^\infty(\overline{\Omega}) \cap CR(\Omega)$.

Or, have $f \in C^\omega(\partial\Omega) \cap CR(\partial\Omega_{CR})$, want $F \in \mathcal{O}(\overline{\Omega})$.



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Sphere in $\mathbb{C}^n \times \mathbb{R}$

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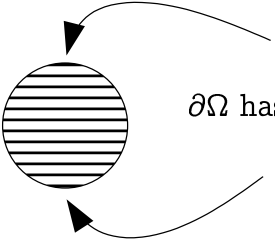

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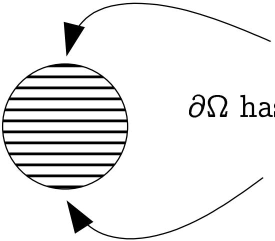

$\partial\Omega$ has CR singularities at the "poles"

Ω has a natural foliation by the copies of $\mathbb{C}^n$ (intersected with the ball).

Most trouble happens at the CR singularities.

Global counterexample for C^∞

Hartogs theorem works in the C^ω case in $\mathbb{C}^n \times \mathbb{R}$ (first proved by Severi for $n = 1$ and Brown, and then Bochner, and most recently generalized by Henkin and Michel).

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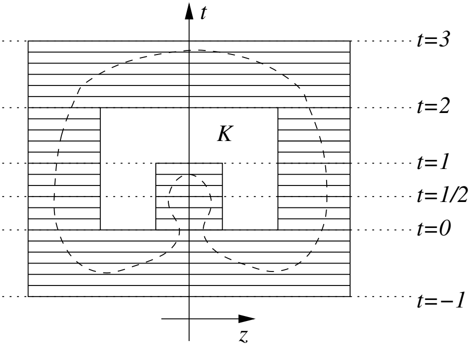
But only in C^ω, not C^∞!

Counterexample picture:



Local situation

Consider $(z, s) \in \mathbb{C}^n \times \mathbb{R}$. Define M by

$$s = \rho(z, \bar{z})$$

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Counterexample 2, $n \geq 1$:
Suppose M is given by $s = \|z\|^4$.
Define f by $\sqrt{s}$.
f is CR and C^ω on M: $\sqrt{s} = \|z\|^2$ on M.
F must be $\sqrt{s}$ which is not even C^1 at the origin.

Codimension 2 CR singularities

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Bishop ('65) first studied such nondegenerate M in $\mathbb{C}^2$:

$$w = z\bar{z} + \lambda(z^2 + \bar{z}^2) + E(z, \bar{z}).$$

$\lambda \geq 0$ is the Bishop invariant.

$0 \leq \lambda < \frac{1}{2}$: elliptic $\quad \lambda = \frac{1}{2}$: parabolic $\quad \frac{1}{2} < \lambda \leq \infty$: hyperbolic

Why elliptic? Because $\{z\bar{z} + \lambda(z^2 + \bar{z}^2) = \text{const}\}$ gives ellipses.

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Studied extensively (elliptic): Moser-Webster, Moser, Kenig-Webster, Gong, Huang-Krantz, Huang, Huang-Yin, etc...
Mostly interested in normal form.

Start with a holomorphically flat M:

$$w = \sum_{j,k} (a_{jk} z_j \bar{z}_k + b_{jk} z_j z_k + \bar{b}_{jk} \bar{z}_j \bar{z}_k) + E(z, \bar{z})$$

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Far less understood (elliptic again nicest): Huang-Yin, Burcea, Gong-L., Dolbeault-Tomassini-Zaitsev, Coffman, Slapar, etc...

Previous work on real-analytic extension

Harris ('78) provides a complete (but difficult to apply) criterion for f on an arbitrary CR singular M to be a restriction of a holomorphic function in C^ω case.

In particular, it is not always true for functions CR on M_{CR}.

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In L.-Minor-Shroff-Son-Zhang we proved that if a real-analytic CR singular manifold M is an image of a real-analytic CR map

$$f\colon N \subset \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$$

from a CR submanifold N that is a diffeomorhism onto $f(N) = M$, then there exists a real-analytic function vanishing on all CR directions (so CR on M_{CR}) that is not a restriction of a holomorphic function.

Local extension

Theorem (L.-Noell-Ravisankar)

Let $M \subset \mathbb{C}^{n+1}$, $n \geq 2$, be a holomorphically-flat real codimension 2 real-analytic submanifold with a nondegenerate CR singularity at $0 \in M$.
Suppose $f \in C^\omega(M) \cap CR(M_{CR})$. Then there exists a neighbourhood U of $0 \in \mathbb{C}^{n+1}$ and $F \in \mathcal{O}(U)$ such that $F|_{M \cap U} = f$.

