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The 2D magnetohydrodynamic equations with magnetic diffusion

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Abstract

This paper examines the initial-value problem for the two-dimensional magnetohydrodynamic equation with only magnetic diffusion (without velocity dissipation). Whether or not its classical solutions develop finite time singularities is a difficult problem and remains open. This paper establishes two main results. The first result features a regularity criterion in terms of the magnetic field. This criterion comes naturally from our approach to obtain a global bound for the vorticity. Due to the lack of velocity dissipation, it is difficult to conclude the boundedness of the vorticity from the vorticity equation itself. Instead we derive and involve a new equation for the combined quantity of the vorticity and a singular integral operator on the tensor product of the magnetic field. This criterion may be verifiable. Our second main result is a weaker version of the small data global existence result, which is shown by the bootstrap argument.

Keywords: magnetohydrodynamic equations, global wellposedness, partial dissipation
Mathematics Subject Classification numbers: 76W05, 76D03, 35Q35

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1. Introduction

This paper examines the initial-value problem for the 2D incompressible magnetohydrodynamic (MHD) equations

$$\begin{cases} u_t + u \cdot \nabla u = -\nabla p + b \cdot \nabla b, \\ b_t + u \cdot \nabla b = b \cdot \nabla u + \eta \Delta b, \\ \nabla \cdot u = 0, \quad \nabla \cdot b = 0, \\ (u, b)(x, 0) = (u_0(x), b_0(x)), \end{cases} \tag{1.1}$$

where $u = u(x, t)$ denotes the velocity of the fluid, $b = b(x, t)$ the magnetic field and $p = p(x, t)$ the scalar pressure. The parameter $\eta \geqslant 0$ denotes the magnetic diffusivity. Due to the lack of the velocity dissipation, the global well-posedness issue is extremely difficult and remains open.

The work presented here contributes to the efforts towards the resolution of the global regularity problem on the 2D MHD equations with partial or fractional dissipation. We recall some of the recent developments in this direction. For the convenience of the description, we write the 2D MHD equations with general partial dissipation as follows,

$$\begin{cases} u_t + u \cdot \nabla u = b \cdot \nabla b - \nabla p + \nu_1 \partial_{x_1x_1} u + \nu_2 \partial_{x_2x_2} u, \\ b_t + u \cdot \nabla b = b \cdot \nabla u + \eta_1 \partial_{x_1x_1} b + \eta_2 \partial_{x_2x_2} b, \\ \nabla \cdot u = 0, \nabla \cdot b = 0. \end{cases} \tag{1.2}$$

The two extreme cases, (1.2) with $\nu_1 > 0, \nu_2 > 0, \eta_1 > 0, \eta_2 > 0$ and (1.2) with $\nu_1 = \nu_2 = \eta_1 = \eta_2 = 0$, are either too easy or too difficult. When $\nu_1 = \nu_2 = \eta_1 = \eta_2 = 0$, (1.2) becomes completely inviscid and the global regularity issue remains outstandingly open. When $\nu_1 > 0, \nu_2 > 0, \eta_1 > 0$ and $\eta_2 > 0$, the global regularity can be established in a similar way as that for the 2D Navier-Stokes equations (see, e.g. [24]). Mathematically it is very natural to study the intermediate cases. In addition, some of the partial dissipation cases do have strong physical backgrounds (see, e.g. [3, 26]). The global regularity issue on these cases has attracted considerable interests in the last few years and progress has been made for some cases (see, e.g. [4–9, 12, 13, 15, 16, 17–20, 22, 23, 27, 31, 33–37, 39–43]). Equation (1.2) with $\nu_1 = 0, \nu_2 > 0, \eta_1 > 0, \eta_2 = 0$ and (1.2) with $\nu_1 > 0, \nu_2 = 0, \eta_1 = 0, \eta_2 > 0$ were recently examined by Cao and Wu and shown to possess global classical solutions for any sufficiently smooth data [6]. Some partial results have also been obtained for the case (1.2) with $\nu_1 > 0, \nu_2 = 0, \eta_1 > 0, \eta_2 = 0$ ([4]).

Another prominent partial dissipation case is when there is only velocity dissipation (no magnetic diffusion), namely, (1.2) with $\nu_1 = \nu_2 > 0$ and $\eta_1 = \eta_2 = 0$. The global well-posedness for this case is open. The velocity dissipation alone is not enough to prove global bounds in any Sobolev space. Very recent efforts are devoted to global solutions near an equilibrium and progress has been made ([15, 22, 27, 37, 43]). The pioneering work of Lin *et al* in this direction reformulated the system in Lagrangian coordinates and estimated the Lagrangian velocity through the anisotropic Littlewood–Paley theory and anisotropic Besov space techniques [22]. The paper of Ren *et al* obtained this global well-posedness near an equilibrium without resorting to the Lagrangian coordinates and rigorously confirmed an numerical observation by establishing explicit large-time decay rates [27]. Wu *et al* recently considered the MHD equations with a velocity damping term and obtained the global solutions near an equilibrium [37]. The approach in [37] is completely different from those in [22, 27, 43]. Wu *et al* [37] offers a systematic new procedure for diagonalizing the system of linearized equations and converting the differential equations into an integral form.

When there is only magnetic diffusion and no velocity dissipation, namely, $\nu_1 = \nu_2 = 0$ and $\eta_1 = \eta_2 > 0$ in (1.2), the global regularity problem remains open. But we do have global H^1-bound for (u, b) in this case [6, 20], which ensures the global existence of weak solutions. It is not clear if such weak solutions are unique or if they can be improved to classical solutions when the initial datum is sufficiently smooth. In addition, the work of Cao *et al* [7] indicates that this case is critical in the sense that a slight more dissipation would yield the global regularity. More precisely, if we replace Δb by $-(-\Delta)^{\beta} b$ with any $\beta > 1$, then the resulting equation does have a global classical solution [7]. A different approach from [7] was later obtained by Jiu and Zhao [18].

The aim of this paper is to gain further understanding of the global regularity problem for the MHD equation with only magnetic diffusion, namely (1.1). We present two main results in hope that they shed light on the eventual resolution of this difficult global regularity problem. The first result features a regularity criterion, which may be verifiable and thus leads to the global regularity.

Theorem 1.1. *Let $s > 2$. Assume $(u_0, b_0) \in H^s(\mathbb{R}^2)$ with $\nabla \cdot u_0 = \nabla \cdot b_0 = 0$. Let (u, b) be the local (in time) solution of (1.1) on $[0, T^*)$. Let $T_0 > T^*$. If there is $\sigma > 0$ and an integer $k_0 > 0$ such that b satisfies*

$$M(T_0) \equiv \int_0^{T_0} \sum_{k \geqslant k_0} 2^{\sigma k} \, \|S_{k-1}(b \otimes b)\|_{L^\infty} \, dt < \infty, \tag{1.3}$$

then the local solution can be extended to $[0, T_0]$. Here $b \otimes b$ denotes the tensor product and S_j denotes the identity approximator defined through the Littlewood–Paley decomposition (see section 2 for details).

We describe the difficulty in dealing with the global regularity problem and explain how the condition (1.3) naturally comes out. The magnetic diffusion does provide certain global regularity, but it fails to produce the crucial global bounds we need when they are applied on the vorticity equation. More precisely, energy estimates do yield the global H^1-bound for (u, b). In addition, by taking advantage of the regularizing effect of the heat kernel, we can also show that, for any $2 < p < \infty$, $2 < q \leqslant \infty$,

$$b \in L^q(0, T; W^{2,p}(\mathbb{R}^2)), \qquad \omega \in L^\infty(0, T; L^p(\mathbb{R}^2))$$

for any $T > 0$, where $\omega = \operatorname{curl} u$. However, it is not clear if $\|\omega\|_{L^\infty(0,T;L^\infty)} < \infty$ for all $T < \infty$. The lack of the global bound for $\|\nabla j\|_{L^\infty(\mathbb{R}^2)}$ with $j = \operatorname{curl} b$ makes it impossible to obtain a global bound directly from the vorticity equation

$$\omega_t + u \cdot \nabla \omega = b \cdot \nabla j. \tag{1.4}$$

To circumvent this difficulty, we consider the combined quantity

$$G = \omega + \mathcal{R}(b \otimes b), \qquad \mathcal{R} = (-\Delta)^{-1} \operatorname{curl} \operatorname{div}.$$

