Regularity of Hölder continuous solutions of the supercritical quasi-geostrophic equation

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Abstract

We present a regularity result for weak solutions of the 2D quasi-geostrophic equation with supercritical \((\alpha < \frac{1}{2})\) dissipation \((-\Delta)^{\alpha}\). If a Leray–Hopf weak solution is Hölder continuous \(\theta \in C^\delta (\mathbb{R}^2)\) with \(\delta > 1 - 2\alpha\) on the time interval \([t_0, t]\), then it is actually a classical solution on \((t_0, t]\).

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1. Introduction

We discuss the surface 2D quasi-geostrophic (QG) equation

\[
\partial_t \theta + u \cdot \nabla \theta + \kappa (-\Delta)^\alpha \theta = 0, \quad x \in \mathbb{R}^2, \quad t > 0,
\]

where \(\alpha > 0\) and \(\kappa \geq 0\) are parameters, and the 2D velocity field \(u = (u_1, u_2)\) is determined from \(\theta\) by the stream function \(\psi\) via the auxiliary relations

\[
(u_1, u_2) = (-\partial x_2 \psi, \partial x_1 \psi), \quad (-\Delta)^{1/2} \psi = -\theta.
\]

Using the notation \(A \equiv (-\Delta)^{1/2}\) and \(\nabla^\perp \equiv (\partial x_2, -\partial x_1)\), the relations in (1.2) can be combined into

\[
u = \nabla^\perp A^{-1} \theta = (-R_2 \theta, R_1 \theta),
\]

where \(R_1\) and \(R_2\) are the usual Riesz transforms in \(\mathbb{R}^2\). The 2D QG equation with \(\kappa > 0\) and \(\alpha = \frac{1}{2}\) arises in geophysical studies of strongly rotating fluids (see [5,16] and references therein) while the inviscid QG equation ((1.1) with \(\kappa = 0\)) was derived to model frontogenesis in meteorology, a formation of sharp fronts between masses of hot and cold air (see [7,10,16]).

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The problem at the center of the mathematical theory concerning the 2D QG equation is whether or not it has a global in time smooth solution for any prescribed smooth initial data. In the subcritical case \( \alpha > \frac{1}{2} \), the dissipative QG equation has been shown to possess a unique global smooth solution for every sufficiently smooth initial data (see [8,17]). In contrast, when \( \alpha \leq \frac{1}{2} \), the issue of global existence and uniqueness is more difficult and has still unanswered aspects. Recently this problem has attracted a significant amount of research ([1–6,9,11–15,18–24]). In Constantin, Córdoba and Wu [6], we proved in the critical case (\( \alpha = \frac{1}{2} \)) the global existence and uniqueness of classical solutions corresponding to any initial data with \( L^\infty \)-norm comparable to or less than the diffusion coefficient \( \kappa \). In a recently posted preprint in arXiv [14], Kiselev, Nazarov and Volberg proved that smooth global solutions exist for any \( C^\infty \) periodic initial data, by removing the \( L^\infty \)-smallness assumption on the initial data of [6]. Caffarelli and Vasseur (arXiv reference [1]) establish the global regularity of the Leray–Hopf type weak solutions (in \( L^\infty((0, \infty) ; L^2) \cap L^2((0, \infty) ; \dot{H}^{1/2}) \)) of the critical 2D QG equation with \( \alpha = \frac{1}{2} \) in general \( \mathbb{R}^n \).

In this paper we present a regularity result of weak solutions of the dissipative QG equation with \( \alpha < \frac{1}{2} \) (the supercritical case). The result asserts that if a Leray–Hopf weak solution \( \theta \) of (1.1) is in the Hölder class \( C^\delta \) with \( \delta > 1 - 2\alpha \) on the time interval \([t_0, t] \), then it is actually a classical solution on \([t_0, t] \). The proof involves representing the functions in Hölder space in terms of the Littlewood–Paley decomposition and using Besov space techniques. When \( \theta \) is in \( C^\delta \), it also belongs to the Besov space \( \dot{B}^{\delta/(1-2\alpha)}_{p,\infty} \) for any \( p \geq 2 \). By taking \( p \) sufficiently large, we have \( \theta \in C^{\delta_1} \cap \dot{B}^{\delta_2}_{p,\infty} \) for \( \delta_1 > 1 - 2\alpha \). The idea is to show that \( \theta \in \dot{C}^{\delta_2} \cap \dot{B}^{\delta_2}_{p,\infty} \) with \( \delta_2 > \delta_1 \). Through iteration, we establish that \( \theta \in \dot{C}^{\gamma} \) with \( \gamma > 1 \). Then \( \theta \) becomes a classical solution.

