

THE TYPE DEFECT OF A SIMPLICIAL COMPLEX

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ABSTRACT. Fix a field k . When Δ is a simplicial complex on n vertices with Stanley-Reisner ideal I_Δ , we define and study an invariant called the *type defect* of Δ . Except when Δ is of a single simplex, the type defect of Δ , $\text{td}(\Delta)$, is the difference $\dim_k \text{Tor}_c^S(S/I_\Delta, k) - c$, where c is the codimension of Δ and $S = k[x_1, \dots, x_n]$. We show that this invariant admits surprisingly nice properties. For example, it is well-behaved when one glues two complexes together along a face. Furthermore, Δ is Cohen-Macaulay if $\text{td}(\Delta) \leq 0$. On the other hand, if Δ is a simple graph (viewed as a one-dimensional complex), then $\text{td}(\Delta') \geq 0$ for every induced subgraph Δ' of Δ if and only if Δ is chordal. Requiring connected induced subgraphs to have type defect zero allows us to define a class of graphs that we call *treeish*, and which we generalize to simplicial complexes. We then extend some of our chordality results to higher dimensions, proving sharp lower bounds for most Betti numbers of ideals with linear resolution, and classifying when equalities occur. As an application, we prove sharp lower bounds for Betti numbers of graded ideals (not necessarily monomial) with linear resolution.

1. INTRODUCTION

Let k be a field, let $S = k[x_1, x_2, \dots, x_n]$, and let I be a homogenous ideal such that S/I is Cohen-Macaulay. The *type* of the quotient S/I is the total Betti number $b_c(S/I) := \dim_k \text{Tor}_c^S(S/I, k)$, where c is the codimension of I . That is, the type of I is the rank of the free module at position c in a minimal free resolution of S/I over S . In the local setting, this invariant is rather classical. In this paper we study some variants of this concept for (not necessarily) Cohen-Macaulay square-free monomial ideals. Roughly speaking, we shall show that these invariants behave very well with respect to the “gluing” of simplicial complexes, and that it is strongly linked to important and much-studied properties of complexes such as chordality, linear resolution, or Cohen-Macaulayness.

Let Δ be a $(d - 1)$ dimensional simplicial complex with n vertices. Throughout, let $c = n - d$, the codimension of Δ . Our key definitions are as follows.

Definition 1.1. We define the *modified type* of Δ to be:

$$\text{mtype}(\Delta) = \dim_k \text{Tor}_c^S(S/I_\Delta, k) - \dim_k \text{Tor}_c^S(S/I_\Delta, k)_c$$

where I_Δ is the Stanley-Reisner ideal of Δ and M_c denotes the degree c component of a graded module M . We also define the *type defect* of Δ , $\text{td}(\Delta)$, to be the difference

between its modified type and codimension:

$$\text{td}(\Delta) = \text{mtype}(\Delta) - c.$$

Except in the case when Δ has one facet, $\text{mtype}(\Delta)$ equals $b_c(S/I)$, see 2.2. However, this small modification (see Section 2) actually makes the results and proofs much more elegant. We summarize some of the key statements below (see Theorems 2.4, 3.3 and Corollary 2.8 for details):

Theorem 1.2. *Let Δ be a $(d - 1)$ -simplicial complex with n vertices.*

- (1) *(Cohen-Macaulayness) If $\text{td}(\Delta) \leq 0$ then Δ is Cohen-Macaulay (over k).*
- (2) *(Chordality) Assume that Δ is a graph with vertex set V . Then Δ is chordal if and only if $\text{td}(\Delta|_W) \geq 0$ for any $W \subseteq V$.*
- (3) *(Gluing) Suppose that Δ is obtained by gluing two Cohen-Macaulay complexes Δ_1 and Δ_2 of dimension $(d - 1)$ along an $(\ell - 1)$ -dimensional face. Then*

$$\text{td}(\Delta) = \text{td}(\Delta_1) + \text{td}(\Delta_2) - d + \ell + \binom{n - \ell}{n - d + 1}.$$