It is shown here that G satisfies

$$\begin{aligned} \partial_t G + u \cdot \nabla G = &- [\mathcal{R}, u \cdot \nabla](b \otimes b) - 2 \sum_{k=1}^{2} \mathcal{R}(\partial_k b \otimes \partial_k b) \\ &+ \mathcal{R}(\nabla u \, (b \otimes b)) + \mathcal{R}((b \otimes b) \, (\nabla u)^{\mathsf{T}}), \end{aligned} \tag{1.5}$$

where $\nabla u \, (b \otimes b)$ denotes the standard multiplication of two matrices ∇u and $b \otimes b$, and $(\nabla u)^{\mathsf{T}}$ denotes the transpose of ∇u. Although (1.5) looks more complex than the vorticity equation (1.4), some of the terms on the right of (1.5) are less regularity demanding and can be suitably bounded via commutator estimates. Since the singular integral type operator $\mathcal{R}$ is not bounded in L^∞, we intend to bound G in the Besov space $B^0_{\infty,1}$ (see section 2 for its definition). We remark that $B^0_{\infty,1}$ is a natural choice due to the fact that $B^0_{\infty,1} \subset L^\infty$ is slightly smaller than L^∞ and the operator $\mathcal{R}$ functions well in $B^0_{\infty,1}$. However, if the setup is $B^0_{\infty,1}$, the estimate of G in this space generates a multiplication factor in terms of ∇u, namely

$$\|G\|_{L^\infty_t B^0_{\infty,1}} \leqslant C\,(\|G(0)\|_{B^0_{\infty,1}} + \|f\|_{L^1_t B^0_{\infty,1}})\left(1 + \int_0^t \|\nabla u\|_{L^\infty}\,\mathrm{d}\tau\right), \tag{1.6}$$

where $G(0)$ denotes the initial data of G and f denotes the right-hand side of (1.5). We need (1.3) in order to show that

$$\|f\|_{L^1([0,T_0];B^0_{\infty,1})} < \infty.$$

In fact, (1.3) is needed only in controlling part of the last two terms in f. As a consequence, we can then show that

$$\|\omega\|_{L^\infty([0,T_0];B^0_{\infty,1})} < \infty,$$

which is sufficient for further higher regularity of $(u,\, b)$. Thus the local solution can be extended to $[0,\, T_0]$.

Our second result elucidates a basic fact on the 2D MHD equations (1.1) with or even without a magnetic diffusion. Given any fixed time $T > 0$. We can find sufficiently small data such that (1.1) always possesses a unique solution on $[0,\, T]$. More precisely, we have the following theorem for (1.1).

Theorem 1.2. *Consider (1.1) with $\eta > 0$. Assume $(u_0, b_0) \in H^s(\mathbb{R}^2)$ with $s > 2$ and $\nabla\cdot u_0 = \nabla\cdot b_0 = 0$. Let $T > 0$. Then, there exists $\delta = \delta(T) > 0$ such that, if*

$$\|b_0\|_{H^s} < \delta, \tag{1.7}$$

then (1.1) has a unique solution $(u,\, b)$ on $[0,\, T]$ satisfying

$$(u,b) \in L^\infty([0,T];H^s(\mathbb{R}^2)) \quad \text{and} \quad b \in L^2([0,T];H^{s+1}(\mathbb{R}^2)).$$

We remark that theorem 1.2 requires only the smallness of b_0 (u_0 needs not be small). For (1.1) without magnetic diffusion, namely (1.1) with $\eta = 0$, we need the smallness of both u_0 and b_0.

Theorem 1.3. *Consider the inviscid MHD equation, namely (1.1) with $\eta = 0$. Let $s > 2$. Assume $(u_0, b_0) \in H^s(\mathbb{R}^2)$ with $\nabla\cdot u_0 = \nabla\cdot b_0 = 0$. Let $T > 0$. Then there exists $\delta = \delta(T) > 0$ such that, if*

$$\|(u_0, b_0)\|_{H^s} \leqslant \delta,$$

then the inviscid MHD equation has a unique solution $(u,\, b)$ on $[0,\, T]$ with

$$u \in C([0,T];H^s(\mathbb{R}^2)), \qquad b \in C([0,T];H^s(\mathbb{R}^2)).$$

We note that the results in theorems 1.2 and 1.3 are weaker than the standard small data global regularity results, which require the choice of the smallness fits all time T. However, for (1.1), it is not even clear whether sufficiently *small* data would yield global classical solutions. The Lorentz forcing term and the lack of dissipation in the velocity equation make it very difficult to control the Sobolev norms of the solutions. Even if u is uniformly Lipschitz in time, namely

$$\|\nabla u(\cdot, t)\|_{L^\infty} \leqslant C,$$

the Sobolev norm of b depends on the time integral of $\|\nabla u\|_{L^\infty}$, which still grows without bound. This simple reasoning indicates that the issue of small data global well-posedness may be as difficult as the well-posedness issue for general initial data.

Theorems 1.2 and 1.3 are proven through the bootstrap principle and their proofs are not very difficult. A good reference for the abstract bootstrap principle is the book by Tao [30, p 20].

The rest of this paper is divided into four sections. Section 2 makes several preparations including presenting the Littlewood–Paley decomposition, functional spaces and related inequalities. Section 3 provides the global H^1 bound and the L^qL^p estimates on the solution and its derivatives by making use of the regularizing effects of the heat kernel. Section 4 proves theorem 1.1 while section 5 proves theorems 1.2 and 1.3. Throughout the rest of this paper, C stands for a generic constant. The L^p- norm of a function f is denoted by $\|f\|_{L^p}$, and the Sobolev norm by $\|f\|_{W^{s,p}}$.

2. Preparations

This section includes several parts. It recalls the Littlewood–Paley theory, introduces the Besov spaces, provides Bernstein inequalities as well as a commutator estimate.

We start with the definitions of some of the functional spaces and related facts that will be used in the subsequent sections. Materials on Besov space and related facts presented here can be found in several books and many papers (see, e.g. [1, 2, 25, 28, 32]).

2.1. Fourier transform and the Littlewood–Paley theory

We start with several notations. $\mathcal{S}$ denotes the usual Schwarz class and $\mathcal{S}'$ its dual, the space of tempered distributions. $\mathcal{S}_0$ denotes a subspace of $\mathcal{S}$ defined by

$$\mathcal{S}_0 = \left\{ \phi \in \mathcal{S} : \int_{\mathbb{R}^d} \phi(x)\, x^\gamma \,\mathrm{d}x = 0,\ |\gamma| = 0, 1, 2, \cdots \right\}$$

and $\mathcal{S}_0'$ denotes its dual. $\mathcal{S}_0'$ can be identified as

$$\mathcal{S}_0' = \mathcal{S}'/\mathcal{S}_0^\perp = \mathcal{S}'/\mathcal{P},$$

where $\mathcal{P}$ denotes the space of multinomials. On the Schwartz class, we can define the Fourier transform and its inverse via

$$\hat{f}(\xi) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} f(x) \mathrm{e}^{-\mathrm{i}x\xi} \mathrm{d}x, \quad f(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \hat{f}(\xi) \mathrm{e}^{\mathrm{i}x\xi} \mathrm{d}\xi.$$

To introduce the Littlewood–Paley decomposition, we write for each $j \in \mathbb{Z}$

$$A_j = \{\xi \in \mathbb{R}^d :\ 2^{j-1} \leqslant |\xi| < 2^{j+1}\}.$$

The Littlewood–Paley decomposition asserts the existence of a sequence of functions $\{\Phi_j\}_{j\in\mathbb{Z}} \in \mathcal{S}$ such that

$$\operatorname{supp} \widehat{\Phi}_j \subset A_j, \qquad \widehat{\Phi}_j(\xi) = \widehat{\Phi}_0(2^{-j}\xi) \quad \text{or} \quad \Phi_j(x) = 2^{jd}\Phi_0(2^j x),$$

and

$$\sum_{j=-\infty}^{\infty} \widehat{\Phi}_j(\xi) = \begin{cases} 1, & \text{if } \xi \in \mathbb{R}^d \setminus \{0\}, \\ 0, & \text{if } \xi = 0. \end{cases}$$

Therefore, for a general function $\psi \in \mathcal{S}$, we have

$$\sum_{j=-\infty}^{\infty} \widehat{\Phi}_j(\xi)\widehat{\psi}(\xi) = \widehat{\psi}(\xi) \quad \text{for } \xi \in \mathbb{R}^d \setminus \{0\}.$$