The results of this paper can be easily extended to a more general form of the quasi-geostrophic equation in which \( x \in \mathbb{R}^n \) and \( u \) is a divergence-free vector field determined by \( \theta \) through a singular integral operator.

The rest of this paper is divided into two sections. Section 2 provides the definition of Besov spaces and necessary tools. Section 3 states and proves the main result.

2. Besov spaces and related tools

This section provides the definition of Besov spaces and several related tools. We start with a some notation. Denote by \( \mathcal{S}(\mathbb{R}^n) \) the usual Schwarz class and \( \mathcal{S}'(\mathbb{R}^n) \) the space of tempered distributions. \( \hat{f} \) denotes the Fourier transform of \( f \), namely

\[
\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-i\mathbf{x} \cdot \xi} f(x) \, dx.
\]

The fractional Laplacian \((-\Delta)^\alpha\) can be defined through the Fourier transform

\[
(-\Delta)^\alpha f = |\xi|^{2\alpha} \hat{f}(\xi).
\]

Let

\[
\mathcal{S}_0 = \left\{ \phi \in \mathcal{S}, \int_{\mathbb{R}^n} \phi(x) x^\gamma \, dx = 0, \ |\gamma| = 0, 1, 2, \ldots \right\}.
\]

Its dual \( \mathcal{S}'_0 \) is given by

\[
\mathcal{S}'_0 = \mathcal{S}'/\mathcal{S}'_0 = \mathcal{S}'/\mathcal{P},
\]

where \( \mathcal{P} \) is the space of polynomials. In other words, two distributions in \( \mathcal{S}' \) are identified as the same in \( \mathcal{S}'_0 \) if their difference is a polynomial.

It is a classical result that there exists a dyadic decomposition of \( \mathbb{R}^n \), namely a sequence \( \{\Phi_j\} \in \mathcal{S}(\mathbb{R}^n) \) such that

\[
\text{supp } \Phi_j \subset A_j, \quad \Phi_j(\xi) = \Phi_0(2^{-j} \xi) \quad \text{or} \quad \Phi_j(x) = 2^{jn} \Phi_0(2^j x)
\]

and

\[
\sum_{k=\infty}^{\infty} \Phi_k(\xi) = \begin{cases} 
1 & \text{if } \xi \in \mathbb{R}^n \setminus \{0\}, \\
0 & \text{if } \xi = 0.
\end{cases}
\]
where
\[ A_j = \{ \xi \in \mathbb{R}^n : 2^{j-1} < |\xi| < 2^{j+1} \}. \]

As a consequence, for any \( f \in \mathcal{S}_0' \),
\[ \sum_{k=-\infty}^{\infty} \Phi_k * f = f. \tag{2.1} \]

For notational convenience, set
\[ \Delta_j f = \Phi_j * f, \quad j = 0, \pm 1, \pm 2, \ldots \tag{2.2} \]

**Definition 2.1.** For \( s \in \mathbb{R} \) and \( 1 \leq p, q \leq \infty \), the homogeneous Besov space \( \dot{B}^s_{p,q} \) is defined by
\[ \dot{B}^s_{p,q} = \{ f \in \mathcal{S}_0' : \| f \|_{\dot{B}^s_{p,q}} < \infty \}, \]
where
\[ \| f \|_{\dot{B}^s_{p,q}} = \begin{cases} \left( \sum_j (2^{js} \| \Delta_j f \|_{L^p})^q \right)^{1/q} & \text{for } q < \infty, \\ \sup_j 2^{js} \| \Delta_j f \|_{L^p} & \text{for } q = \infty. \end{cases} \]

For \( \Delta_j \) defined in (2.2) and \( S_j \equiv \sum_{k<j} \Delta_k \),
\[ \Delta_j \Delta_k = 0 \quad \text{if } |j - k| \geq 2 \quad \text{and} \quad \Delta_j (S_{k-1} f - \Delta_k f) = 0 \quad \text{if } |j - k| \geq 3. \]

The following proposition lists a few simple facts that we will use in the subsequent section.

