Chapter 7

Power series methods

7.1 Power series

Note: 1 or 1.5 lecture , §3.1 in [EP], §5.1 in [BD]

Many functions can be written in terms of a power series

$$\sum_{k=0}^{\infty} a_k (x - x_0)^k.$$

If we assume that a solution of a differential equation is written as a power series, then perhaps we can use a method reminiscent of undetermined coefficients. That is, we will try to solve for the numbers a_k. Before we can carry out this process, let us review some results and concepts about power series.

7.1.1 Definition

As we said, a *power series* is an expression such as

$$\sum_{k=0}^{\infty} a_k (x - x_0)^k = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + a_3(x - x_0)^3 + \cdots , \tag{7.1}$$

where $a_0, a_1, a_2, \ldots, a_k, \ldots$ and x_0 are constants. Let

$$S_n(x) = \sum_{k=0}^{n} a_k (x - x_0)^k = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + a_3(x - x_0)^3 + \cdots + a_n(x - x_0)^n,$$

denote the so-called *partial sum*. If for some x, the limit

$$\lim_{n\to\infty} S_n(x) = \lim_{n\to\infty} \sum_{k=0}^{n} a_k (x - x_0)^k$$

exists, then we say that the series (7.1) *converges* at x. Note that for $x = x_0$, the series always converges to a_0. When (7.1) converges at any other point $x \neq x_0$, we say that (7.1) is a *convergent power series*. In this case we write

$$\sum_{k=0}^{\infty} a_k (x - x_0)^k = \lim_{n\to\infty} \sum_{k=0}^{n} a_k (x - x_0)^k.$$

If the series does not converge for any point $x \neq x_0$, we say that the series is *divergent*.

Example 7.1.1: The series

$$\sum_{k=0}^{\infty} \frac{1}{k!} x^k = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots$$

is convergent for any x. Recall that $k! = 1 \cdot 2 \cdot 3 \cdots k$ is the factorial. By convention we define $0! = 1$. In fact, you may recall that this series converges to e^x.

We say that (7.1) *converges absolutely* at x whenever the limit

$$\lim_{n\to\infty} \sum_{k=0}^{n} |a_k|\,|x - x_0|^k$$

exists. That is, if the series $\sum_{k=0}^{\infty} |a_k|\,|x - x_0|^k$ is convergent. Note that if (7.1) converges absolutely at x, then it converges at x. However, the opposite is not true.

Example 7.1.2: The series

$$\sum_{k=1}^{\infty} \frac{1}{k} x^k$$

converges absolutely at any $x \in (-1, 1)$. It converges at $x = -1$, as $\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$ converges (conditionally) by the alternating series test. But the power series does not converge absolutely at $x = -1$, because $\sum_{k=1}^{\infty} \frac{1}{k}$ does not converge. The series diverges at $x = 1$.

7.1.2 Radius of convergence

If a series converges absolutely at some x_1, then for all x such that $|x - x_0| \leq |x_0 - x_1|$ we have that $|a_k(x - x_0)^k| \leq |a_k(x_1 - x_0)^k|$ for all k. As the numbers $|a_k(x_1 - x_0)^k|$ sum to some finite limit, summing smaller positive numbers $|a_k(x - x_0)^k|$ must also have a finite limit. Therefore, the series must converge absolutely at x. We have the following result.

Theorem 7.1.1. *For a power series* (7.1)*, there exists a number ρ (we allow $\rho = \infty$) called the* radius of convergence *such that the series converges absolutely on the interval* $(x_0 - \rho, x_0 + \rho)$ *and diverges for $x < x_0 - \rho$ and $x > x_0 + \rho$. We write $\rho = \infty$ if the series converges for all x.*

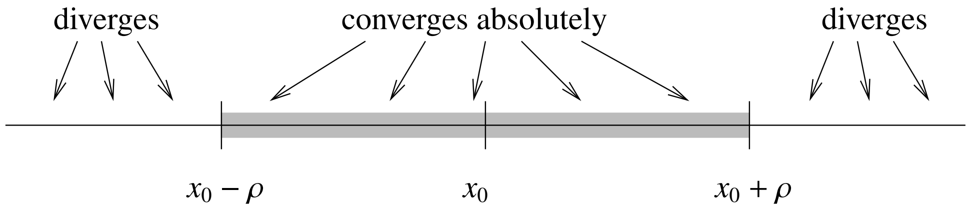

Figure 7.1: Convergence of a power series.