In addition, if $\psi \in \mathcal{S}_0$, then

$$\sum_{j=-\infty}^{\infty} \widehat{\Phi}_j(\xi)\widehat{\psi}(\xi) = \widehat{\psi}(\xi) \quad \text{for any } \xi \in \mathbb{R}^d.$$

That is, for $\psi \in \mathcal{S}_0$,

$$\sum_{j=-\infty}^{\infty} \Phi_j * \psi = \psi$$

and hence

$$\sum_{j=-\infty}^{\infty} \Phi_j * f = f, \qquad f \in \mathcal{S}_0'$$

in the sense of weak-$*$ topology of $\mathcal{S}_0'$. For notational convenience, we define

$$\mathring{\Delta}_j f = \Phi_j * f, \qquad j \in \mathbb{Z}. \tag{2.1}$$

We now choose $\Psi \in \mathcal{S}$ such that

$$\widehat{\Psi}(\xi) = 1 - \sum_{j=0}^{\infty} \widehat{\Phi}_j(\xi), \quad \xi \in \mathbb{R}^d.$$

Then, for any $\psi \in \mathcal{S}$,

$$\Psi * \psi + \sum_{j=0}^{\infty} \Phi_j * \psi = \psi$$

and hence

$$\Psi * f + \sum_{j=0}^{\infty} \Phi_j * f = f$$

in $\mathcal{S}'$ for any $f \in \mathcal{S}'$. We set

$$\Delta_j f = \begin{cases} 0, & \text{if } j \leqslant -2, \\ \Psi * f, & \text{if } j = -1, \\ \Phi_j * f, & \text{if } j = 0, 1, 2, \cdots. \end{cases} \tag{2.2}$$

For notational convenience, we write Δ_j for $\mathring{\Delta}_j$ when there is no confusion. They are different for $j \leqslant -1$. As provided below, the homogeneous Besov spaces are defined in terms of $\mathring{\Delta}_j$ while the inhomogeneous Besov spaces are defined in Δ_j. Besides the Fourier localization operators Δ_j, the partial sum S_j is also a useful notation. For an integer j,

$$S_j \equiv \sum_{k=-1}^{j-1} \Delta_k,$$

where Δ_k is given by (2.2). For any $f \in \mathcal{S}'$, the Fourier transform of $S_j f$ is supported on the ball of radius 2^j and

$$S_j f \rightharpoonup f \qquad \text{in } \mathcal{S}'.$$

In addition, for two tempered distributions u and v, we also recall the notion of paraproducts

$$T_u v = \sum_j S_{j-1} u \Delta_j v, \qquad R(u,v) = \sum_{|i-j|\leqslant 2} \Delta_i u \Delta_j v$$

and Bony's decomposition

$$uv = T_u v + T_v u + R(u,v). \tag{2.3}$$

2.2. Besov spaces

Definition 2.1. For $s \in \mathbb{R}$ and $1 \leqslant p, q \leqslant \infty$, the homogeneous Besov space $\mathring{B}^s_{p,q}$ consists of $f \in \mathcal{S}'_0$ satisfying

$$\|f\|_{\mathring{B}^s_{p,q}} \equiv \|2^{js}\|\mathring{\Delta}_j f\|_{L^p}\|_{l^q} < \infty.$$

Definition 2.2. The inhomogeneous Besov space $B^s_{p,q}$ with $1 \leqslant p, q \leqslant \infty$ and $s \in \mathbb{R}$ consists of functions $f \in \mathcal{S}'$ satisfying

$$\|f\|_{B^s_{p,q}} \equiv \|2^{js}\|\Delta_j f\|_{L^p}\|_{l^q} < \infty.$$

2.3. Bernstein inequalities

Bernstein's inequalities are useful tools in dealing with Fourier localized functions and these inequalities trade integrability for derivatives. The following proposition provides Bernstein type inequalities for fractional derivatives.

Proposition 2.3. *Let $\alpha \geqslant 0$. Let $1 \leqslant p \leqslant q \leqslant \infty$.*

(1) *If f satisfies*

$$\operatorname{supp} \widehat{f} \subset \{\xi \in \mathbb{R}^d : |\xi| \leqslant K 2^j\},$$

for some integer j and a constant $K > 0$, then

$$\|\Lambda^\alpha f\|_{L^q(\mathbb{R}^d)} \leqslant C_1\, 2^{\alpha j + jd\left(\frac{1}{p} - \frac{1}{q}\right)} \|f\|_{L^p(\mathbb{R}^d)},$$

where C_1 is a constant depending on K, α, p and q only.

(2) *If f satisfies*

$$\operatorname{supp} \widehat{f} \subset \{\xi \in \mathbb{R}^d : \ K_1 2^j \leqslant |\xi| \leqslant K_2 2^j\}$$

for some integer j and constants $0 < K_1 \leqslant K_2$, then

$$C_1\, 2^{\alpha j} \|f\|_{L^q(\mathbb{R}^d)} \leqslant \|\Lambda^\alpha f\|_{L^q(\mathbb{R}^d)} \leqslant C_2\, 2^{\alpha j + jd\left(\frac{1}{p}-\frac{1}{q}\right)} \|f\|_{L^p(\mathbb{R}^d)},$$

where C_2 is a constant depending on K_1, K_2, α, p and q only.

2.4. Commutator estimate and propagation of a Besov norm

The following commutator involving a standard singular integral operator will also be used (see, e.g. [29]).

Lemma 2.4. *Let $\mathcal{R}$ denote a standard singular integral operator, say Riesz transform or $\mathcal{R} = (-\Delta)^{-1}\mathrm{curl}\,\mathrm{div}$. Let $1 < p \leqslant \infty$. For any integer k, for $0 \leqslant s_1, s_2 \leqslant 1$ and $s_1 + s_2 \leqslant 1$, we have*

$$\|\Delta_k([\mathcal{R}, u\cdot\nabla]\theta)\|_{L^p} \leqslant C_s 2^{(1-s_1-s_2)k} \|\Lambda^{s_1} u\|_{L^{p_1}} \|\Lambda^{s_2}\theta\|_{L^{p_2}} \tag{2.4}$$

where $1 < p_1, p_2 \leqslant \infty$ and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$.

In addition, we need the following estimate on the propagation of the Besov norm $B^0_{q,1}$ of solutions to a linear equation (see, e.g. [1, 14, 25]).

Lemma 2.5. *Consider the linear equation*

$$\begin{cases} \partial_t\theta + u\cdot\nabla\theta + \nu(-\Delta)^\alpha\theta = f, \\ \theta(x,0) = \theta_0(x), \end{cases} \tag{2.5}$$

where $\nu \geqslant 0$ and $\alpha \in (0,1)$. Then, there exists $C > 0$ such that

$$\|\theta\|_{L^\infty_t B^0_{q,1}} \leqslant C\,(\|\theta_0\|_{B^0_{q,1}} + \|f\|_{L^1_t B^0_{q,1}})\left(1 + \int_0^t \|\nabla u\|_{L^\infty}\,\mathrm{d}\tau\right),$$

where $q \in [1,\infty]$.

3. Preliminary bounds

This section proves some of the *a priori* bounds to be used in the subsequent section. It is divided into two subsections. The first subsection contains the global H^1-bound while the second subsection proves the global bounds for $\|\omega\|_{L^p(\mathbb{R}^2)}$ and $\|b\|_{W^{2,p}(\mathbb{R}^2)}$ for $p \in (2,\infty)$.

3.1. Global H^1-bound for (u, b)

This subsection provides the global H^1-bound. This bound has been known before (see, e.g. [6, 20]), but it is presented here for the sake of completeness. For the rest of this paper, as defined before, $\omega = \mathrm{curl}\, u$ and $j = \mathrm{curl}\, b$.