In particular if $\ell \geq d - 1$, then

$$\text{td}(\Delta) = \text{td}(\Delta_1) + \text{td}(\Delta_2).$$

In the second half of this work, we focus on extending the chordality part of Theorem 1.2 to complexes of higher dimensions. The main difficulty here is that there are many proposed definitions of chordality in general (for an incomplete list, see [2], [3], [8], [10], [14]). Thus we focus our attention to complexes with a *linear resolution*, a property that often appears whatever definition of chordality one uses. Here we are able to prove that for such a complex, $\text{td}(\Delta) \geq 0$. In fact, we prove a number of lower bounds for *all* the Betti numbers of S/I_Δ up to the codimension and characterize when any of these inequalities become equality. The full result reads:

Theorem 1.3. *Let Δ be a simplicial complex such that $I = I_\Delta$ has a linear resolution. Let c be the codimension of I and let s be its generating degree (the degree of any minimal generator of I , or the size of any minimal non-face). Then for each j such that $1 \leq j \leq c$ we have*

$$b_j(S/I_\Delta) \geq \binom{s + j - 2}{j - 1} \sum_{i=j-1}^{c-1} \binom{s + i - 1}{s + j - 2}.$$

Furthermore, the following are equivalent:

- (1) *Equality occurs for some j (between 1 and c).*
- (2) *Δ is Cohen-Macaulay (so Δ is bi-Cohen-Macaulay in the sense of [9]).*
- (3) *Δ has exactly $\binom{s+c-1}{c}$ maximal facets of size $n - c$.*
- (4) *Δ has exactly $\binom{s+c-1}{s}$ minimal non-faces (necessarily of size s).*

This allows us to completely classify complexes with linear resolution and type defect zero as follows.

Corollary 1.4. *Let Δ be a simplicial complex such that $I = I_\Delta$ has a linear resolution. Then $\text{td}(\Delta) = 0$ if and only if Δ is a tree of $d - 1$ -simplices, glued along codimension one faces.*

Another noteworthy consequence is that the inequalities in Theorem 1.3 hold for any homogenous ideals with linear resolution, monomial or not, see Theorem 4.6.

The article is organized as follows. Section 2 deals with preparatory materials. Here we explain the connection between type and Cohen-Macaulayness, Theorem 2.4, which is really just a reformulation of a famous result in commutative algebra. We also prove the gluing results, 2.7 and 2.8, and discuss the modified type of 2-Cohen-Macaulay complexes.

In Section 3 we consider the case when Δ is a graph, which we view as a 1-dimensional simplicial complex. In particular, we show in Theorem 3.3 that a graph G is chordal if and only if the type defect of every induced subgraph is nonnegative. Motivated by this surprising connection, we then characterize the graphs for which every connected subgraph has zero type defect. This set of graphs, which we call *treeish*, contains all trees, as well as graphs that can be obtained by adding triangles to a tree in a specific way.

We also consider a natural generalization of treeish graphs to simplicial complexes, ones in which certain induced subcomplexes have type defect zero.

We then turn our attention to higher dimensional versions of our chordality results in Section 4. Throughout this section we assume that I_Δ has linear resolution. We prove Theorem 1.3 and its consequences, including the lower bounds for Betti numbers for general graded ideals, Theorem 4.6 here.

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2. BACKGROUND AND PRELIMINARY RESULTS

Let Δ be a $(d - 1)$ -dimensional simplicial complex with vertex set V and associated Stanley-Reisner ideal I_Δ . Recall that I_Δ is the ideal generated by monomials corresponding to non-faces of Δ . For $W \subseteq V$, we write $\Delta|_W$ to denote the subcomplex of Δ induced on the set W . The *codimension* of Δ is the number of vertices in the complement of a maximal facet of Δ . As Δ is $(d - 1)$ -dimensional on n vertices, we have:

$$\text{codim}(\Delta) = n - d.$$

For ease of notation we shall write $b_{i,j}(M)$ for $\dim_k \text{Tor}_i^S(M, k)_j$. We first recall Hochster's Formula, which expresses the multigraded Betti numbers in terms of the homologies of induced subcomplexes of Δ .

Theorem 2.1 (Hochster's Formula). *The multigraded Betti numbers $b_{i,j}$ of I_Δ are given by:*

$$b_{i,j}(I_\Delta) = \sum_{|W|=j} \dim(\tilde{H}_{j-i-2}(\Delta|_W)).$$

If we resolve the quotient instead, we get:

$$b_{i,j}(S/I_\Delta) = b_{i-1,j}(I_\Delta) = \sum_{|W|=j} \dim(\tilde{H}_{j-i-1}(\Delta|_W)).$$

Hochster's formula immediately implies several facts about the modified type and type defect.