See Figure 7.1. In Example 7.1.1 the radius of convergence is $\rho = \infty$ as the series converges everywhere. In Example 7.1.2 the radius of convergence is $\rho = 1$. We note that $\rho = 0$ is another way of saying that the series is divergent.

A useful test for convergence of a series is the *ratio test*. Suppose that

$$\sum_{k=0}^{\infty} c_k$$

is a series such that the limit

$$L = \lim_{n\to\infty} \left| \frac{c_{k+1}}{c_k} \right|$$

exists. Then the series converges absolutely if $L < 1$ and diverges if $L > 1$.

Let us apply this test to the series (7.1). That is we let $c_k = a_k(x - x_0)^k$ in the test. We let

$$L = \lim_{n\to\infty} \left| \frac{c_{k+1}}{c_k} \right| = \lim_{n\to\infty} \left| \frac{a_{k+1}(x-x_0)^{k+1}}{a_k(x-x_0)^k} \right| = \lim_{n\to\infty} \left| \frac{a_{k+1}}{a_k} \right| |x - x_0|.$$

Define A by

$$A = \lim_{n\to\infty} \left| \frac{a_{k+1}}{a_k} \right|.$$

Then if $1 > L = A|x - x_0|$ the series (7.1) converges absolutely. If $A = 0$, then the series always converges. If $A > 0$, then the series converges absolutely if $|x - x_0| < 1/A$, and diverges if $|x - x_0| > 1/A$. That is, the radius of convergence is $1/A$. Let us summarize.

Theorem 7.1.2. *Let*

$$\sum_{k=0}^{\infty} a_k(x - x_0)^k$$

be a power series such that

$$A = \lim_{n\to\infty} \left| \frac{a_{k+1}}{a_k} \right|$$

exists. If $A = 0$, then the radius of convergence of the series is ∞. Otherwise the radius of convergence is $1/A$.

Example 7.1.3: Suppose we have the series

$$\sum_{k=0}^{\infty} 2^{-k}(x-1)^k.$$

First we compute,

$$A = \lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \left|\frac{2^{-k-1}}{2^{-k}}\right| = 2^{-1} = 1/2.$$

Therefore the radius of convergence is 2, and the series converges absolutely on the interval $(-1, 3)$.

The ratio test does not always apply. That is the limit of $\left|\frac{a_{k+1}}{a_k}\right|$ might not exist. There exist more sophisticated ways of finding the radius of convergence, but those would be beyond the scope of this chapter.

7.1.3 Analytic functions

Functions represented by series are called *analytic functions*. Not every function is analytic, although the majority of the functions you have seen in calculus are.

An analytic function $f(x)$ is equal to its *Taylor series** near a point x_0. That is, for x near x_0 we have

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k, \tag{7.2}$$

where $f^{(k)}(x_0)$ denotes the k^{th} derivative of $f(x)$ at the point x_0.

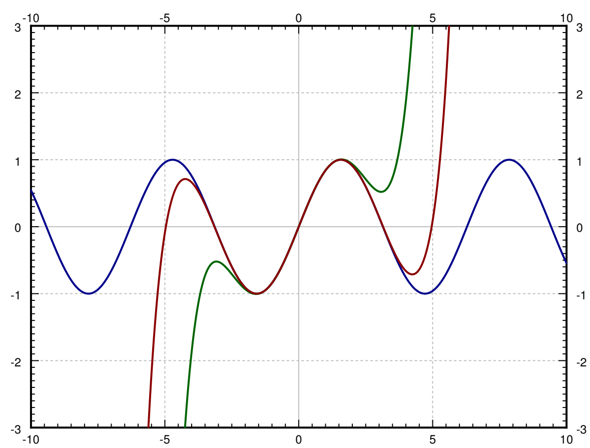

Figure 7.2: The sine function and its Taylor approximations around $x_0 = 0$ of 5^{th} and 9^{th} degree.