**Proposition 2.2.** Assume that \( s \in \mathbb{R} \) and \( p, q \in [1, \infty] \).

1. If \( 1 \leq q_1 \leq q_2 \leq \infty \), then \( \dot{B}^s_{p,q_1} \subset \dot{B}^s_{p,q_2} \).
2. (Besov embedding) If \( 1 \leq p_1 \leq p_2 \leq \infty \) and \( s_1 = s_2 + n(\frac{1}{p_1} - \frac{1}{p_2}) \), then \( \dot{B}^{s_1}_{p_1,q_1}(\mathbb{R}^n) \subset \dot{B}^{s_2}_{p_2,q_2}(\mathbb{R}^n) \).
3. If \( 1 < p < \infty \), then
\[ \dot{B}^s_{p,\min(p,2)} \subset W^{s,p} \subset \dot{B}^s_{p,\max(p,2)}, \]
where \( W^{s,p} \) denotes a standard homogeneous Sobolev space.

We will need a Bernstein type inequality for fractional derivatives.

**Proposition 2.3.** Let \( \alpha \geq 0 \). Let \( 1 \leq p \leq q \leq \infty \).

1. If \( f \) satisfies
\[ \text{supp} \hat{f} \subset \{ \xi \in \mathbb{R}^n : |\xi| \leq K 2^j \}, \]
for some integer \( j \) and a constant \( K > 0 \), then
\[ \| (-\Delta)^{\alpha} f \|_{L^q(\mathbb{R}^n)} \leq C_1 2^{2\alpha j + jn(\frac{1}{p} - \frac{1}{q})} \| f \|_{L^p(\mathbb{R}^n)}. \tag{2.3} \]
2. If \( f \) satisfies
\[ \text{supp} \hat{f} \subset \{ \xi \in \mathbb{R}^n : K_1 2^j \leq |\xi| \leq K_2 2^j \} \]
for some integer \( j \) and constants \( 0 < K_1 \leq K_2 \), then
\[ C_1 2^{2\alpha j} \| f \|_{L^q(\mathbb{R}^n)} \leq \| (-\Delta)^{\alpha} f \|_{L^q(\mathbb{R}^n)} \leq C_2 2^{2\alpha j + jn(\frac{1}{p} - \frac{1}{q})} \| f \|_{L^p(\mathbb{R}^n)}, \]
where \( C_1 \) and \( C_2 \) are constants depending on \( \alpha, p \) and \( q \) only.
The following proposition provides a lower bound for an integral that originates from the dissipative term in the process of $L^p$ estimates (see [21,4]).

**Proposition 2.4.** Assume either $\alpha \geq 0$ and $p = 2$ or $0 \leq \alpha < 1$ and $2 < p < \infty$. Let $j$ be an integer and $f \in S'$. Then
\[
\int_{\mathbb{R}^n} |\Delta_j f|^p - 2 \Delta_j f A^{2\alpha} \Delta_j f \, dx \geq C 2^{2j} \|\Delta_j f\|_{L^p}^p
\]
for some constant $C$ depending on $n$, $\alpha$ and $p$.

3. The main theorem and its proof

**Theorem 3.1.** Let $\theta$ be a Leray–Hopf weak solution of (1.1), namely
\[
\theta \in L^\infty([0, \infty); L^2(\mathbb{R}^2)) \cap L^2([0, \infty); \dot{H}^\alpha(\mathbb{R}^2)). \tag{3.1}
\]
Let $\delta > 1 - 2\alpha$ and let $0 < t_0 < t < \infty$. If
\[
\theta \in L^\infty([t_0, t]; C^\delta(\mathbb{R}^2)), \tag{3.2}
\]
then
\[
\theta \in C^\infty((t_0, t] \times \mathbb{R}^2).
\]