Corollary 2.2. *Let Δ be a $(d-1)$ -complex with n vertices and $c = n - d$.*

- (1) *$\text{mtype}(\Delta) = b_c(S/I_\Delta) > 0$ if and only if Δ is not a simplex.*
- (2) *When Δ is a simplex, $\text{mtype}(\Delta) = \text{td}(\Delta) = 0$.*
- (3) *If Δ is Gorenstein, $\text{td}(\Delta) = 0$ if and only if Δ is a $(d-1)$ simplex or the boundary of a d -simplex.*

Proof. For (1), we need to show that $\dim_k \text{Tor}_c^S(S/I_\Delta, k)_c = 0$ if and only if Δ is not a simplex. By Hochster's formula, this is

$$b_{c,c}(S/I_\Delta) = \sum_{|W|=c} \dim(\tilde{H}_{-1}(\Delta|_W))$$

This sum is 0 if and only if $\Delta|_W$ is non-empty for each subset W of size $c = n - d$. This clearly happens if and only if $n > d$, namely, Δ is not a simplex.

Part (2) is clear. For (3), as S/I_Δ is Gorenstein, $b_c(S/I_\Delta) = 1$. Thus either Δ is a simplex, or $c = 1$, which means Δ is a boundary of a simplex. \square

Remark 2.3. Let Δ be a $(d-1)$ -dimensional simplicial complex on n vertices with more than one facet. We have

$$(1) \quad \text{mtype}(\Delta) = \sum_{k \geq n-d} b_{n-d,k}(S/I_\Delta) = \sum_{k \geq n-d} \sum_{|W|=k} \dim(\tilde{H}_{k-n+d-1}(\Delta|_W)).$$

Substituting $k = n - d + j$ changes Equation 1 to

$$(2) \quad \text{mtype}(\Delta) = \sum_{j \geq 0} \sum_{|W|=n-d+j} \dim(\tilde{H}_{j-1}(\Delta|_W)).$$

We can also use the Alexander dual form of Hochster's Formula to get

$$\text{mtype}(\Delta) = \sum_{j \geq 0} \sum_{|W|=n-d+j, W \notin \Delta} \dim(\tilde{H}_{n-d-2}(\text{link}_{\Delta^\vee}(V \setminus W))).$$

If we write $c = n - d$ for the codimension, the above can be written as

$$\text{mtype}(\Delta) = \sum_{j \geq 0} \sum_{|W|=c+j, W \notin \Delta} \dim(\tilde{H}_{c-2}(\text{link}_{\Delta^\vee}(V \setminus W))).$$

A rather non-trivial fact about type defect is the following.

Theorem 2.4. *If $\text{td}(\Delta) \leq 0$ (i.e., if $\text{mtype}(\Delta) \leq \text{codim}(\Delta)$), then Δ is Cohen-Macaulay.*

Proof. If Δ is a simplex, then it is Cohen-Macaulay. Otherwise, the modified type is just $b_c(S/I_\Delta)$, and the assertion is a consequence of the classical Syzygy Theorem, see for example [4, Corollary 9.5.6]. \square

Observation 2.5. In the glueing result below we need to consider complexes inside a bigger set of vertices. If Δ is Cohen-Macaulay, one can compute $\text{mtype}(\Delta)$ using $\text{Tor}_c^{S'}(k[\Delta], k)$ with $S = k[x_1, \dots, x_n, y_1, \dots, y_m]$ where the y_i s are vertices not in Δ . This is because Tor_c is the last non-zero Tor, and this number is unchanged if we throw in more variables. Note that it is not true if we drop the Cohen-Macaulayness assumption. For example, if Δ be a union of disjoint edges $\{xy\}$ and $\{zw\}$. Then $k[\Delta] = k[x, y, z, w]/(xz, xw, yw, yw)$ with $c = 2$ and $\text{mtype}(\Delta) = 3$. If we introduce a new vertex t , then we have $S' = k[x, y, z, w, t]$, $k[\Delta] = S'/(xz, xw, yw, yw, t)$, the new codimension is $c' = 3$, and $\dim \text{Tor}_3^{S'}(k[\Delta], k) = 5$. This is the main reason why in the glueing Theorem 2.7 we need to assume the complexes are Cohen-Macaulay.

