Extensions of CR functions from CR singular submanifolds of codimension 2

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joint work with Alan Noell and Sivaguru Ravisankar

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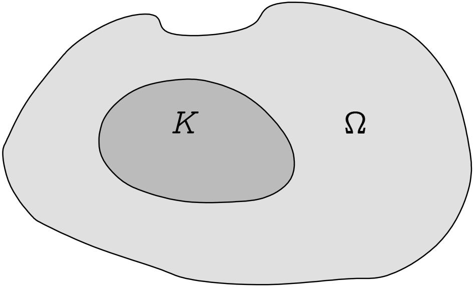
Hartogs phenomenon

Theorem (Hartogs)

Let $\Omega \subset \mathbb{C}^n$, $n \geq 2$, be a domain, and $K \subset\subset \Omega$ be compact with $\Omega \setminus K$ connected. If $f \in \mathcal{O}(\Omega \setminus K)$, then there exists a unique $F \in \mathcal{O}(\Omega)$ such that $F|_{\Omega \setminus K} = f$.



There are no hypotheses on the geometry of Ω, only a mild clearly required topological requirement on $\Omega \setminus K$.
Furthermore, K can be "as large as we want."

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A smooth function $f \colon M \to \mathbb{C}$ on a CR submanifold is a *CR function* if $vf = 0$ for all $v \in T^{0,1} M$.
We will write $f \in CR(M)$.

Bochner-Hartogs

Theorem (Bochner-Hartogs)

Let $\Omega \subset \mathbb{C}^n$, $n \geq 2$, be a bounded domain with smooth connected boundary. If $f \in C^\infty(\partial\Omega) \cap CR(\partial\Omega)$, then there exists a unique $F \in C^\infty(\overline{\Omega}) \cap \mathcal{O}(\Omega)$ such that $F|_{\partial\Omega} = f$.

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Theorem (Severi)

Let $\Omega \subset \mathbb{C}^n$, $n \geq 2$, be a bounded domain with real-analytic connected boundary. If $f \in C^\omega(\partial\Omega) \cap CR(\partial\Omega)$, then there exists a unique $F \in \mathcal{O}(\overline{\Omega})$ such that $F|_{\partial\Omega} = f$.

Local extension CR case

A smooth CR function f on a strictly pseudoconvex smooth hypersurface $M \subset \mathbb{C}^{n+1}$ extends to one side.

If Levi-form has eigenvalues of both signs, then to both sides, so to a neighbourhood.

If f and M is real-analytic, then no need to check the Levi-form, f always extends to a neighbourhood.

Use coordinates $(z, w) \in \mathbb{C}^n \times \mathbb{C}$ consider the hypersurface X given by

$$\operatorname{Im} w = 0.$$

Let $w = s + it$. Then parametrize X using $(z, s) \in \mathbb{C}^n \times \mathbb{R}$.

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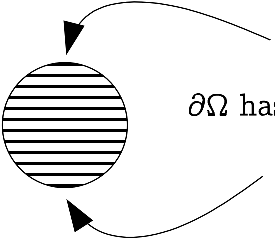
The CR vectors on X are $\frac{\partial}{\partial \bar{z}_j}$.

A function $f(z, s)$ is CR if it is holomorphic for fixed s.

Sphere in $\mathbb{C}^n \times \mathbb{R}$

$$\Omega = \{(z, s) \in \mathbb{C}^n \times \mathbb{R} : \|z\|^2 + s^2 < 1\}$$

Have $f \in C^\infty(\partial\Omega) \cap CR(\partial\Omega_{CR})$, want $F \in C^\infty(\overline{\Omega}) \cap CR(\Omega)$.