**Proof.** First, we notice that (3.1) and (3.2) imply that
\[
\theta \in L^\infty([t_0, t]; B^{\delta_1}_{p, \infty} \cap C^{\delta_2}(\mathbb{R}^2))
\]
for any $p \geq 2$ and $\delta_1 = \delta(1 - \frac{2}{p})$. In fact, for any $\tau \in [t_0, t]$,
\[
\|\theta(\cdot, \tau)\|_{B^{\delta_1}_{p, \infty}} = \sup_j 2^{\delta_1 j} \|\Delta_j \theta\|_{L^p} \\
\leq \sup_j 2^{\delta_1 j} \|\Delta_j \theta\|_{L^\infty} 2^{-\frac{2}{p}} \|\Delta_j \theta\|_{L^2} \\
\leq \|\theta(\cdot, \tau)\|_{C^\delta} 2^{\delta_1} \|\theta(\cdot, \tau)\|_{L^2}.
\]
Since $\delta > 1 - 2\alpha$, we have $\delta_1 > 1 - 2\alpha$ when
\[
p > p_0 \equiv \frac{2\delta}{\delta - (1 - 2\alpha)}.
\]
Next, we show that
\[
\theta \in L^\infty([t_0, t]; B^{\delta_1}_{p, \infty} \cap C^{\delta_2})(\mathbb{R}^2)
\]
implies
\[
\theta(\cdot, t) \in B^{\delta_2}_{p, \infty} \cap C^{\delta_2}
\]
for some $\delta_2 > \delta_1$ to be specified. Let $j$ be an integer. Applying $\Delta_j$ to (1.1), we get
\[
\partial_t \Delta_j \theta + \kappa A^{2\alpha} \Delta_j \theta = -\Delta_j (u \cdot \nabla \theta). \tag{3.3}
\]
By Bony’s notion of paraproduct,
\[
\Delta_j (u \cdot \nabla \theta) = \sum_{|j-k| \leq 2} \Delta_j (S_{k-1} u \cdot \nabla \Delta_k \theta) + \sum_{|j-k| \leq 2} \Delta_j (\Delta_k u \cdot \nabla S_{k-1} \theta) \\
+ \sum_{k \geq j-1} \sum_{|k-l| \leq 1} \Delta_j (\Delta_k u \cdot \nabla \Delta_l \theta). \tag{3.4}
\]
Multiplying (3.3) by \( p|\Delta_j \theta|^{p-2} \Delta_j \theta \), integrating with respect to \( x \), and applying the lower bound
\[
\int_{\mathbb{R}^d} |\Delta_j f|^{p-2} \Delta_j f A^{2a} \Delta_j f \, dx \geq C 2^{2a} |\Delta_j f|_{L^p}^p
\]
of Proposition 2.4, we obtain
\[
\frac{d}{dt} |\Delta_j \theta|_{L^p}^p + C \kappa 2^{2a} |\Delta_j \theta|_{L^p}^p \leq I_1 + I_2 + I_3,
\]
where \( I_1, I_2 \) and \( I_3 \) are given by
\[
I_1 = -p \sum_{|j-k| \leq 2} |\Delta_j \theta|^{p-2} \Delta_j \theta \cdot \Delta_j (S_{k-1} u \cdot \nabla \Delta_k \theta) \, dx,
\]
\[
I_2 = -p \sum_{|j-k| \leq 2} |\Delta_j \theta|^{p-2} \Delta_j \theta \cdot \Delta_j (\Delta_k u \cdot \nabla S_{k-1} \theta) \, dx,
\]
\[
I_3 = -p \sum_{k \geq j-1} |\Delta_j \theta|^{p-2} \Delta_j \theta \cdot \sum_{|k-l| \leq 1} \Delta_j (\Delta_k u \cdot \nabla \Delta_l \theta) \, dx.
\]
We first bound \( I_2 \). By H"older’s inequality
\[
I_2 \leq C |\Delta_j \theta|_{L^p}^{p-1} \sum_{|j-k| \leq 2} |\Delta_k u|_{L^p} \|\nabla S_{k-1} \theta\|_{L^\infty}.
\]
Applying Bernstein’s inequality, we obtain
\[
I_2 \leq C |\Delta_j \theta|_{L^p}^{p-1} \sum_{|j-k| \leq 2} |\Delta_k u|_{L^p} \sum_{m \leq k-1} 2^m |\Delta_m \theta|_{L^\infty}
\]
\[
\leq C |\Delta_j \theta|_{L^p}^{p-1} \sum_{|j-k| \leq 2} |\Delta_k u|_{L^p} 2^{(1-\delta_1)k} \sum_{m \leq k-1} 2^{(m-k) (1-\delta_1)} 2^{m \delta_1} |\Delta_m \theta|_{L^\infty}.
\]
Thus, for \( 1 - \delta_1 > 0 \), we have
\[
I_2 \leq C |\Delta_j \theta|_{L^p}^{p-1} \|\theta\|_{C^1} \sum_{|j-k| \leq 2} |\Delta_k u|_{L^p} 2^{(1-\delta_1)k}.
\]
We now estimate \( I_1 \). The standard idea is to decompose it into three terms: one with commutator, one that becomes zero due to the divergence-free condition and the rest. That is, we rewrite \( I_1 \) as
\[
I_1 = - p \sum_{|j-k| \leq 2} |\Delta_j \theta|^{p-2} \Delta_j \theta \cdot [\Delta_j, S_{k-1} u \cdot \nabla] \Delta_k \theta \, dx - p \int |\Delta_j \theta|^{p-2} \Delta_j \theta \cdot (S_j u \cdot \nabla \Delta_j \theta) \, dx
\]
\[
= I_{11} + I_{12} + I_{13},
\]
where we have used the simple fact that \( \sum_{|k-j| \leq 2} \Delta_k \Delta_j \theta = \Delta_j \theta \), and the brackets \([\;]\) represent the commutator, namely
\[
[\Delta_j, S_{k-1} u \cdot \nabla] \Delta_k \theta = \Delta_j (S_{k-1} u \cdot \nabla \Delta_k \theta) - S_{k-1} u \cdot \nabla \Delta_j \Delta_k \theta.
\]
Since \( u \) is divergence free, \( I_{12} \) becomes zero. \( I_{12} \) can also be handled without resort to the divergence-free condition. In fact, integrating by parts in \( I_{12} \) yields
\[
I_{12} = \int |\Delta_j \theta|^{p} \nabla \cdot S_j u \, dx \leq |\Delta_j \theta|_{L^p}^{p} \|\nabla \cdot S_j u\|_{L^\infty}.
\]
By Bernstein’s inequality,
\[ |I_{12}| \leq \| \Delta_j \|_{L^p}^p \sum_{m \leq j-1} 2^m \| \Delta_m u \|_{L^\infty} \]
\[ = \| \Delta_j \|_{L^p}^p 2^{(1-\delta_1)j} \sum_{m \leq j-1} 2^{(1-\delta_1)(m-j)} 2^m \| \Delta_m u \|_{L^\infty}. \]