Observation 2.6. If Δ is 2-Cohen-Macaulay and $(d-1)$ -dimensional with more than one facets, then

$$\text{mtype}(\Delta) = \dim(\tilde{H}_{d-1}(\Delta)).$$

This is simply because of Hochster's formula and the fact that $\tilde{H}_{j-c-1}(\Delta|_W) = 0$ unless $|W| = n$. Note that 2-Cohen-Macaulay complexes include matroid complexes and triangulations of spheres.

Our first result concerns how the type and type defect behave when Cohen-Macaulay complexes are glued together. It gives a more precise version of [12, Lemma 3.5]. We need this extension, as we need to know what happens when one of the complexes we glued is a simplex.

If Δ and Δ' are two complexes on disjoint vertex sets, and E and E' are faces of Δ and Δ' , respectively, we write $\Delta \sqcup_{E=E'} \Delta'$ to denote the complex obtained by identifying E and E' .

Theorem 2.7. *Let Δ and Δ' be Cohen-Macaulay complexes of dimension $(d - 1)$, and let E and E' be $(\ell - 1)$ -dimensional faces of Δ and Δ' , respectively. Then*

$$\text{mtype}(\Delta \sqcup_{E=E'} \Delta') = \text{mtype}(\Delta) + \text{mtype}(\Delta') + \binom{n - \ell}{c + 1}$$

Here $c = n - d$, where n is the number of vertices in $\Delta \sqcup_{E=E'} \Delta'$.

Proof. Consider the short exact sequence

$$0 \rightarrow k[\Delta \cup \Delta'] \rightarrow k[\Delta] \oplus k[\Delta'] \rightarrow k[\Delta \cap \Delta'] \rightarrow 0.$$

Let $\Sigma = \Delta \cup \Delta'$, $E = \Delta \cap \Delta'$ and taking $\text{Tor}^S(-, k)$ of this sequence yields:

$$\cdots \rightarrow \text{Tor}_{i+1}^S(k[\Gamma]) \rightarrow \text{Tor}_i^S(k[\Sigma]) \rightarrow \text{Tor}_i^S(k[\Delta]) \oplus \text{Tor}_i^S(k[\Delta']) \rightarrow \text{Tor}_i^S(k[E]) \rightarrow \cdots$$

As $\text{Tor}_i^S(k[\Delta]) \oplus \text{Tor}_i^S(k[\Delta'])$ is zero for all $i > c$, we have an exact sequence:

$$0 \rightarrow \text{Tor}_{c+1}^S(k[\Gamma]) \rightarrow \text{Tor}_c^S(k[\Sigma]) \rightarrow \text{Tor}_c^S(k[\Delta]) \oplus \text{Tor}_c^S(k[\Delta']).$$

Now we argue in as in [12]. $\text{Tor}_i^S(k[\Delta \cap \Delta'])$ only exists in degree i , as $\Delta \cap \Delta'$ is a simplex. On the other hand, by Hochster's formula, for any complex Δ with more than one facet, $\text{Tor}_c^S(k[\Delta])_c = 0$ (where c is the codimension). So the above is a short exact sequence (the last map is surjective) in every degree $j \neq c$, and in degree c it becomes:

$$0 \rightarrow 0 \rightarrow 0 \rightarrow \text{Tor}_c^S(k[\Delta])_c \oplus \text{Tor}_c^S(k[\Delta'])_c$$

Which implies that $\dim_k \text{Tor}_c^S(k[\Sigma])$ equals

$$\dim_k \text{Tor}_{c+1}^S(k[\Gamma]) + \dim_k \text{Tor}_c^S(k[\Delta]) \oplus \text{Tor}_c^S(k[\Delta']) - \dim_k \text{Tor}_c^S(k[\Delta])_c \oplus \text{Tor}_c^S(k[\Delta'])_c.$$