*Named after the English mathematician Sir Brook Taylor (1685 – 1731).

For example, sine is an analytic function and its Taylor series around $x_0 = 0$ is given by

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}.$$

In Figure 7.2 on the facing page we plot $\sin(x)$ and the truncations of the series up to degree 5 and 9. You can see that the approximation is very good for x near 0, but gets worse for larger x. This is what will happen in general. To get good approximation far away from x_0 you will need to take more and more terms of the Taylor series.

7.1.4 Manipulating power series

One of the main properties of power series that we will use is that we can differentiate them term by term. That is Suppose that $\sum a_k(x-x_0)^k$ is a convergent power series. Then for x in the radius of convergence we have

$$\frac{d}{dx}\left[\sum_{k=0}^{\infty} a_k(x-x_0)^k\right] = \sum_{k=1}^{\infty} k a_k (x-x_0)^{k-1}.$$

Notice that the term corresponding to $k = 0$ disappeared as it was constant. The radius of convergence of the differentiated series is the same as that of the original.

Example 7.1.4: Let us show that the exponential $y = e^x$ solves $y' = y$. First write

$$y = e^x = \sum_{k=0}^{\infty} \frac{1}{k!} x^k.$$

Now differentiate

$$y' = \sum_{k=1}^{\infty} k \frac{1}{k!} x^{k-1} = \sum_{k=1}^{\infty} \frac{1}{(k-1)!} x^{k-1}.$$

For convenience we *reindex* the series by simply replacing k with $k+1$. The series does not change, what changes is simply how we write it. After reindexing the series starts at $k = 0$ again.

$$\sum_{k=1}^{\infty} \frac{1}{(k-1)!} x^{k-1} = \sum_{k=0}^{\infty} \frac{1}{k!} x^k.$$

That was precisely the power series for e^x that we started with, so we showed that $\frac{d}{dx} e^x = e^x$.

Convergent power series can be added and multiplied together, and multiplied by constants using the following rules. Firstly, we can add series by adding term by term,

$$\left(\sum_{k=0}^{\infty} a_k(x-x_0)^k\right) + \left(\sum_{k=0}^{\infty} b_k(x-x_0)^k\right) = \sum_{k=0}^{\infty} (a_k+b_k)(x-x_0)^k.$$

We can multiply by constants,

$$\alpha\left(\sum_{k=0}^{\infty} a_k(x-x_0)^k\right) = \sum_{k=0}^{\infty} \alpha a_k(x-x_0)^k.$$

We can also multiply series together,

$$\left(\sum_{k=0}^{\infty} a_k(x-x_0)^k\right)\left(\sum_{k=0}^{\infty} b_k(x-x_0)^k\right) = \sum_{k=0}^{\infty} c_k(x-x_0)^k,$$

where $c_k = a_0b_k + a_1b_{k-1} + \cdots + a_kb_0$. The radius of convergence of the sum or the product is at least the minimum of the radii of convergence of the two series involved.

7.1.5 Power series for rational functions

Note that a series for a function only defines the function on an interval. For example, for $-1 < x < 1$ we have

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \cdots$$

This series is called the *geometric series*. The ratio test tells us that the radius of convergence is 1. The series diverges for $x \leq -1$ and $x \geq 1$, even though $\frac{1}{1-x}$ is defined for all $x \neq 1$.

Notice that polynomials are simply finite power series. That is a polynomial is a power series where the a_k beyond a certain point are all zero. We can always expand a polynomial as a power series about any point x_0 by writing the polynomial as a polynomial of $(x - x_0)$. For example, let us write $2x^2 - 3x + 4$ as a power series around $x_0 = 1$:

$$2x^2 - 3x + 4 = 3 + (x-1) + 2(x-1)^2.$$

In other words $a_0 = 3$, $a_1 = 1$, $a_2 = 2$, and all other $a_k = 0$. To do this, we know that $a_k = 0$ for all $k \geq 3$. So we write $a_0 + a_1(x-1) + a_2(x-1)^2$, we expand, and we solve for a_0, a_1, and a_2. We could have also differentiated at $x = 1$ and used the Taylor series formula (7.2).