Proposition 3.1. *If (u, b) solves system (1.1), then, for any $t > 0$,*

$$\|\omega(t)\|^2_{L^2} + \|j(t)\|^2_{L^2} + \int_0^t \|\nabla j(s)\|^2_{L^2}\,\mathrm{d}s \leqslant (\|\omega_0\|^2_{L^2} + \|j_0\|^2_{L^2})\mathrm{e}^{C(\|u_0\|^2_{L^2} + \|b_0\|^2_{L^2})} \tag{3.1}$$

and consequently

$$\|u(t)\|_{H^1}^2 + \|b(t)\|_{H^1}^2 + \int_0^t \|b(s)\|_{H^2}^2 \, \mathrm{d}s$$
$$\leqslant C(\|\omega_0\|_{L^2}^2 + \|j_0\|_{L^2}^2)\mathrm{e}^{C(\|u_0\|_{L^2}^2 + \|b_0\|_{L^2}^2)} + \|u_0\|_{L^2}^2 + \|b_0\|_{L^2}^2 . \tag{3.2}$$

Proof. It follows easily from (1.1) that, for any $t > 0$,

$$\|u(t)\|_{L^2}^2 + \|b(t)\|_{L^2}^2 + 2\int_0^t \|\nabla b(s)\|_{L^2}^2 \, \mathrm{d}s = \|u_0\|_{L^2}^2 + \|b_0\|_{L^2}^2 . \tag{3.3}$$

To prove (3.1), we employ the equations of ω and j,

$$\omega_t + u \cdot \nabla\omega = b \cdot \nabla j, \tag{3.4}$$

$$j_t + u \cdot \nabla j = b \cdot \nabla\omega + \Delta j + Q(\nabla u, \nabla b), \tag{3.5}$$

where

$$Q(\nabla u, \nabla b) = 2\partial_1 b_1(\partial_1 u_2 + \partial_2 u_1) - 2\partial_1 u_1(\partial_1 b_2 + \partial_2 b_1).$$

Taking the L^2-inner products of (3.4) with ω and of (3.5) with j, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|\omega\|_{L^2}^2 + \|j\|_{L^2}^2) + 2\|\nabla j\|_{L^2}^2 = 2\int Q(\nabla u, \nabla b) j.$$

By the Hölder inequality and the Gagliardo–Nirenberg inequality,

$$\begin{aligned}\int Q(\nabla u, \nabla b) j &\leqslant C\|\nabla u\|_{L^2}\|\nabla b\|_{L^4}\|j\|_{L^4} \\ &\leqslant C\|\omega\|_{L^2}\|j\|_{L^4}^2 \\ &\leqslant C\|\omega\|_{L^2}\|j\|_{L^2}\|\nabla j\|_{L^2}.\end{aligned}$$

By Young's inequality,

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|\omega\|_{L^2}^2 + \|j\|_{L^2}^2) + 2\|\nabla j\|_{L^2}^2 \leqslant C\|\omega\|_{L^2}^2\|j\|_{L^2}^2 + \|\nabla j\|_{L^2}^2 .$$

In particular,

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|\omega\|_{L^2}^2 + \|j\|_{L^2}^2) + \|\nabla j\|_{L^2}^2 \leqslant C\|j\|_{L^2}^2 (\|\omega\|_{L^2}^2 + \|j\|_{L^2}^2).$$

Recalling the global L^2-bound in (3.1)–(3.3) then follow from Gronwall's inequality. This completes the proof of proposition 3.1. □

3.2. Bounds for $\|\omega\|_{L^p(\mathbb{R}^2)}$ *and* $\|b\|_{W^{2,p}(\mathbb{R}^2)}$ *with* $p \in [2, \infty)$

In order to obtain the desired global bounds, we need to use a regularization property involving the heat operator. Let $K_t(x) = (4\pi t)^{-\frac{d}{2}}\mathrm{e}^{-\frac{|x|^2}{4t}}$ and write

$$\mathrm{e}^{t\Delta} f = K_t(x) * f.$$

Then the following lemma holds (see, e.g. [21]).

Lemma 3.2. *(Maximal $L_t^q L_x^p$ regularity for the heat kernel) Define the operator A by*

$$Af \equiv \int_0^t \mathrm{e}^{(t-s)\Delta}\, \Delta f(s)\,\mathrm{d}s.$$

Let $p, q \in (1, \infty)$. Then A is bounded from $L^p(0, T; L^q(\mathbb{R}^d))$ to $L^p(0, T; L^q(\mathbb{R}^d))$ for every $T \in (0, \infty]$.

We are ready to prove the desired bounds.

Proposition 3.3. *Assume that (u_0, b_0) satisfies the conditions in theorem 1.1. Let $p \in [2, \infty)$ and $q \in (1, \infty)$. Then the corresponding solution (u, b) of (1.1) obeys, for any $T > 0$,*

$$\|w\|_{L^q(0,T;L^p)} \leqslant C, \qquad \|b\|_{L^q(0,T;W^{2,p})} \leqslant C,$$

where C is a constant depending on p, q, T and the initial data only.

Proof. We write the second equation in (1.1) as

$$b_t - \Delta b = \operatorname{div} f$$

with $f_i = b_i u - u_i b$ $(i = 1, 2)$. The global bound in proposition 3.1 and Sobolev's inequality indicate, for any $p \in [2, \infty)$,

$$f_i \in L^\infty(0, T; L^p).$$

Resorting to the heat kernel, we further write

$$b(x,t) = \mathrm{e}^{t\Delta} b_0 + \int_0^t \mathrm{e}^{(t-s)\Delta} \operatorname{div} f(s, \cdot)\mathrm{d}s. \tag{3.6}$$

For any $p \in [2, \infty)$ and $p' > 1$ satisfying $\frac{1}{p} + \frac{1}{p'} = 1$,

$$\begin{aligned}\|b\|_{L^\infty(0,T;L^\infty)} &\leqslant C(\|K_t\|_{L^\infty(0,T;L^1)}\|b_0\|_{L^\infty} + \|\nabla K_t\|_{L^1(0,T;L^{p'})}\|f\|_{L^\infty(0,T;L^p)})\\ &\leqslant C(\|b_0\|_{L^\infty} + \|f\|_{L^\infty(0,T;L^p)})\\ &\leqslant C(\|b_0\|_{L^\infty} + \|u\|_{L^\infty(0,T;L^{2p})}\|b\|_{L^\infty(0,T;L^{2p})})\\ &\leqslant C(\|b_0\|_{H^{s-1}} + \|u\|_{L^\infty(0,T;H^1)}\|b\|_{L^\infty(0,T;H^1)})\\ &\leqslant C,\end{aligned} \tag{3.7}$$

where C is a constant depending on p, T and the initial data only. By (3.6) and lemma 3.2,

$$\begin{aligned}\|\nabla b\|_{L^q(0,T;L^p)} &\leqslant C(\|K_t\|_{L^q(0,T;L^1)}\|\nabla b_0\|_{L^p} + \|f\|_{L^q(0,T;L^p)})\\ &\leqslant C(\|\nabla b_0\|_{L^p} + \|f\|_{L^\infty(0,T;L^p)})\\ &\leqslant C(\|b_0\|_{H^2} + \|u\|_{L^\infty(0,T;H^1)}\|b\|_{L^\infty(0,T;H^1)})\\ &\leqslant C.\end{aligned} \tag{3.8}$$

Multiplying (3.4) by $|\omega|^{p-2}\omega$ with $p \in [2, \infty)$, we obtain

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\|\omega\|_{L^p}^p = \int b \cdot \nabla j \cdot \omega|\omega|^{p-2} \leqslant \|b\|_{L^\infty}\|\nabla j\|_{L^p}\|\omega\|_{L^p}^{p-1}.$$

Therefore, for $q \in (1, \infty)$ and $\frac{1}{q'} + \frac{1}{q} = 1$,

$$\begin{aligned}\|\omega(t)\|_{L^p} &\leqslant \|\omega_0\|_{L^p} + \int_0^t \|b(s)\|_{L^\infty}\|\nabla j(s)\|_{L^p}\,\mathrm{d}s\\ &\leqslant C\|\omega_0\|_{L^p} + \|b\|_{L_t^{q'}L^\infty}\|\nabla j\|_{L_t^qL^p}\\ &\leqslant C(\|\omega_0\|_{L^p} + \|\Delta b\|_{L_t^qL^p}).\end{aligned} \tag{3.9}$$

In addition, applying Δ to (3.6) and using lemma 3.2, we have

$$\begin{aligned}\|\Delta b\|_{L_t^qL^p} &\leqslant C(\|\Delta b_0\|_{L^p} + \|b\cdot\nabla u - u\cdot\nabla b\|_{L_t^qL^p})\\ &\leqslant C\left(\|\Delta b_0\|_{L^p} + \|b\|_{L_t^\infty L^\infty}\|\omega\|_{L_t^qL^p} + \|u\|_{L_t^2L^{2p}}\|\nabla b\|_{L_t^{\frac{2q}{2-q}}L^{2p}}\right)\\ &\leqslant C\left(\|\Delta b_0\|_{L^p} + \|b\|_{L_t^\infty L^\infty}\|\omega\|_{L_t^qL^p} + \|u\|_{L_t^2H^1}\|\nabla b\|_{L_t^{\frac{2q}{2-q}}L^{2p}}\right).\end{aligned} \tag{3.10}$$