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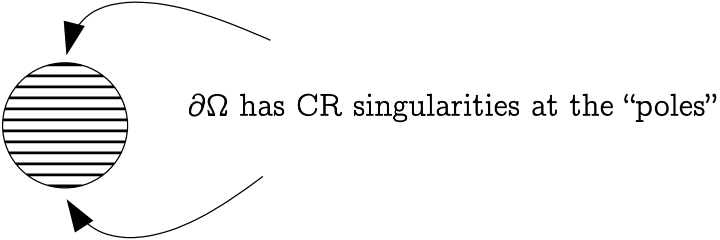

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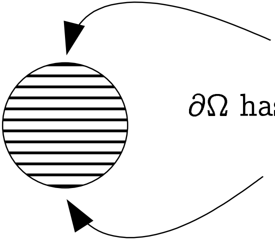

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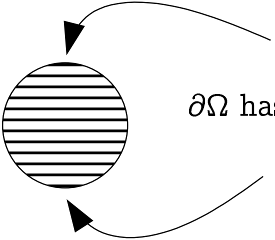

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Most trouble happens at the CR singularities.

We cannot expect F to be a restriction of a holomorphic function. Any C^∞ function depending only on s is CR.

Local situation

Consider $(z, s) \in \mathbb{C}^n \times \mathbb{R}$. Define M by

$$s = \rho(z, \bar{z})$$

and H by

$$s \geq \rho(z, \bar{z}).$$

Have $f \in C^\infty(M) \cap CR(M_{CR})$,
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Counterexample:
Suppose M is given by $s = ||z||^4$.
Define f by $\sqrt{s}$.
f is CR and C^ω on M: $\sqrt{s} = ||z||^2$ on M.
F must be $\sqrt{s}$ which is not C^∞ on H.

Codimension 2 CR singularities

A CR singularity of codim 2 in $\mathbb{C}^{n+1}$ can be put in the form

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Bishop ('65) first studied such nondegenerate M in $\mathbb{C}^2$:

$$w = z\bar{z} + \lambda(z^2 + \bar{z}^2) + E(z, \bar{z}).$$

$\lambda \geq 0$ is the Bishop invariant.

$0 \leq \lambda < \frac{1}{2}$: elliptic $\quad \lambda = \frac{1}{2}$: parabolic $\quad \frac{1}{2} < \lambda \leq \infty$: hyperbolic

Why elliptic? Because $\{z\bar{z} + \lambda(z^2 + \bar{z}^2) = \text{const}\}$ gives ellipses.

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Studied extensively (elliptic): Moser-Webster, Moser, Kenig-Webster, Gong, Huang-Krantz, Huang, Huang-Yin, etc... Mostly interested in normal form.

Start with M

$$w = \sum_{j,k} (a_{jk} z_j \bar{z}_k + b_{jk} z_j z_k + \bar{b}_{jk} \bar{z}_j \bar{z}_k) + E(z, \bar{z})$$

For flat M, we arrange $[a_{jk}]$ to be Hermitian and E real-valued.

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Nondegenerate M is *elliptic* if M intersected with $\{w = \text{const}\}$ are boundaries of domains shrinking to zero, then $[a_{jk}]$ must be definite (WLOG positive) and we can diagonalize

$$w = \sum_j (z_j \bar{z}_j + \lambda_j (z_j^2 + \bar{z}_j^2)) + E(z, \bar{z})$$

and $0 \leq \lambda_j < \frac{1}{2}$.

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Far less understood (elliptic again nicest): Huang-Yin, Burcea, Gong-L., Dolbeault-Tomassini-Zaitsev, Slapar, etc...

The local theorem

Theorem (L.-Noell-Ravisankar)

Suppose H *and* M *are closed submanifolds of* $U = \{(z,w) \in \mathbb{C}^n \times \mathbb{C} : \|z\| < \delta_z, |w| < \delta_w\}$ *given by*

$$M : w = \sum \left(|z_j|^2 + \lambda_j(z_j^2 + \bar{z}_j^2)\right) + E(z,\bar{z}),$$

$$H : \operatorname{Re} w \geq \sum \left(|z_j|^2 + \lambda_j(z_j^2 + \bar{z}_j^2)\right) + E(z,\bar{z}), \quad \operatorname{Im} w = 0.$$

E *is real-valued, smooth, and* $O(3)$, $0 \leq \lambda_j < \frac{1}{2}$ *for all* j *and* $\delta_z, \delta_w > 0$ *"small enough." Suppose* $f : M \to \mathbb{C}$ *is* C^∞ *and either*