We can use the geometric series together with rules for addition and multiplication of power series to expand rational functions around a point, as long as the denominator is not zero at x_0. Note that as for polynomials, we could equivalently use the Taylor series expansion (7.2).

Example 7.1.5: Expand $\frac{x}{1+2x+x^2}$ as a power series around the origin and find the radius of convergence.

First, write $1 + 2x + x^2 = (1 + x)^2 = (1 - (-x))^2$. Now we compute

$$\begin{aligned}\frac{x}{1+2x+x^2} &= x\left(\frac{1}{1-(-x)}\right)^2 \\ &= x\left(\sum_{k=0}^{\infty}(-1)^k x^k\right) \\ &= x\left(\sum_{k=0}^{\infty} c_k x^k\right) \\ &= \sum_{k=0}^{\infty} c_k x^{k+1},\end{aligned}$$

where using the formula for product of product of series we obtain, $c_0 = 1$, $c_1 = -1 - 1 = -2$, $c_2 = 1 + 1 + 1 = 3$, etc.... Therefore

$$\frac{x}{1+2x+x^2} = \sum_{k=1}^{\infty}(-1)^{k+1} k x^k = x - 2x^2 + 3x^3 - 4x^4 + \cdots$$

The radius of convergence is at least 1. We use the ratio test

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\frac{(-1)^{k+2}(k+1)}{(-1)^{k+1}k}\right| = \lim_{k\to\infty}\frac{k+1}{k} = 1.$$

So the radius of convergence is actually equal to 1.

7.1.6 Exercises

***Exercise* 7.1.1:** *Is the power series* $\sum_{k=0}^{\infty} e^k x^k$ *convergent? If so, what is the radius of convergence?*

***Exercise* 7.1.2:** *Is the power series* $\sum_{k=0}^{\infty} k x^k$ *convergent? If so, what is the radius of convergence?*

***Exercise* 7.1.3:** *Is the power series* $\sum_{k=0}^{\infty} k! x^k$ *convergent? If so, what is the radius of convergence?*

***Exercise* 7.1.4:** *Is the power series* $\sum_{k=0}^{\infty} \frac{1}{(2k)!}(x-10)^k$ *convergent? If so, what is the radius of convergence?*

***Exercise* 7.1.5:** *Determine the Taylor series for* $\sin x$ *around the point* $x_0 = \pi$.

Exercise* 7.1.6*: *Determine the Taylor series for* $\ln x$ *around the point* $x_0 = 1$*, and find the radius of convergence.*

Exercise* 7.1.7*: *Determine the Taylor series and its radius of convergence of* $\dfrac{1}{1+x}$ *around* $x_0 = 0$*.*

Exercise* 7.1.8*: *Determine the Taylor series and its radius of convergence of* $\dfrac{x}{4-x^2}$ *around* $x_0 = 0$*. Hint: you will not be able to use the ratio test.*

Exercise* 7.1.9*: *Expand* $x^5 + 5x + 1$ *as a power series around* $x_0 = 5$*.*

Exercise* 7.1.10*: *Suppose that the ratio test applies to a series* $\displaystyle\sum_{k=0}^{\infty} a_k x^k$*. Show, using the ratio test, that the radius of convergence of the differentiated series is the same as that of the original series.*

7.2 Series solutions of linear second order ODEs

Note: 1 or 1.5 lecture , §3.1 in [EP], §5.2 and §5.3 in [BD]

Suppose we have a linear second order homogeneous ODE of the form

$$p(x)y'' + q(x)y' + r(x)y = 0. \tag{7.3}$$

Suppose that $p(x)$, $q(x)$, and $r(x)$ are polynomials. We will try a solution of the form

$$y = \sum_{k=0}^{\infty} a_k (x - x_0)^k \tag{7.4}$$

and solve for the a_k to try to obtain a solution defined in some interval around x_0.