For \( 1 - \delta_1 > 0 \),
\[ |I_{12}| \leq C \| \Delta_j \|_{L^p}^p 2^{(1-\delta_1)j} \| u \|_{C^{\delta_1}} \leq C \| \Delta_j \|_{L^p}^p 2^{(1-2\delta_1)j} \| \theta \|_{B^{\delta_1}_{p,\infty}} \| u \|_{C^{\delta_1}}. \]

We now bound \( I_{11} \) and \( I_{13} \). By Hölder’s inequality,
\[ |I_{11}| \leq p \| \Delta_j \|_{L^p}^{p-1} \sum_{|j-k| \leq 2} \| [\Delta_j, S_{k-1} u \cdot \nabla] \Delta_k \theta \|_{L^p}. \]

To bound the commutator, we have by the definition of \( \Delta_j \)
\[ [\Delta_j, S_{k-1} u \cdot \nabla] \Delta_k \theta = \int \Phi_j(x-y)(S_{k-1}(u)(x) - S_{k-1}(u)(y)) : \nabla \Delta_k \theta(y) dy. \]

Using the fact that \( \theta \in C^{\delta_1} \) and thus
\[ \| S_{k-1}(u)(x) - S_{k-1}(u)(y) \|_{L^\infty} \leq \| u \|_{C^{\delta_1}} |x-y|^{\delta_1}, \]
we obtain
\[ \| [\Delta_j, S_{k-1} u \cdot \nabla] \Delta_k \theta \|_{L^p} \leq 2^{-\delta_1 j} \| u \|_{C^{\delta_1}} 2^k \| \Delta_k \theta \|_{L^p}. \]

Therefore,
\[ |I_{11}| \leq C p \| \Delta_j \|_{L^p}^{p-1} 2^{-\delta_1 j} \| u \|_{C^{\delta_1}} \sum_{|j-k| \leq 2} 2^k \| \Delta_k \theta \|_{L^p}. \]

The estimate for \( I_{13} \) is straightforward. By Hölder’s inequality,
\[ |I_{13}| \leq p \| \Delta_j \|_{L^p}^{p-1} \sum_{|j-k| \leq 2} \| S_{k-1} u - S_j u \|_{L^p} \| \nabla \Delta_j \theta \|_{L^\infty} \]
\[ \leq C p \| \Delta_j \|_{L^p}^{p-1} 2^{(1-\delta_1)j} \| \theta \|_{C^{\delta_1}} \sum_{|j-k| \leq 2} \| \Delta_k u \|_{L^p}. \]