(i) $n > 1$ *and* f *is a CR function on* M_{CR}*, or*

(ii) $n = 1$ *and for every* $0 < c < \delta_w$*, there exists a continuous function on* $H \cap \{w = c\}$*, holomorphic on* $(H \setminus M) \cap \{w = c\}$ *extending* $f|_{M \cap \{w=c\}}$

Then there exists an $F \in C^\infty(H) \cap CR(H \setminus M)$*, and* $F|_M = f$*. Furthermore,* F *has a formal power series at 0 in* z *and* w*. If* M *and* f *are* C^ω*, then* F *is a restriction of a holomorphic function defined in a neighborhood of* H *in* $\mathbb{C}^{n+1}$*.*

Current work in C^ω...

Theorem (L.-Noell-Ravisankar)

Let $M \subset \mathbb{C}^{n+1}$, $n \geq 2$, be a holomorphically-flat real codimension 2 real-analytic submanifold with a nondegenerate CR singularity at $0 \in M$.
Suppose $f \in C^\omega(M) \cap CR(M \setminus \{0\})$. Then there exists a neighbourhood U of $0 \in \mathbb{C}^{n+1}$ and $F \in \mathcal{O}(U)$ such that $F|_{M \cap U} = f$.

The global theorem

Theorem (L.-Noell-Ravisankar)

Suppose $\Omega \subset \mathbb{C}^n \times \mathbb{R}$ *is a bounded domain with smooth boundary. Let* $(z, s) \in \mathbb{C}^n \times \mathbb{R}$ *be the coordinates. Suppose all CR singularities of* $\partial\Omega$ *are nondegenerate and elliptic. Suppose* $f \colon \partial\Omega \to \mathbb{C}$ *is smooth and either*

(i) $n > 1$ *and* f *is a CR function on* $(\partial\Omega)_{CR}$*, or*

(ii) $n = 1$ *and for every* $c \in \mathbb{R}$ *where* $\Omega \cap \{s = c\}$ *is nonempty, there exists a continuous function on* $\overline{\Omega} \cap \{s = c\}$*, holomorphic on* $\Omega \cap \{s = c\}$ *extending* $f|_{\partial\Omega \cap \{s=c\}}$.

Then there exists $F \in C^\infty(\overline{\Omega}) \cap CR(\Omega)$ *and* $F|_{\partial\Omega} = f$*. Furthermore, if* $\partial\Omega$ *and* f *are real-analytic, then* F *is a restriction of a holomorphic function defined in a neighborhood of* $\overline{\Omega}$ *in* $\mathbb{C}^{n+1}$.

The global theorem for real-analytic

Theorem (L.-Noell-Ravisankar)

Suppose $\Omega \subset \mathbb{C}^n \times \mathbb{R}$, $n \geq 2$, *is a bounded domain with connected real-analytic boundary. Suppose all CR singularities of* $\partial\Omega$ *are nondegenerate. Suppose* $f \in C^\omega(\partial\Omega) \cap CR((\partial\Omega)_{CR})$.
Then there exists F *holomorphic on a neighbourhood of* $\overline{\Omega}$ *in* $\mathbb{C}^{n+1}$, *such that* $F|_{\partial\Omega} = f$.

Levi-flat Plateau problem

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Our global theorem has an immediate corollary, giving a singular solution for certain M. Here is the real-analytic case.

Corollary

Suppose $\Omega \subset \mathbb{C}^n \times \mathbb{R}$, $n > 1$, is a bounded domain with connected real-analytic boundary, and $M = f(\partial\Omega) \subset \mathbb{C}^{n+1}$ is the image of a smooth map f that is CR on $(\partial\Omega)_{CR}$. Suppose all CR singularities of $\partial\Omega$ are nondegenerate. Then there exists a holomorphic map F to $\mathbb{C}^{n+1}$ whose restriction to $\partial\Omega$ is f. $F(\overline{\Omega})$ is a Levi-flat wherever it is nonsingular.

Thank you