The point x_0 is called an *ordinary point* if $p(x_0) \neq 0$. That is, the functions

$$\frac{q(x)}{p(x)} \qquad \text{and} \qquad \frac{r(x)}{p(x)} \tag{7.5}$$

are defined for x near x_0. If $p(x_0) = 0$, then we say x_0 is a *singular point*. Handling singular points is harder than ordinary points and so we will focus only on ordinary points.

Example 7.2.1: Let us start with a very simple example

$$y'' - y = 0.$$

Let us try a power series solution near $x_0 = 0$, which is an ordinary point. Every point is an ordinary point in fact, as the equation is constant coefficient. We already know we should obtain exponentials or the hyperbolic sine and cosine, but let us pretend we do not know this.

We try

$$y = \sum_{k=0}^{\infty} a_k x^k.$$

If we differentiate, the $k = 0$ term is a constant and hence disappears. We therefore get

$$y' = \sum_{k=1}^{\infty} k a_k x^{k-1}.$$

We differentiate yet again to obtain (now the $k = 1$ term disappears)

$$y'' = \sum_{k=2}^{\infty} k(k-1) a_k x^{k-2}.$$

We reindex the series (replace k with $k+2$) to obtain

$$y'' = \sum_{k=0}^{\infty} (k+2)\,(k+1)\,a_{k+2}x^k.$$

Now we plug y and y'' into the differential equation

$$\begin{aligned} 0 = y'' - y &= \left(\sum_{k=0}^{\infty} (k+2)\,(k+1)\,a_{k+2}x^k \right) - \left(\sum_{k=0}^{\infty} a_k x^k \right) \\ &= \sum_{k=0}^{\infty} \Big((k+2)\,(k+1)\,a_{k+2}x^k - a_k x^k \Big) \\ &= \sum_{k=0}^{\infty} \big((k+2)\,(k+1)\,a_{k+2} - a_k \big)\, x^k. \end{aligned}$$

As $y'' - y$ is supposed to be equal to 0, we know that the coefficients of the resulting series must be equal to 0. Therefore,

$$(k+2)\,(k+1)\,a_{k+2} - a_k = 0, \qquad \text{or} \qquad a_{k+2} = \frac{a_k}{(k+2)(k+1)}.$$

The above equation is called a *recurrence relation* for the coefficients of the power series. It did not matter what a_0 or a_1 was, they can be arbitrary. But once we pick a_0 and a_1, then all other coefficients are determined by the recurrence relation.

So let us see what the coefficients must be. First, a_0 and a_1 are arbitrary

$$a_2 = \frac{a_0}{2}, \quad a_3 = \frac{a_1}{(3)(2)}, \quad a_4 = \frac{a_2}{(4)(3)} = \frac{a_0}{(4)(3)(2)}, \quad a_5 = \frac{a_3}{(5)(4)} = \frac{a_1}{(4)(3)(2)}, \quad \ldots$$

So we note that for even k, that is $k = 2n$ we get

$$a_k = a_{2n} = \frac{a_0}{(2n)!}, \tag{7.6}$$

and for odd k, that is $k = 2n+1$ we have

$$a_k = a_{2n+1} = \frac{a_1}{(2n+1)!}. \tag{7.7}$$

Let us write down the series

$$y = \sum_{k=0}^{\infty} a_k x^k = \sum_{n=0}^{\infty} \left(\frac{a_0}{(2n)!}\, x^{2n} + \frac{a_1}{(2n+1)!}\, x^{2n+1} \right) = a_0 \sum_{n=0}^{\infty} \frac{1}{(2n)!}\, x^{2n} + a_1 \sum_{n=0}^{\infty} \frac{1}{(2n+1)!}\, x^{2n+1}.$$

Now we recognize the two series as the hyperbolic sine and cosine. Therefore,

$$y = a_0 \cosh x + a_1 \sinh x.$$

Of course, in general we will not be able to recognize the series that appears, since usually there will not be any elementary function that matches it. In that case we will be content with the series.