Since Γ is just the simplex $E = E'$, note that $k[\Delta \cap \Delta'] = S/\langle x_i : x_i \notin E \rangle$. This quotient is resolved by the Koszul resolution, meaning that the rank of Tor_{c+1}^S is $\binom{n-\ell}{c+1}$. Finally, note that as Δ is Cohen-Macaulay,

$$\dim_k \text{Tor}_c^S(k[\Delta]) - \dim_k \text{Tor}_c^S(k[\Delta])_c = \text{mtype}(\Delta),$$

and the same equality holds for Δ' (see 2.5). \square

Corollary 2.8. *Let Δ and Δ' be Cohen-Macaulay complexes of dimension $d - 1$. Let F and F' be $(\ell - 1)$ dimensional faces of Δ and Δ' , respectively. Then*

$$\text{td}(\Delta \sqcup_{F=F'} \Delta') = \text{td}(\Delta) + \text{td}(\Delta') - d + \ell + \binom{n - \ell}{c + 1}.$$

In particular, if $\ell \geq d - 1$, then

$$\text{td}(\Delta \sqcup_{F=F'} \Delta') = \text{td}(\Delta) + \text{td}(\Delta').$$

Proof. The assertions follow from Theorem 2.7. One just needs to take care of the codimensions in the definition of type defect: if Δ has p vertices and Δ' has m vertices, then $n = p + m - \ell$, so the codimensions are $p - d, m - d, p + m - \ell - d$ respectively. \square

3. CHORDAL GRAPHS AND TREEISH COMPLEXES

Throughout this section, let $\Delta = G$ be a simple graph on n vertices, viewed as a 1-dimensional simplicial complex. In this case we can simplify Definition 2.3 as follows.

$$\text{mtype}(G) = \dim(\tilde{H}_1(G)) + \sum_{|W|=n-1} \dim(\tilde{H}_0(G_W)).$$

Note that the term $\dim(\tilde{H}_1(G))$ is simply the number of basic cycles of G . Assume G is connected. Let e be the number of edges of G . Then any spanning tree of G has $n - 1$ edges, and any edge not in a given spanning tree corresponds to a basic cycle of G . Thus, $\dim(\tilde{H}_1(G)) = e - n + 1$. For any graph H , let $C(H)$ be the number of connected components of H . Also, if W is a subset of vertices of G with $|W| = n - 1$, let $W = V(G) \setminus \{v\}$. Then $\dim(\tilde{H}_0(G_W)) = C(G - v) - 1$. Thus, the above expression becomes:

$$(3) \quad \text{mtype}(G) = e - n + 1 + \sum_{v \in V(G)} (C(G - v) - 1),$$

provided G is connected.

More generally, for any simple graph G , the dimension of $\tilde{H}_1(G)$ is given by $e - n + C(G)$, and so

$$(4) \quad \text{mtype}(G) = e - n + C(G) + \sum_{v \in V(G)} (C(G - v) - 1),$$

Observation 3.1. If G is 2-connected, meaning the removal of any one vertex cannot disconnect the graph, we have $\text{mtype}(G) = e - n + 1$. Note that this can also be seen via Observation 2.6. Thus, If G is a 2-connected graph with e edges and n vertices, then $\text{td}(G) = (e - n + 1) + 1 - n + 1 = e - 2n + 3$.

Proposition 3.2. *If T is a tree, then $\text{td}(T) = 0$.*

Proof. Let $v \in T$, and suppose T has n vertices and e edges. Then the number of connected components of $G - v$ is $\deg(v)$. Since T has no cycles,

$$\text{mtype}(T) = \sum_{v \in V(T)} \deg(v) - 1 = 2e - n = 2(n - 1) - n = n - 2.$$

As $\text{codim}(T) = n - 1 - 1 = n - 2$, we get $\text{td}(T) = 0$. □

The property of *chordality* is one of the most widely-studied graph properties. It turns out that chordality is equivalent to a requirement on the type defect, as shown by the next theorem.

Theorem 3.3. *G is chordal iff $\text{td}(G|_W) \geq 0$ for any $W \subseteq V$.*

Our proofs use the following classical result of Dirac:

Theorem 3.4. [6] *Let G be a chordal graph. Then there exists a vertex v of G such that $G|_{N(v)}$ is complete, where we write $N(v)$ to mean the set of neighbors of v . Such a vertex is called simplicial.*

Proof of Theorem 3.3. First assume that G is a chordal graph, let v be a simplicial vertex of G , and let $N(v)$ be its neighbors. Then $\text{td}(G \setminus v) \geq 0$, by induction, the base case being immediate. Adding v to $G \setminus v$ increases the codimension by 1, so we need to show that the type increases by at least 1.