The global extension

Corollary (L.-Noell-Ravisankar)

Suppose $\Omega \subset \mathbb{C}^n \times \mathbb{R}$, $n \geq 2$, is a bounded domain with connected real-analytic boundary and all CR singularities of $\partial\Omega$ are nondegenerate. Suppose $f \in C^\omega(\partial\Omega) \cap CR\big((\partial\Omega)_{CR}\big)$. Then there exists F holomorphic on a neighbourhood of $\overline{\Omega}$ in $\mathbb{C}^{n+1}$, such that $F|_{\partial\Omega} = f$.

Proof is to follow Severi's example: apply the local extension and then apply the Hartogs theorem (in this case Hartogs for $\mathbb{C}^n \times \mathbb{R}$).

Levi-flat Plateau problem

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Our global theorem has an immediate corollary, giving a singular solution for certain M. Here is the real-analytic case.

Corollary

Suppose $\Omega \subset \mathbb{C}^n \times \mathbb{R}$, $n > 1$, is a bounded domain with connected real-analytic boundary, and $M = f(\partial\Omega) \subset \mathbb{C}^{n+1}$ is the image of a C^ω map f that is CR on $(\partial\Omega)_{CR}$. Suppose all CR singularities of $\partial\Omega$ are nondegenerate.
Then there exists a holomorphic map F to $\mathbb{C}^{n+1}$ whose restriction to $\partial\Omega$ is f. $F(\overline{\Omega})$ is a Levi-flat wherever it is nonsingular.

“Proof” of local extension

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4) Extend along these families to find a holomorphic function in neighbourhood of a large attached disc.
5) Show that this holomorphic function is actually a polynomial.
6) Use the above model case to obtain a formal solution in general and show that it converges.

The smooth case

In the elliptic case we have also the extension for smooth maps. For $n > 1$ and a nondegenerate M is given by

$$w = \sum_{j,k} (a_{jk} z_j \bar{z}_k + b_{jk} z_j z_k + \bar{b}_{jk} \bar{z}_j \bar{z}_k) + E(z, \bar{z})$$

for a real valued E. Then M is *elliptic* if M intersected with $\{w = \text{const}\}$ are boundaries of domains shrinking to zero, then $[a_{jk}]$ must be definite (WLOG positive) and we can diagonalize

$$w = \sum_{j} (z_j \bar{z}_j + \lambda_j (z_j^2 + \bar{z}_j^2)) + E(z, \bar{z})$$

and $0 \leq \lambda_j < \frac{1}{2}$.

The local theorem

Theorem (L.-Noell-Ravisankar)

Suppose H and M are closed submanifolds of $U = \{(z, w) \in \mathbb{C}^n \times \mathbb{C} : \|z\| < \delta_z, |w| < \delta_w\}$ given by

$$M : w = \sum \left(|z_j|^2 + \lambda_j (z_j^2 + \bar{z}_j^2) \right) + E(z, \bar{z}),$$

$$H : \operatorname{Re} w \geq \sum \left(|z_j|^2 + \lambda_j (z_j^2 + \bar{z}_j^2) \right) + E(z, \bar{z}), \quad \operatorname{Im} w = 0.$$

E is real-valued, smooth, and $O(3)$, $0 \leq \lambda_j < \frac{1}{2}$ for all j and $\delta_z, \delta_w > 0$ "small enough." Suppose $f : M \to \mathbb{C}$ is C^∞ and either

(i) *$n > 1$ and f is a CR function on M_{CR}, or*

(ii) *$n = 1$ and for every $0 < c < \delta_w$, there exists a continuous function on $H \cap \{w = c\}$, holomorphic on $(H \setminus M) \cap \{w = c\}$ extending $f|_{M \cap \{w=c\}}$*

Then there exists an $F \in C^\infty(H) \cap CR(H \setminus M)$, and $F|_M = f$. Furthermore, F has a formal power series at 0 in z and w. If M and f are C^ω, then F is a restriction of a holomorphic function defined in a neighborhood of H in $\mathbb{C}^{n+1}$.

The global theorem

Theorem (L.-Noell-Ravisankar)

Suppose $\Omega \subset \mathbb{C}^n \times \mathbb{R}$ is a bounded domain with smooth boundary. Let $(z, s) \in \mathbb{C}^n \times \mathbb{R}$ be the coordinates. Suppose all CR singularities of $\partial\Omega$ are nondegenerate and elliptic. Suppose $f : \partial\Omega \to \mathbb{C}$ is smooth and either

(i) *$n > 1$ and f is a CR function on $(\partial\Omega)_{CR}$, or*

(ii) *$n = 1$ and for every $c \in \mathbb{R}$ where $\Omega \cap \{s = c\}$ is nonempty, there exists a continuous function on $\overline{\Omega} \cap \{s = c\}$, holomorphic on $\Omega \cap \{s = c\}$ extending $f|_{\partial\Omega \cap \{s=c\}}$.*

Then there exists $F \in C^\infty(\overline{\Omega}) \cap CR(\Omega)$ and $F|_{\partial\Omega} = f$. Furthermore, if $\partial\Omega$ and f are real-analytic, then F is a restriction of a holomorphic function defined in a neighborhood of $\overline{\Omega}$ in $\mathbb{C}^{n+1}$.

Thank you