Example 7.2.2: Let us do a more complex example. Suppose we wish to solve *Airy's equation*[†], that is

$$y'' - xy = 0,$$

near the point $x_0 = 0$. Note that $x_0 = 0$ is an ordinary point.

We try

$$y = \sum_{k=0}^{\infty} a_k x^k.$$

We differentiate twice (as above) to obtain

$$y'' = \sum_{k=2}^{\infty} k\,(k-1)\,a_k x^{k-2}.$$

Now we plug into the equation

$$\begin{aligned} 0 = y'' - xy &= \left(\sum_{k=2}^{\infty} k\,(k-1)\,a_k x^{k-2}\right) - x\left(\sum_{k=0}^{\infty} a_k x^k\right) \\ &= \left(\sum_{k=2}^{\infty} k\,(k-1)\,a_k x^{k-2}\right) - \left(\sum_{k=0}^{\infty} a_k x^{k+1}\right). \end{aligned}$$

Now we reindex to make things easier to sum

$$\begin{aligned} 0 = y'' - xy &= \left(2a_2 + \sum_{k=1}^{\infty} (k+2)\,(k+1)\,a_{k+2} x^k\right) - \left(\sum_{k=1}^{\infty} a_{k-1} x^k\right). \\ &= 2a_2 + \sum_{k=1}^{\infty} \Big((k+2)\,(k+1)\,a_{k+2} - a_{k-1}\Big)\,x^k. \end{aligned}$$

Again $y'' - xy$ is supposed to be 0 so first we notice that $a_2 = 0$ and also

$$(k+2)\,(k+1)\,a_{k+2} - a_{k-1} = 0, \qquad \text{or} \qquad a_{k+2} = \frac{a_{k-1}}{(k+2)(k+1)}.$$

Now we jump in steps of three. First we notice that since $a_2 = 0$ we must have that, $a_5 = 0$, $a_8 = 0$, $a_{11} = 0$, etc.... In general $a_{2+3n} = 0$.

The constants a_0 and a_1 are arbitrary and we obtain

$$a_3 = \frac{a_0}{(3)(2)}, \quad a_4 = \frac{a_1}{(4)(3)}, \quad a_6 = \frac{a_3}{(6)(5)} = \frac{a_0}{(6)(5)(3)(2)}, \quad a_7 = \frac{a_4}{(7)(6)} = \frac{a_1}{(7)(6)(4)(3)}, \quad \ldots$$

[†]Named after the English mathematician Sir George Biddell Airy (1801 – 1892).

For a_k where k is a multiple of 3, that is $k = 3n$ we notice that

$$a_{3n} = \frac{a_0}{(2)(3)(5)(6)\cdots(3n-1)(3n)}.$$

For a_k where $k = 3n+1$, we notice

$$a_{3n+1} = \frac{a_1}{(3)(4)(6)(7)\cdots(3n)(3n+1)}.$$

In other words, if we write down the series for y we notice that it has two parts

$$\begin{aligned} y &= \left(a_0 + \frac{a_0}{6}x^3 + \frac{a_0}{180}x^6 + \cdots + \frac{a_0}{(2)(3)(5)(6)\cdots(3n-1)(3n)}x^{3n} + \cdots\right) \\ &\quad + \left(a_1 x + \frac{a_1}{12}x^4 + \frac{a_1}{504}x^7 + \cdots + \frac{a_1}{(3)(4)(6)(7)\cdots(3n)(3n+1)}x^{3n+1} + \cdots\right) \\ &= a_0\left(1 + \frac{1}{6}x^3 + \frac{1}{180}x^6 + \cdots + \frac{1}{(2)(3)(5)(6)\cdots(3n-1)(3n)}x^{3n} + \cdots\right) \\ &\quad + a_1\left(x + \frac{1}{12}x^4 + \frac{1}{504}x^7 + \cdots + \frac{1}{(3)(4)(6)(7)\cdots(3n)(3n+1)}x^{3n+1} + \cdots\right). \end{aligned}$$