By inserting (3.10) into (3.9) and applying the Gronwall inequality, we obtain the desired bound for $\|w\|_{L^q(0,T;L^p)}$. By Sobolev embedding, $u\in L^q(0,T;L^\infty)$. As a consequence,

$$\begin{aligned}\|\mathrm{div}\, f\|_{L^q(0,T;L^p)} &\leqslant \|u\cdot\nabla b\|_{L^q(0,T;L^p)} + \|b\cdot\nabla u\|_{L^q(0,T;L^p)}\\ &\leqslant \|u\|_{L^{2q}(0,T;L^\infty)}\|j\|_{L^{2q}(0,T;L^p)} + \|b\|_{L^{2q}(0,T;L^\infty)}\|\omega\|_{L^{2q}(0,T;L^p)}\\ &\leqslant C.\end{aligned}$$

By (3.6) and lemma 3.2 again, we obtain, for any $p\in[2,\infty)$, $q\in(1,\infty)$,

$$\begin{aligned}\|\Delta b\|_{L^q(0,T;L^p)} &\leqslant C(\|\Delta b_0\|_{L^p} + \|\mathrm{div}\, f\|_{L^q(0,T;L^p)})\\ &\leqslant C.\end{aligned} \tag{3.11}$$

This completes the proof of proposition 3.3. □

We also need the global bound in the following proposition.

Proposition 3.4. *Assume that (u_0, b_0) satisfies the conditions in theorem 1.1. Let (u, b) be the corresponding solution of (1.1). Let $p\in[2,\infty)$. Then, for any $T>0$,*

$$\|j\|_{L^\infty(0,T;L^p)} \leqslant C, \qquad \|w\|_{L^\infty(0,T;L^p)} \leqslant C,$$

where C depends on p, T and the initial data.

Proof. Multiplying (3.5) by $|j|^{p-2}j$, we obtain after integration by parts

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\|j\|_{L^p}^p + (p-1)\||j|^{\frac{p-2}{2}}|\nabla j|\|_{L^2}^2 = K_1 + K_2, \tag{3.12}$$

where

$$K_1 = \int b\cdot\nabla\omega\, j|j|^{p-2},$$

$$K_2 = \int Q(\nabla u, \nabla b) j|j|^{p-2}.$$

The estimate for K_1 is given by

$$\begin{aligned} K_1 &= -(p-1)\int b\cdot\nabla j|j|^{p-2}\omega \\ &\leqslant (p-1)\|b\|_{L^\infty}\||j|^{\frac{p-2}{2}}|\nabla j|\|_{L^2}\|\omega\|_{L^p}\|j\|_{L^p}^{\frac{p-2}{2}} \\ &\lesssim \frac{p-1}{4}\||j|^{\frac{p-2}{2}}|\nabla j|\|_{L^2}^2+(p-1)\|b\|_{L^\infty}^2\|\omega\|_{L^p}^2\|j\|_{L^p}^{p-2}, \end{aligned} \tag{3.13}$$

where $A\lesssim B$ means $A\leqslant C\,B$ for a constant C. By integration by parts,

$$\begin{aligned} \int\partial_1 b_1\partial_1 u_2\, j|j|^{p-2} &= -\int u_2\partial_{11}b_1 j|j|^{p-2}-(p-1)\int u_2\partial_1 b_1|j|^{p-2}\partial_1 j, \\ \int\partial_1 b_1\partial_2 u_1\, j|j|^{p-2} &= -\int u_1\partial_{12}b_1 j|j|^{p-2}-(p-1)\int u_1\partial_1 b_1|j|^{p-2}\partial_2 j, \\ -\int\partial_1 b_2\partial_1 u_1\, j|j|^{p-2} &= \int u_1\partial_{11}b_2 j|j|^{p-2}+(p-1)\int u_1\partial_1 b_2|j|^{p-2}\partial_1 j, \\ -\int\partial_2 b_1\partial_1 u_1\, j|j|^{p-2} &= \int u_1\partial_{12}b_1 j|j|^{p-2}+(p-1)\int u_1\partial_2 b_1|j|^{p-2}\partial_1 j. \end{aligned}$$

Therefore,

$$\begin{aligned} K_2 = {} &(p-1)\int u_1\partial_1 b_2\partial_1 j|j|^{p-2}+(p-1)\int u_1\partial_2 b_1\partial_1 j|j|^{p-2} \\ &-(p-1)\int u_1\partial_1 b_1\partial_2 j|j|^{p-2}-(p-1)\int u_2\partial_1 b_1\partial_1 j|j|^{p-2} \\ &+\int u\cdot\nabla(\partial_1 b_2)|j|^{p-2}j. \end{aligned}$$

Integrating by parts in the last term yields

$$\begin{aligned} K_2 = {} &-(p-1)\int u_2\partial_1 b_1\partial_1 j|j|^{p-2}-(p-1)\int u_1\partial_1 b_1\partial_2 j|j|^{p-2} \\ &+(p-1)\int u_1\partial_2 b_1\partial_1 j|j|^{p-2}-(p-1)\int u_2\partial_1 b_2\partial_2 j|j|^{p-2}. \end{aligned}$$

Therefore, by the Hölder and the Young inequalities,

$$\begin{aligned} K_2 &\leqslant 4(p-1)\|u\|_{L^\infty}\||j|^{\frac{p-2}{2}}|\nabla j|\|_{L^2}\|\nabla b\|_p\|j\|_{L^p}^{\frac{p-2}{2}} \\ &\lesssim 4(p-1)\|u\|_{L^\infty}\||j|^{\frac{p-2}{2}}|\nabla j|\|_{L^2}\|j\|_p\|j\|_{L^p}^{\frac{p-2}{2}} \\ &\lesssim \frac{p-1}{4}\||j|^{\frac{p-2}{2}}|\nabla j|\|_{L^2}^2+C(p-1)\|u\|_{L^\infty}^2\|j\|_{L^p}^p\,. \end{aligned} \tag{3.14}$$

Inserting (3.13) with (3.14) in (3.12), we obtain

$$\begin{aligned} &\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\|j\|_{L^p}^p+(p-1)\||j|^{\frac{p-2}{2}}|\nabla j|\|_{L^2}^2 \\ &\qquad\lesssim (p-1)\|b\|_{L^\infty}^2\|\omega\|_{L^p}^2\|j\|_{L^p}^{p-2}+(p-1)\|u\|_{L^\infty}^2\|j\|_{L^p}^p \\ &\qquad\lesssim (p-1)\|b\|_{L^\infty}^2\|\omega\|_{L^p}^p+(p-1)(\|u\|_{L^\infty}^2+\|b\|_{L^\infty}^2)\|j\|_{L^p}^p\,. \end{aligned}$$

Gronwall's inequality yields the desired result. The bound

$$\|w\|_{L^\infty(0,T;L^p)} \leqslant C$$

follows from the vorticity equation and the bound for j. This proves proposition 3.4. □

4. Proof of theorem 1.1

This section proves theorem 1.1. The main effort is devoted to proving a global *a priori* bound for $\|\omega\|_{B^0_{\infty,1}}$. Due to the lack of a global bound on ∇j in L^∞, the global bound for $\|\omega\|_{L^\infty}$ does not follow directly from the vorticity equation

$$\omega_t + u\cdot\nabla\omega = \text{curl div}\,(b\otimes b). \tag{4.1}$$

To overcome this difficulty, we consider the combined quantity

$$G = \omega + \mathcal{R}(b\otimes b), \qquad \mathcal{R} = (-\Delta)^{-1}\text{curl div} \tag{4.2}$$

in order to eliminate the regularity-demanding term curl div$(b\otimes b)$ in (4.1). More details will be provided in the following proof.