If v has only one neighbor w , then the number of components of $G \setminus w$ is one greater than the number of components of $(G \setminus v) \setminus w$. Thus $\text{mtype}(G) = \text{mtype}(G \setminus v) + 1$.

If v has at least two neighbors, then $\text{rank } \tilde{H}_1(G) \geq \text{rank } \tilde{H}_1(G) + 1$.

For the converse, assume G is not chordal. Then there exists a set $W \subseteq V$ such that $G|_W$ is a cycle and $|W| \geq 4$. Using Observation 3.1, $\text{mtype}(G|_W) = 1$, meaning $\text{td}(G|_W) = 1 - (|W| - 2) \leq -1$. \square

Corollary 3.5. *G is chordal iff $\text{td}(G|_W) \geq 0$ for any $W \subseteq V$ such that $G|_W$ is connected.*

Proof. It is straightforward to see that the type defect of a graph is greater than or equal to the sum of the type defects of its connected components. Thus $\text{td}(G|_W) \geq 0$ for all induced subgraphs $G|_W$ of G if and only if this holds for all connected subgraphs $G|_W$, and the result follows from Theorem 3.3. \square

Now we turn our attention to graphs for which every connected induced subgraph has type defect zero. By Theorem 3.3, such graphs must necessarily be chordal. However, note that many chordal graphs have positive type defect (for example, K_n for $n > 3$).

Definition 3.6. Call a graph *treeish* if it can be constructed recursively via the following rules.

- (1) An edge or a triangle is treeish.
- (2) If G is treeish and v is a vertex of G , adding a new vertex w and a new edge $\{v, w\}$ results in a treeish graph.
- (3) If G is treeish, and $\{v, w\}$ is an edge of G , then the graph obtained from G by adding a new vertex z and the edges $\{v, z\}$ and $\{w, z\}$ is treeish.

Clearly any tree is treeish, as it can be constructed from a single vertex by repeated application of rule (2). Also, any treeish graph is easily seen to be chordal.

Theorem 3.7. *The following are equivalent.*

- (1) G is treeish.
- (2) $\text{td}(G|_W) = 0$ for any $W \subseteq V$ such that $G|_W$ is connected.
- (3) G is chordal, and the number of triangles of G is the rank of $\tilde{H}_1(G)$.

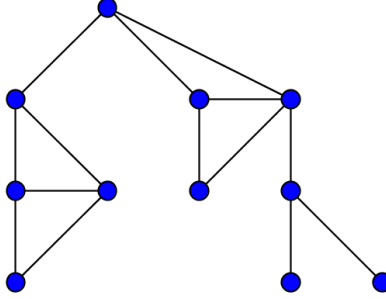


FIGURE 1. A treeish graph.

Proof. (1) \Rightarrow (2):

Let W be a subset of vertices such that $G|_W$ is connected. Then $G|_W$ is easily seen to be treeish; simply restrict the operations described in Definition 3.6 to the vertices in W . So, it suffices to show that $\text{td}(G) = 0$ for any treeish graph. However, this follows immediately from Theorem 2.7.

(2) \Rightarrow (3):

First suppose that G is not chordal. Then there exists a subset W of vertices of G such that $|W| \geq 4$ and $G|_W$ is a cycle. Then Equation 3 shows that $\text{mtype}(G|_W) = 1$, meaning $\text{td}(G|_W) = 1 - (|W| - 2) < 0$. So we can assume that G is chordal. Let v be a simplicial vertex of G as guaranteed by Theorem 3.4. By induction on the number of vertices of G (the base case being straightforward), we have that $G - v$ is chordal, and that $\text{rank}(\tilde{H}_1(G - v))$ is the number of triangles of $G - v$. Let W be the set of neighbors of v . We claim that $|W| = 1$ or 2 . Indeed, suppose $|W| \geq 3$. Then $G|_{W \cup \{v\}}$ would be a complete graph on $m \geq 4$ vertices, meaning

$$\text{td}(G|_{W \cup \{v\}}) = \binom{m}{2} - m + 1 - (m - 2) = \frac{m(m - 1)}{2} - 2m + 3 = \frac{m^2 - 5m + 6}{2} > 0 \text{ if } m \geq 4.$$