We define

$$\begin{aligned} y_1(x) &= 1 + \frac{1}{6}x^3 + \frac{1}{180}x^6 + \cdots + \frac{1}{(2)(3)(5)(6)\cdots(3n-1)(3n)}x^{3n} + \cdots, \\ y_2(x) &= x + \frac{1}{12}x^4 + \frac{1}{504}x^7 + \cdots + \frac{1}{(3)(4)(6)(7)\cdots(3n)(3n+1)}x^{3n+1} + \cdots, \end{aligned}$$

and write the general solution to the equation as $y(x) = a_0 y_1(x) + a_1 y_2(x)$. Notice from the power series that $y_1(0) = 1$ and $y_2(0) = 0$. Also, $y_1'(0) = 0$ and $y_2'(0) = 1$. If we obtained a solution that satisfies the initial conditions $y(0) = a_0$ and $y'(0) = a_1$.

The functions y_1 and y_2 cannot be written in terms of the elementary functions that you know. See Figure 7.3 for the plot of the solutions y_1 and y_2. These functions have very intersting properties. For example, they are oscillatory for negative x and for positive x they grow without bound.

Sometimes at least on of the solutions turns out to be a polynomial.

Example 7.2.3: Let us find a solution to the so-called *Hermite's equation of order n*[‡] is the equation

$$y'' - 2xy' + 2ny = 0.$$

Let us find a solution around the point $x_0 = 0$. We try

$$y = \sum_{k=0}^{\infty} a_k x^k.$$

[‡]Named after the French mathematician Charles Hermite (1822–1901).

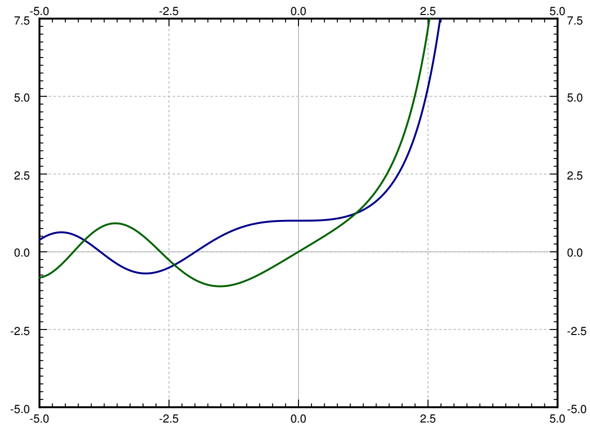

Figure 7.3: The two solutions y_1 and y_2 to Airy's equation.

We differentiate (as above) to obtain

$$y' = \sum_{k=1}^{\infty} k a_k x^{k-1},$$
$$y'' = \sum_{k=2}^{\infty} k\,(k-1)\,a_k x^{k-2}.$$

Now we plug into the equation

$$\begin{aligned}
0 = y'' - 2xy' + 2ny &= \left(\sum_{k=2}^{\infty} k\,(k-1)\,a_k x^{k-2}\right) - 2x\left(\sum_{k=1}^{\infty} k a_k x^{k-1}\right) + 2n\left(\sum_{k=0}^{\infty} a_k x^k\right) \\
&= \left(\sum_{k=2}^{\infty} k\,(k-1)\,a_k x^{k-2}\right) - \left(\sum_{k=1}^{\infty} 2k a_k x^k\right) + \left(\sum_{k=0}^{\infty} 2n a_k x^k\right) \\
&= \left(2a_2 + \sum_{k=1}^{\infty} (k+2)\,(k+1)\,a_{k+2} x^k\right) - \left(\sum_{k=1}^{\infty} 2k a_k x^k\right) + \left(2na_0 + \sum_{k=1}^{\infty} 2n a_k x^k\right) \\
&= 2a_2 + 2na_0 + \sum_{k=1}^{\infty} \big((k+2)\,(k+1)\,a_{k+2} - 2ka_k + 2na_k\big) x^k.
\end{aligned}$$

As $y'' - 2xy' + 2ny = 0$ we have

$$(k+2)\,(k+1)\,a_{k+2} + (-2k+2n)a_k = 0, \qquad \text{or} \qquad a_{k+2} = \frac{(2k-2n)}{(k+2)(k+1)} a_k.$$

This recurrence relation actually includes $a_2 = -na_0$ (which comes about from $2a_2 + 2na_0 = 0$).