Proof of theorem 1.1. As mentioned previously, a key step is to control $\|\omega\|_{B^0_{\infty,1}}$ and this is achieved through the consideration of a new quantity G, as defined in (4.2). We first derive the equation for G. Multiplying the i-th component of the magnetic equation by b_j, we have

$$(b_i)_t b_j + u\cdot\nabla b_i b_j = b\cdot\nabla u_i b_j + \Delta b_i b_j. \tag{4.3}$$

Similarly,

$$(b_j)_t b_i + u\cdot\nabla b_j b_i = b\cdot\nabla u_j b_i + \Delta b_j b_i. \tag{4.4}$$

Adding (4.3) to (4.4), the (i, j)-th component of $b\otimes b$ satisfies

$$(b_ib_j)_t + u\cdot\nabla(b_ib_j) = (b_ib_j)\,(\nabla u)^\intercal + \nabla u\,(b_ib_j) + \Delta(b_ib_j) - 2\sum_{k=1}^{2}\partial_k b_i\partial_k b_j,$$

or simply

$$(b\otimes b)_t + u\cdot\nabla(b\otimes b) = \nabla u\,(b\otimes b) + (b\otimes b)\,(\nabla u)^\intercal + \Delta(b\otimes b) - 2\sum_{k=1}^{2}(\partial_k b\otimes\partial_k b). \tag{4.5}$$

Applying $\mathcal{R} = (-\Delta)^{-1}$curl div to (4.5) yields to

$$\begin{aligned}((\mathcal{R}(b\otimes b))_t + u\cdot\nabla\mathcal{R}(b\otimes b) &= -[\mathcal{R}, u\cdot\nabla](b\otimes b)\\ &+\mathcal{R}(\nabla u\,(b\otimes b) + (b\otimes b)\,(\nabla u)^\intercal) - \text{curl div}\,(b\otimes b) - 2\sum_{k=1}^{2}\mathcal{R}(\partial_k b\otimes\partial_k b).\end{aligned} \tag{4.6}$$

Adding (4.6)–(4.1) and setting $G = \omega + \mathcal{R}(b\otimes b)$, we get

$$\begin{aligned}\partial_t G + u\cdot\nabla G = &-[\mathcal{R}, u\cdot\nabla](b\otimes b) - 2\sum_{k=1}^{2}\mathcal{R}(\partial_k b\otimes\partial_k b)\\ &+\mathcal{R}(\nabla u\,(b\otimes b)) + \mathcal{R}((b\otimes b)\,(\nabla u)^\intercal).\end{aligned} \tag{4.7}$$

According to lemma 2.5,

$$\|G\|_{L_t^\infty B^0_{\infty,1}} \leqslant C\,(\|G_0\|_{B^0_{\infty,1}} + \|f\|_{L_t^1 B^0_{\infty,1}})\left(1 + \int_0^t \|\nabla u\|_{L^\infty}\,\mathrm{d}\tau\right), \tag{4.8}$$

where

$$f = -[\mathcal{R}, u\cdot\nabla](b\otimes b) - 2\sum_{k=1}^{2}\mathcal{R}(\partial_k b\otimes\partial_k b) + \mathcal{R}((b\otimes b)(\nabla u)^{\mathrm{T}}) + \mathcal{R}(\nabla u(b\otimes b)).$$

If

$$\|f\|_{L_t^1 B^0_{\infty,1}} < \infty, \tag{4.9}$$

then we would be able to obtain a global bound for $\|\omega\|_{B^0_{\infty,1}}$, which would imply global regularity. In fact, if we have (4.8) with (4.9), then

$$\begin{aligned}\|\omega\|_{B^0_{\infty,1}} &\leqslant \|G\|_{B^0_{\infty,1}} + \|b\otimes b\|_{B^0_{\infty,1}} \leqslant \|b\otimes b\|_{B^0_{\infty,1}} + C\left(1 + \int_0^t \|\nabla u\|_{L^\infty}\,\mathrm{d}\tau\right)\\ &\leqslant \|b\otimes b\|_{B^0_{\infty,1}} + C\left(1 + \int_0^t (\|u\|_{L^2} + \|\omega\|_{B^0_{\infty,1}})\,\mathrm{d}\tau\right).\end{aligned}$$

Gronwall's inequality and $\|b\otimes b\|_{B^0_{\infty,1}} \leqslant \|b\ \|^2_{B^\epsilon_{\infty,1}} < \infty$ $(0<\epsilon<1)$ then imply that

$$\|\omega\|_{B^0_{\infty,1}} < \infty$$

and then higher regularities follow.

It then suffices to check (4.9). The terms in f can be estimated as follows. By the commutator estimate in lemma 2.4, for $s\in(0,1)$,

$$\|[\mathcal{R}, u\cdot\nabla](b\otimes b)\|_{B^0_{\infty,1}} \leqslant C\,\|\Lambda^s u\|_{B^0_{\infty,1}}\,\|\Lambda^{1-s}(b\otimes b)\|_{B^0_{\infty,1}}.$$

For any $s\in(0,1)$, $\|\Lambda^s u\|_{B^0_{\infty,1}}$ can be bounded by $\|\omega\|_{L^q}$ for some large $q\in(2,\infty)$. In fact, by Bernstein's inequality,

$$\begin{aligned}\|\Lambda^s u\|_{B^0_{\infty,1}} &\leqslant \|\Delta_{-1}\Lambda^s u\|_{L^\infty} + \sum_{k=0}^{\infty}\|\Delta_k\Lambda^s u\|_{L^\infty}\\ &\leqslant C\|u\|_{L^2} + C\sum_{k=0}^{\infty} 2^{(s-1)k}\,\|\Delta_k\omega\|_{L^\infty}\\ &\leqslant C\|u\|_{L^2} + C\sum_{k=0}^{\infty} 2^{\left(s-1+\frac{2}{q}\right)k}\,\|\Delta_k\omega\|_{L^q}\\ &\leqslant C\|u\|_{L^2} + C\,\|\omega\|_{L^q}\sum_{k=0}^{\infty} 2^{\left(s-1+\frac{2}{q}\right)k}.\end{aligned}$$

Therefore, if we choose $q\in(2,\infty)$ such that $s-1+\frac{2}{q}<0$, then

$$\|\Lambda^s u\|_{B^0_{\infty,1}} \leqslant C(\|u\|_{L^2} + \|\omega\|_{L^q}) < \infty.$$

The regularity of b also implies that, for $s \in (0,1)$ close to 1,

$$\|\Lambda^{1-s}(b\otimes b)\|_{B^0_{\infty,1}} < \infty.$$

In fact, in a similar fashion as above, if $\tilde{q} \in (2,\infty)$ such that $-s+\epsilon+\frac{2}{\tilde{q}}<0$

$$\|\Lambda^{1-s}(b\otimes b)\|_{B^0_{\infty,1}} \leqslant C\,\|\Lambda^{1-s}b\,\|^2_{B^\epsilon_{\infty,1}} \leqslant C\,(\|b\,\|^2_{L^2}+\|\nabla b\,\|^2_{L^{\tilde{q}}}).$$

Therefore,

$$\|[\mathcal{R},u\cdot\nabla](b\otimes b)\|_{L^1_tB^0_{\infty,1}} \leqslant C\,(\|u\|_{L^\infty_tL^2}+\|\omega\|_{L^\infty_tL^q})\,(\|b\,\|^2_{L^2_tL^2}+\|\nabla b\,\|^2_{L^2_tL^{\tilde{q}}})<\infty,$$

where we have used the bound in proposition 3.3. We now estimate the second term in f. For any $\epsilon>0$,

$$\left\|\sum_{k=1}^{2}\mathcal{R}(\partial_k b\otimes\partial_k b)\right\|_{B^0_{\infty,1}} \leqslant \|\nabla b\,\|^2_{B^\epsilon_{\infty,1}}.$$

By Bernstein's inequality,

$$\begin{aligned}
\|\nabla b\|_{B^\epsilon_{\infty,1}} &\leqslant C\|b\|_{L^2}+C\sum_{k=0}^{\infty}2^{\epsilon k}\,\|\Delta_k\nabla b\|_{L^\infty}\\
&\leqslant C\|b\|_{L^2}+C\sum_{k=0}^{\infty}2^{k\left(\epsilon+1+\frac{2}{q}\right)}\|\Delta_k b\|_{L^q}\\
&\leqslant C\|b\|_{L^2}+C\sum_{k=0}^{\infty}2^{k\left(\epsilon+1+\frac{2}{q}-\gamma\right)}\|\Lambda^\gamma\Delta_k b\|_{L^q}\\
&\leqslant C\|b\|_{L^2}+C\,\|\Lambda^\gamma b\|_{L^q},
\end{aligned}$$

where $\epsilon+1+\frac{2}{q}<\gamma<2$. Therefore, by proposition 3.3,

$$\left\|\sum_{k=1}^{2}\mathcal{R}(\partial_k b\otimes\partial_k b)\right\|_{L^1_tB^0_{\infty,1}} \leqslant \|\nabla b\|^2_{L^2_tB^\epsilon_{\infty,1}} \leqslant C\,\|b\|^2_{L^2_tL^2}+C\,\|\Lambda^\gamma b\|^2_{L^2_tL^q}<\infty.$$