Thus we may assume that $|W| = 1$ or 2 . If $|W| = 1$, then $\text{rank}(\tilde{H}_1(G)) = \text{rank}(\tilde{H}_1(G - v))$, and G clearly has the same number of triangles as $G - v$. If $|W| = 2$, then G has one more triangle than $G - v$ (namely $G|_{W \cup \{v\}}$). However, this triangle cannot be written as a sum of any other elements in $\tilde{H}_1(G)$, as it is the only triangle containing the vertex v . Thus $\text{rank}(\tilde{H}_1(G)) = \text{rank}(\tilde{H}_1(G - v)) + 1$.

(3) \Rightarrow (1):

Here we use Theorem 3.4 in the same way as in the previous case. Indeed, let v be a simplicial vertex of the graph G , and let W be its neighbors. If $|W| \geq 3$, then it is easy to see that $\text{rank}(\tilde{H}_1(G))$ is strictly less than the number of triangles of G : If $\{a, b, c\} \subseteq W$, then the homology element corresponding to the boundary of the triangle abc is a linear combination of the boundaries of the triangles abv , acv , and bcv . Thus we have $|W| = 1$ or 2 . By induction, $G - v$ is treeish. But then the addition

of v and the corresponding edge(s) either corresponds to rule (2) or rule (3), meaning G is treeish. \square

Theorem 3.8. *Let Σ be a flag complex with vertex set V . Then Σ is the clique complex of some chordal graph if and only if $\text{td}(\Sigma|_W) \geq 0$ for every $W \subseteq V$.*

Proof. First suppose that Σ is the clique complex of some chordal graph G . If Σ is not Cohen-Macaulay, Fact 2.4 gives that $\text{td}(\Sigma) \geq 0$. So we can assume that Σ is Cohen-Macaulay. Let v be a simplicial vertex of G and $N(v)$ its neighbors. If we set $\Delta = \Sigma \setminus v$ and let Δ' denote the full simplex on $\{v\} \cup N(v)$, then $\Sigma = \Delta \cup \Delta'$, and $\Delta \cap \Delta'$ is the simplex on $N(v)$. Because Δ' is a facet of Σ , which is Cohen-Macaulay, it follows that Δ and Δ' are of the same dimension. Since Δ is an induced subcomplex of Σ , Δ must be Cohen-Macaulay as well. By induction, each of Δ and Δ' must have nonnegative type defect. Thus, by Corollary 2.8, $\text{td}(\Sigma) = \text{td}(\Delta) + \text{td}(\Delta') \geq 0$. Now let $W \subseteq V$. Then Σ_W is the clique complex of $G|_W$, which is chordal.

For the converse, let G be a graph that is not chordal. Then G must contain a chordless cycle on vertex set W , where $|W| \geq 4$. Then, as in the proof of Theorem 3.3, $\text{td}(\Sigma|_W) \leq -1$. \square

A natural question is how to extend the treeish graphs of Definition 3.6 to arbitrary simplicial complexes. The most natural generalization is immediate, given Theorem 2.7 and Observation 2.6.

Observation 3.9. Let Δ be a triangulation of a $(d - 1)$ -dimensional sphere. If $\text{td}(\Delta) = 0$, then Δ must be the boundary of the d -simplex.

Proof. As spheres are 2-Cohen-Macaulay, Observation 2.6 gives that $\text{mtype}(\Delta) = \dim \tilde{H}_{d-1}(\Delta) = 1$. If $\text{td}(\Delta) = 0$, then the codimension of Δ must be 1. But clearly this can only happen if Δ is the boundary of the d -simplex. \square

Definition 3.10. We call a pure $(d - 1)$ -dimensional simplicial complex *treeish* if it can be constructed recursively via the following rules.

- (1) A single $(d - 1)$ -simplex or a boundary of a d -simplex is treeish.
- (2) If Σ is treeish and F is a $(d - 1)$ -simplex, then $\Sigma \sqcup_{E'=E} F$ is treeish, where E' and E are $(d - 2)$ -dimensional faces of Σ and F , respectively.
- (3) If Σ is treeish and Δ is the boundary of a d -simplex, then $\Sigma \sqcup_{E'=E} \Delta$ is treeish, where E' and E are $(d - 2)$ or $(d - 1)$ -dimensional faces of Σ and Δ , respectively.