We now bound \( I_3 \). By Hölder’s inequality and Bernstein’s inequality,
\[ |I_3| \leq p \| \Delta_j \|_{L^p}^{p-1} \| \Delta_j \nabla \cdot \left( \sum_{k \geq j-1} \sum_{|j-k| \leq 1} \Delta_j u \Delta_k \theta \right) \|_{L^p} \]
\[ \leq p \| \Delta_j \|_{L^p}^{p-1} 2^j \| u \|_{C^{\delta_1}} \sum_{k \geq j-1} 2^{-\delta_1 k} \| \Delta_k \theta \|_{L^p}. \]  \hspace{1cm} (3.6)

Inserting the estimates for \( I_1, I_2 \) and \( I_3 \) in (3.5) and eliminating \( p \| \Delta_j \|_{L^p}^{p-1} \) from both sides, we get
\[
\frac{d}{dt} \| \Delta_j \theta \|_{L^p} + C \kappa 2^{2\alpha j} \| \Delta_j \theta \|_{L^p} \leq C 2^{(1-2\delta_1)j} \| \theta \|_{B^{\delta_1}_{p,\infty}} \| u \|_{C^{\delta_1}} + C 2^{-\delta_1 j} \| u \|_{C^{\delta_1}} \sum_{|j-k| \leq 2} 2^k \| \Delta_k \theta \|_{L^p} \]
\[ + C \| \theta \|_{C^{\delta_1}} \sum_{|j-k| \leq 2} \| \Delta_k u \|_{L^p} 2^{(1-\delta_1)k} + C 2^{1-\delta_1 j} \| \theta \|_{C^{\delta_1}} \sum_{|j-k| \leq 2} \| \Delta_k u \|_{L^p} \]
\[ + C 2^j \| u \|_{C^{\delta_1}} \sum_{k \geq j-1} 2^{-\delta_1 k} \| \Delta_k \theta \|_{L^p}. \]  \hspace{1cm} (3.7)

The terms on the right can be further bounded as follows.
\[ C 2^{-\delta_1 j} \| u \|_{\dot{B}_{p, \infty}^{\delta_1}} \sum_{|j-k| \leq 2} 2^k \| \Delta_k \theta \|_{L^p} = C 2^{(1-2\delta_1)j} \| u \|_{\dot{B}_{p, \infty}^{\delta_1}} \sum_{|j-k| \leq 2} 2^{\delta_1 k} \| \Delta_k \theta \|_{L^p} 2^{(k-j)(1-\delta_1)} \leq C 2^{(1-2\delta_1)j} \| u \|_{\dot{B}_{p, \infty}^{\delta_1}}, \]

\[ \| \Delta_k \theta \|_{L^p} 2^{(1-\delta_1)k} = C 2^{(1-2\delta_1)j} \| \theta \|_{\dot{B}_{p, \infty}^{\delta_1}} \sum_{|j-k| \leq 2} 2^{\delta_1 k} \| \Delta_k \theta \|_{L^p} 2^{(k-j)(1-2\delta_1)} \leq C 2^{(1-2\delta_1)j} \| \theta \|_{\dot{B}_{p, \infty}^{\delta_1}}, \]

\[ C 2^{(1-\delta_1)j} \| \theta \|_{\dot{B}_{p, \infty}^{\delta_1}} \sum_{|j-k| \leq 2} \| \Delta_k \theta \|_{L^p} 2^{(1-\delta_1)k} = C 2^{(1-2\delta_1)j} \| \theta \|_{\dot{B}_{p, \infty}^{\delta_1}} \sum_{|j-k| \leq 2} 2^{\delta_1 k} \| \Delta_k \theta \|_{L^p} 2^{(k-j)(1-2\delta_1)} \leq C 2^{(1-2\delta_1)j} \| \theta \|_{\dot{B}_{p, \infty}^{\delta_1}} \| u \|_{\dot{B}_{p, \infty}^{\delta_1}}, \]

and

\[ C 2^j \| u \|_{\dot{B}_{p, \infty}^{\delta_1}} \sum_{k \geq j} 2^{-\delta_1 k} \| \Delta_k \theta \|_{L^p} = C 2^{(1-2\delta_1)j} \| u \|_{\dot{B}_{p, \infty}^{\delta_1}} \sum_{k \geq j} 2^{-2\delta_1 (k-j)} 2^{\delta_1 k} \| \Delta_k \theta \|_{L^p} \leq C 2^{(1-2\delta_1)j} \| u \|_{\dot{B}_{p, \infty}^{\delta_1}} \| \theta \|_{\dot{B}_{p, \infty}^{\delta_1}}. \]