We bound the last two terms in f. Their estimates are similar and we shall handle one of them. By Bernstein's inequality,

$$\begin{aligned}
\|\mathcal{R}(\nabla u(b\otimes b))\|_{B^0_{\infty,1}} &\leqslant C\,\|\Delta_{-1}(\nabla u(b\otimes b))\|_{L^2}+\sum_{k=0}^{\infty}\|\Delta_k(\nabla u(b\otimes b))\|_{L^\infty}\\
&\leqslant C\,\|\omega\|_{L^2}\,\|b\|^2_{L^\infty}+\sum_{k=0}^{\infty}\|\Delta_k(\nabla u(b\otimes b))\|_{L^\infty}.
\end{aligned}$$

Following the notion of paraproducts, we write

$$\begin{aligned}
\Delta_k(\nabla u(b\otimes b)) = &\sum_{|k-m|\leqslant 2}\Delta_k(S_{m-1}\nabla u\,\Delta_m(b\otimes b))+\sum_{|k-m|\leqslant 2}\Delta_k(\Delta_m\nabla u\,S_{m-1}(b\otimes b))\\
&+\sum_{m\geqslant k-1}\Delta_k(\Delta_m\nabla u\widetilde{\Delta}_m(b\otimes b)),
\end{aligned}\tag{4.10}$$

where $\widetilde{\Delta}_m = \Delta_{m+1} + \Delta_m + \Delta_{m-1}$. By Bernstein's inequality,

$$\begin{aligned}\sum_{|k-m|\leqslant 2} \|\Delta_k(S_{m-1}\nabla u\, \Delta_m(b\otimes b))\|_{L^\infty} &\leqslant \sum_{|k-m|\leqslant 2} \|S_{m-1}\nabla u\|_{L^\infty}\, \|\Delta_m(b\otimes b)\|_{L^\infty}\\ &\leqslant C \sum_{|k-m|\leqslant 2} 2^{\frac{2}{q}m}\, \|S_{m-1}\omega\|_{L^q}\, \|\Delta_m(b\otimes b)\|_{L^\infty}\\ &\leqslant C\, \|\omega\|_{L^q} \sum_{|k-m|\leqslant 2} 2^{\frac{2}{q}m}\, \|\Delta_m(b\otimes b)\|_{L^\infty}.\end{aligned}$$

By Bernstein's inequality and the Hardy-Littlewood-Sobolev inequality, the third term in (4.10) can be bounded by

$$\begin{aligned}\sum_{m\geqslant k-1} \|\Delta_k(\Delta_m\nabla u\widetilde{\Delta}_m(b\otimes b))\|_{L^\infty} &= \sum_{m\geqslant k-1} \|\Delta_k\Lambda^{\frac{2}{q}}\Lambda^{-\frac{2}{q}}(\Delta_m\nabla u\widetilde{\Delta}_m(b\otimes b))\|_{L^\infty}\\ &\leqslant \sum_{m\geqslant k-1} 2^{\frac{2}{q}k}\, \|\Lambda^{-\frac{2}{q}}(\Delta_m\nabla u\widetilde{\Delta}_m(b\otimes b))\|_{L^\infty}\\ &\leqslant C \sum_{m\geqslant k-1} 2^{\frac{2}{q}k}\, \|\Delta_m\nabla u\|_{L^q}\, \|\widetilde{\Delta}_m(b\otimes b)\|_{L^\infty}\\ &\leqslant C\, \|\omega\|_{L^q} \sum_{m\geqslant k-1} 2^{\frac{2}{q}(k-m)}\, 2^{\frac{2}{q}m}\, \|\widetilde{\Delta}_m(b\otimes b)\|_{L^\infty}.\end{aligned}$$

The condition (1.3) is needed to handle the second term in (4.10). As in the estimate of the first term, we have

$$\sum_{|k-m|\leqslant 2} \|\Delta_k(\Delta_m\nabla u\, S_{m-1}(b\otimes b))\|_{L^\infty} \leqslant C\, \|\omega\|_{L^q} \sum_{|k-m|\leqslant 2} 2^{\frac{2}{q}m}\, \|S_{m-1}(b\otimes b)\|_{L^\infty}.$$

Combining the estimates above, we have

$$\begin{aligned}\|\mathcal{R}(\nabla u(b\otimes b))\|_{B^0_{\infty,1}} &\leqslant C\, \|\omega\|_{L^2}\, \|b\|^2_{L^\infty} + C\, \|\omega\|_{L^q} \sum_{k\geqslant 0}\sum_{|k-m|\leqslant 2} 2^{\frac{2}{q}m}\, \|\Delta_m(b\otimes b)\|_{L^\infty}\\ &\quad + C\, \|\omega\|_{L^q} \sum_{k\geqslant 0}\sum_{m\geqslant k-1} 2^{\frac{2}{q}(k-m)}\, 2^{\frac{2}{q}m}\, \|\widetilde{\Delta}_m(b\otimes b)\|_{L^\infty}\\ &\quad + C\, \|\omega\|_{L^q} \sum_{k\geqslant 0}\sum_{|k-m|\leqslant 2} 2^{\frac{2}{q}m}\, \|S_{m-1}(b\otimes b)\|_{L^\infty}\\ &\leqslant C\, \|\omega\|_{L^2}\, \|b\|^2_{L^\infty} + C\, \|\omega\|_{L^q}\, \|b\|_{L^\infty}\, \|b\|_{B^{\frac{2}{q}}_{\infty,1}}\\ &\quad + C\, \|\omega\|_{L^q} \sum_{k\geqslant 0} 2^{\frac{2}{q}k}\, \|S_{k-1}(b\otimes b)\|_{L^\infty}. \qquad (4.11)\end{aligned}$$

We provide some details for the last inequality, namely

$$\sum_{k\geqslant 0}\sum_{|k-m|\leqslant 2} 2^{\frac{2}{q}m}\, \|\Delta_m(b\otimes b)\|_{L^\infty} \leqslant C\, \|b\|_{L^\infty}\, \|b\|_{B^{\frac{2}{q}}_{\infty,1}}, \qquad (4.12)$$

$$\sum_{k\geqslant 0}\sum_{m\geqslant k-1} 2^{\frac{2}{q}(k-m)}\, 2^{\frac{2}{q}m}\, \|\widetilde{\Delta}_m(b\otimes b)\|_{L^\infty} \leqslant C\, \|b\|_{L^\infty}\, \|b\|_{B^{\frac{2}{q}}_{\infty,1}}. \qquad (4.13)$$

In fact, by the paraproduct decomposition,

$$\begin{aligned}\sum_{k\geqslant 0}\sum_{|k-m|\leqslant 2} 2^{\frac{2}{q}m}\,\|\Delta_m(b\otimes b)\|_{L^\infty} \leqslant\ & C\sum_{k\geqslant 0}2^{\frac{2}{q}k}\sum_{|k-l|\leqslant 2}\|\Delta_k(S_{l-1}b\otimes\Delta_l b)\|_{L^\infty}\\ &+C\sum_{k\geqslant 0}2^{\frac{2}{q}k}\sum_{|k-l|\leqslant 2}\|\Delta_k(\Delta_l b\otimes S_{l-1}b)\|_{L^\infty}\\ &+C\sum_{k\geqslant 0}2^{\frac{2}{q}k}\sum_{l\geqslant k-1}\|\Delta_k(\Delta_l b\otimes\widetilde{\Delta}_l b)\|_{L^\infty}\\ \leqslant\ & C\sum_{k\geqslant 0}2^{\frac{2}{q}k}\,\|b\|_{L^\infty}\,\|\Delta_k b\|_{L^\infty}\\ &+C\sum_{k\geqslant 0}\sum_{l\geqslant k-1}2^{\frac{2}{q}(k-l)}\|b\|_{L^\infty}\,2^{\frac{2}{q}l}\|\Delta_l b\|_{L^\infty}\\ \leqslant\ & C\,\|b\|_{L^\infty}\,\|b\|_{B^{\frac{2}{q}}_{\infty,1}}.\end{aligned}$$

This proves (4.12). The proof of (4.13) is similar. According to proposition 3.3, the first two terms in (4.11) are time integrable if q is sufficiently large. Due to (1.3), the third term is also time integrable if we choose q large enough, say $\frac{2}{q}<\sigma$. Therefore we have proven (4.9). This completes the proof of theorem 1.1. □

5. Proofs of theorems 1.2 and 1.3

To prove theorem 1.2, we recall the following abstract bootstrap argument or continuity argument (see, e.g. Tao [30, p 20]).