Again a_0 and a_1 are arbitrary.

$$\begin{aligned} a_2 &= \frac{2n}{(2)(1)}a_0, \qquad a_3 = \frac{2(1-n)}{(3)(2)}a_1, \\ a_4 &= \frac{2(2-n)}{(4)(3)}a_2 = \frac{2^2(2-n)(-n)}{(4)(3)(2)(1)}a_0, \\ a_5 &= \frac{2(3-n)}{(5)(4)}a_3 = \frac{2^2(3-n)(1-n)}{(5)(4)(3)(2)}a_1, \quad \ldots \end{aligned}$$

Let us separate the even and odd coefficients. We find that

$$\begin{aligned} a_{2m} &= \frac{2^m(-n)(2-n)\cdots(2m-2-n)}{(2m)!}, \\ a_{2m+1} &= \frac{2^m(1-n)(3-n)\cdots(2m-1-n)}{(2m+1)!}. \end{aligned}$$

Let us write down the two series, one with the even powers and one with the odd.

$$\begin{aligned} y_1(x) &= 1 + \frac{2(-n)}{2!}x^2 + \frac{2^2(-n)(2-n)}{4!}x^4 + \frac{2^3(-n)(2-n)(4-n)}{6!}x^6 + \cdots, \\ y_2(x) &= x + \frac{2(1-n)}{3!}x^3 + \frac{2^2(1-n)(3-n)}{5!}x^5 + \frac{2^3(1-n)(3-n)(5-n)}{7!}x^7 + \cdots. \end{aligned}$$

We then write

$$y(x) = a_0 y_1(x) + a_1 y_2(x). \tag{7.8}$$

We also notice that if n is a positive even integer, then $y_1(x)$ is a polynomial as all the coefficients in the series beyond a certain degree are zero. If n is a positive odd integer, then $y_2(x)$ is a polynomial. For example if $n = 4$, then

$$y_1(x) = 1 + \frac{2(-4)}{2!}x^2 + \frac{2^2(-4)(2-4)}{4!}x^4 = 1 - 4x^2 + \frac{4}{3}x^4. \tag{7.9}$$

7.2.1 Exercises

In the following exercises, when asked to solve an equation using power series methods, you should find the first few terms of the series, and if possible find a general formula for the k^{th} coefficient.

***Exercise* 7.2.1:** *Use power series methods to solve $y'' + y = 0$ at the point $x_0 = 1$.*

***Exercise* 7.2.2:** *Use power series methods to solve $y'' + 4xy = 0$ at the point $x_0 = 0$.*

***Exercise* 7.2.3:** *Use power series methods to solve $y'' - xy = 0$ at the point $x_0 = 1$.*

***Exercise* 7.2.4:** *Use power series methods to solve $y'' + x^2y = 0$ at the point $x_0 = 0$.*

Exercise* 7.2.5*: *The methods work for other orders than second order. Try the methods of this section to solve the first order system* $y' - xy = 0$ *at the point* $x_0 = 0$.

Exercise* 7.2.6** (Chebyshev's equation of order p):*** *a) Solve* $(1 - x^2)y'' - xy' + p^2y = 0$ *using power series methods at* $x_0 = 0$. *b) For what* p *is there a polynomial solution.*

Exercise* 7.2.7*: *Find a polynomial solution to* $(x^2 + 1)y'' - 2xy' + 2y = 0$ *using power series methods.*

Exercise* 7.2.8*: *a) Use power series methods to solve* $(1 - x)y'' + y = 0$ *at the point* $x_0 = 0$. *b) Use the solution to part a) to find a solution for* $xy'' + y = 0$ *around the point* $x_0 = 1$.