We can write (3.7) in the following integral form

\[ \| \Delta_j \theta(t) \|_{L^p} \leq e^{-C \chi 2^{2\delta_1/ (t-t_0)}} \| \Delta_j \theta(t_0) \|_{L^p} + C \int_{t_0}^t e^{-C \chi 2^{2\delta_1/ (t-s)}} 2^{(1-2\delta_1)j} \| \theta \|_{\dot{B}_{p, \infty}^{\delta_1}} \| u \|_{\dot{B}_{p, \infty}^{\delta_1}} \| \theta \|_{\dot{B}_{p, \infty}^{\delta_1}} ds. \]

Multiplying both sides by \( 2^{(2\alpha + 2\delta_1 - 1)j} \) and taking the supremum with respect to \( j \), we get

\[ \| \theta(t) \|_{\dot{B}_{p, \infty}^{\delta_1 + 2\alpha - 1}} \leq \sup_j \left\{ e^{-C \chi 2^{2\delta_1/ (t-t_0)} 2^{(\delta_1 + 2\alpha - 1)j} \| \theta(t_0) \|_{\dot{B}_{p, \infty}^{\delta_1}}} + C \kappa^{-1} \sup_j \left\{ (1 - e^{-C \chi 2^{2\delta_1/ (t-t_0)}}) \right\} \right\} \max_{s \in [t_0, t]} \| \theta(s) \|_{\dot{B}_{p, \infty}^{\delta_1}} \| \theta(s) \|_{\dot{B}_{p, \infty}^{\delta_1}}. \]

Here we have used the fact that

\[ \| u \|_{\dot{B}_{p, \infty}^{\delta_1}} \leq \| \theta \|_{\dot{B}_{p, \infty}^{\delta_1}} \quad \text{and} \quad \| u \|_{\dot{B}_{p, \infty}^{\delta_1}} \leq \| \theta \|_{\dot{B}_{p, \infty}^{\delta_1}}. \]

Therefore, we conclude that if

\[ \theta \in L^\infty([t_0, t]; \dot{B}_{p, \infty}^{\delta_1} \cap \dot{C}^{\delta_1}), \]

then

\[ \theta(\cdot, t) \in \dot{B}_{p, \infty}^{2\delta_1 + 2\alpha - 1}. \]  

(3.8)

Since \( \delta_1 > 1 - 2\alpha \), we have \( 2\delta_1 + 2\alpha - 1 > \delta_1 \) and thus gain regularity. In addition, according to the Besov embedding of Proposition 2.2,

\[ \dot{B}_{p, \infty}^{2\delta_1 + 2\alpha - 1} \subset \dot{B}_{\infty, \infty}^{\delta_2}, \]

where

\[ \delta_2 = 2\delta_1 + 2\alpha - 1 - \frac{2}{p} = \delta_1 + \left( \delta_1 - \left( \frac{1 - 2\alpha + \frac{2}{p}}{p} \right) \right). \]

We have \( \delta_2 > \delta_1 \) when

\[ p > p_1 \equiv \frac{2}{\delta_1 - (1 - 2\alpha)}. \]
Noting that \( \dot{B}^{\delta_2}_{2,\infty} \cap L^\infty = C^{\delta_2} \), we conclude that, for \( p > \max\{ p_0, p_1 \} \),
\[
\theta(\cdot, t) \in \dot{B}^{\delta_2}_{p,\infty} \cap C^{\delta_2}
\]
for some \( \delta_2 > \delta_1 \). The above process can then be iterated with \( \delta_1 \) replaced by \( \delta_2 \). A finite number of iterations allow us to obtain that
\[
\theta(\cdot, t) \in C^{\gamma}
\]
for some \( \gamma > 1 \). The regularity in the spatial variable can then be converted into regularity in time. We have thus established that \( \theta \) is a classical solution to the supercritical QG equation. Higher regularity can be proved by well-known methods. □

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References