Lemma 5.1. *Let $T>0$. Assume that two statements $C(t)$ and $H(t)$ with $t\in[0,T]$ satisfy the following conditions:*

(a) If $H(t)$ holds for some $t\in[0,T]$, then $C(t)$ holds for the same t;
(b) If $C(t)$ holds for some $t_0\in[0,T]$, then $H(t)$ holds for t in a neighborhood of t_0;
(c) If $C(t)$ holds for $t_m\in[0,T]$ and $t_m\to t$, then $C(t)$ holds;
(d) $C(t)$ holds for at least one $t_1\in[0,T]$.

Then $C(t)$ holds for all $t\in[0,T]$.

Proof of theorem 1.2. We use the bootstrap argument. Let $\gamma>0$ be a fixed large number, say

$$\gamma>2\,\|b_0\|_{H^s}.$$

Denote by $H(t)$ the statement that, for $t\in[0,T]$,

$$\|b\|_{L^\infty([0,t];H^s)}+\eta\|b\|_{L^2([0,t];H^{s+1})}\leqslant\gamma \tag{5.1}$$

and $C(t)$ the statement that

$$\|b\|_{L^\infty([0,t];H^s)}+\eta\|b\|_{L^2([0,t];H^{s+1})}\leqslant\frac{\gamma}{2}. \tag{5.2}$$

The conditions (b)–(d) in lemma 5.1 are clearly true and it remains to verify (a) under the smallness condition (1.7). Once this is verified, then the bootstrap argument would imply that $C(t)$, or (5.2) actually holds for any $t\in[0,T]$.

First it is not difficult to show that (5.1) implies

$$\|u\|_{L^\infty([0,t];H^s)} \leqslant C_0(\|u_0\|_{H^s}, \gamma, T, \eta), \tag{5.3}$$

where C_0 will be made explicit later. It follows from the vorticity equation that the vorticity is bounded. In fact,

$$\partial_t \omega + u \cdot \nabla \omega = b \cdot \nabla j$$

yields

$$\begin{aligned} \|\omega(\cdot, t)\|_{L^\infty} &\leqslant \|\omega_0\|_{L^\infty} + \int_0^t \|b \cdot \nabla j\|_{L^\infty} \, d\tau \\ &\leqslant \|u_0\|_{H^s} + \|b\|_{L_t^\infty L^\infty} \int_0^t \|b\|_{H^{s+1}} \, d\tau \\ &\leqslant \|u_0\|_{H^s} + \|b\|_{L_t^\infty H^s} \sqrt{t} \, \|b\|_{L^2([0,t];H^{s+1})} \\ &\leqslant \|u_0\|_{H^s} + \frac{1}{\eta} \sqrt{T} \, \gamma^2 \equiv C_1. \end{aligned} \tag{5.4}$$

A standard H^s-estimate involving the velocity equation yields

$$\partial_t \|u\|_{H^s}^2 \leqslant C \, \|\nabla u\|_{L^\infty} \|u\|_{H^s}^2 + C \|u\|_{H^s} \|b \cdot \nabla b\|_{H^s},$$

or

$$\partial_t \|u\|_{H^s} \leqslant C \, \|\nabla u\|_{L^\infty} \|u\|_{H^s} + C \|b\|_{H^s} \, \|b\|_{H^{s+1}}.$$

Inserting the logarithmic inequality

$$\|\nabla u\|_{L^\infty} \leqslant C(1 + \|u\|_{L^2} + \|\omega\|_{L^\infty} \log(e + \|u\|_{H^s}))$$

and invoking Gronwall's inequality yield (5.3),

$$\begin{aligned} \|u\|_{L^\infty([0,t];H^s)} &\leqslant C_0(\|u_0\|_{H^s}, \gamma, T) \\ &\equiv \exp(\exp(CC_1 T)(C(1 + \|u_0\|_{L^2})T + \gamma^2 \sqrt{T} + \log(e + \|u_0\|_{H^s}))), \end{aligned}$$

where C_1 is specified in (5.4) and C is a pure constant.

We write $J^\sigma = (I - \Delta)^{\sigma/2}$ and recall that $\|f\|_{H^\sigma} \equiv \|J^\sigma f\|_{L^2}$. Applying J^s to the equation of b

$$b_t + u \cdot \nabla b = b \cdot \nabla u + \eta \Delta b$$

and taking the inner product with $J^s b$, we have, by $\nabla \cdot u = 0$,

$$\frac{1}{2} \frac{d}{dt} \|b\|_{H^s}^2 + \eta \|b\|_{H^{s+1}}^2 \leqslant C \, \|J^s(ub)\|_{L^2} \, \|b\|_{H^{s+1}}.$$

By the standard product estimate and the Sobolev embedding,

$$\|J^s(ub)\|_{L^2} \leqslant C \, \|u\|_{H^s} \, \|b\|_{L^\infty} + C \, \|b\|_{H^s} \, \|u\|_{L^\infty} \leqslant C \, \|u\|_{H^s} \, \|b\|_{H^s}.$$

Therefore,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|b\|_{H^s}^2 + \eta\|b\|_{H^{s+1}}^2 \leqslant C\,\|u\|_{H^s}\,\|b\|_{H^s}\,\|b\|_{H^{s+1}} \leqslant \frac{\eta}{2}\|b\|_{H^{s+1}}^2 + C\,\|u\|_{H^s}^2\,\|b\|_{H^s}^2\,.$$

By Gronwall's inequality,

$$\|b(\cdot,t)\|_{H^s} \leqslant \|b_0\|_{H^s}\exp\!\left(C\int_0^t \|u(\cdot,\tau)\|_{H^s}^2\,\mathrm{d}\tau\right).$$

Therefore, by (5.3),

$$\|b(\cdot,t)\|_{H^s} \leqslant \|b_0\|_{H^s}\,C(\|u_0\|_{H^s},\gamma,T,\eta)$$

and

$$\|b(\cdot,t)\|_{H^s} + \eta\|b\|_{L_t^2H^{s+1}} \leqslant \|b_0\|_{H^s}\,C(\|u_0\|_{H^s},\gamma,T,\eta).$$

It is then clear that we can choose sufficiently small $\delta = \delta(T)$ such that

$$\|b\|_{L_t^\infty H^s} + \eta\|b\|_{L_t^2H^{s+1}} \leqslant \frac{\gamma}{2}.$$

This completes the proof of theorem 1.2. □

We now turn to the proof of theorem 1.3.

Proof of theorem 1.3. When the magnetic diffusion term is not present in (1.1), the system is inviscid. We need the smallness of both u_0 and b_0. The proof is proceeded slightly differently from the previous proof. We still use the the bootstrap argument. We set $\gamma > 0$ to be a fixed number satisfying

$$\gamma > 2\|(u_0,b_0)\|_{H^s}$$

and assume that

$$\|u\|_{L^\infty([0,t];H^s)} \leqslant \gamma, \qquad \|b\|_{L^\infty([0,t];H^s)} \leqslant \gamma, \quad t\in[0,T].$$

In particular, by the Sobolev embedding, for $s > 2$,

$$\|\nabla u\|_{L^\infty([0,t];L^\infty)} \leqslant C\,\gamma, \qquad \|\nabla b\|_{L^\infty([0,t];L^\infty)} \leqslant C\,\gamma, \quad t\in[0,T].$$

It then from (1.1) with $\eta = 0$ via standard energy estimates that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|(u,b)\|_{H^s}^2 \leqslant C\,(\|\nabla u\|_{L^\infty} + \|\nabla b\|_{L^\infty})\,\|(u,b)\|_{H^s}^2\,.$$

Consequently,

$$\|(u,b)(t)\|_{H^s}^2 \leqslant \|(u_0,b_0)\|_{H^s}^2\;\mathrm{e}^{C\int_0^t(\|\nabla u\|_{L^\infty}+\|\nabla b\|_{L^\infty})\,\mathrm{d}\tau}.$$

It is then clear that we can choose $\delta = \delta(T)$ such that, if $\|(u_0,b_0)\|_{H^s} < \delta$,

$$\|(u,b)(t)\|_{H^s} \leqslant \frac{\gamma}{2}.$$

The bootstrap argument then implies that the inequality above holds for any $t \in [0, T]$. This completes the proof of theorem 1.3. □

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