Note that it follows from Theorem 2.7 and the above observation that any treeish complex Σ satisfies $\text{td}(\Sigma) = 0$. Moreover, as in the graph case, the treeish property is somewhat hereditary, as shown by the following theorem. Recall that a pure simplicial complex is called *strongly facet-connected* if, for any two facets F and F' , there is a chain of facets $F = F_0, F_1, F_2, \dots, F_t = F'$ such that F_i intersects F_{i+1} in a codimension-one face for each i .

Theorem 3.11. *Let Σ be a treeish complex, and let Δ be an induced subcomplex of Σ that is strongly facet-connected. Then Δ is a treeish complex.*

Proof. We think of building Σ via the steps in Definition 3.10. So, suppose Σ has been constructed via subcomplexes as $\Sigma = \Sigma_1 \cup \Sigma_2 \cup \cdots \cup \Sigma_t$, where each Σ_i is either a full simplex or a simplex boundary, identified along a facet or a codimension-1 face. We prove by induction on t . Let W be the set of vertices of Δ . If $t = 1$, then $\Delta = \Sigma|_W$ is either a simplex or a boundary of a simplex. Suppose $t > 1$. Let $\Sigma' = \Sigma_1 \cup \Sigma_2 \cup \cdots \cup \Sigma_{t-1}$ and $\Delta' = \Sigma'|_W$. By induction Δ' is treeish.

By construction, Σ is obtained from Σ' by adding one or two vertices and taking a cone of the vertex (or boundary of the cone, in the case of two points) over some face F in Σ' . Thus Δ is obtained from Δ' by adding at most two vertices and performing similar operations over some face $F' = F \cap \Delta'$. By the assumption of Δ being strongly facet-connected, F' must be of codimension one in Δ' , and we are done. \square

Corollary 3.12. *If Σ is a treeish complex, then $\text{td}(\Delta) = 0$ for any strongly facet-connected induced subcomplex Δ of Σ .*

Note that this generalizes the (1) \Rightarrow (2) direction of Theorem 3.7, as a graph is connected if and only if it is strongly facet-connected.

Example 3.13. In the above Theorem and Corollary, it is not enough to assume that Δ is pure and connected. To see this, consider the complex with facets $\{1, 2, 3\}$, $\{2, 3, 4\}$, $\{3, 4, 5\}$, $\{3, 5, 6\}$ and induce on the vertex set $W = \{1, 2, 3, 5, 6\}$.

4. IDEALS WITH LINEAR RESOLUTION

In this section we study lower bounds on Betti numbers of a complex with linear resolution.

Theorem 4.1. *Let Δ be a simplicial complex such that $I = I_\Delta$ has a linear resolution. Let c be the codimension of I and s be its generating degree (that is, the degree of any minimal generator of I , or the size of a minimal non-face). Then for each j such that $1 \leq j \leq c$ we have*

$$b_j(S/I_\Delta) \geq \binom{s+j-2}{j-1} \sum_{i=j-1}^{c-1} \binom{s+i-1}{s+j-2}.$$

Furthermore, the following are equivalent:

- (1) Equality occurs for some j (between 1 and c).
- (2) Δ is Cohen-Macaulay.
- (3) Δ has exactly $\binom{s+c-1}{c}$ maximal facets of size $n - c$.
- (4) Δ has exactly $\binom{s+c-1}{s}$ minimal non-faces.

Proof. Let h_0, h_1, \dots be the h -vector of the Alexander dual of Δ , Δ^\vee . By Theorem 4 of [7]

$$b_j(S/I_\Delta) = \text{the coefficient of } t^{j-1} \text{ in } \sum_{i \geq 0} h_i(1+t)^i = \sum_{i \geq j-1} \binom{i}{j-1} h_i.$$

Let $d-1 = \dim \Delta^\vee$. Obviously, $s = n-d$. Since $A = S/I_{\Delta^\vee}$ is Cohen-Macaulay, the h vector of Δ^\vee is the same as the Hilbert function of the Artinian algebra $B = A/(l_1, \dots, l_d)$, where the l_i s form a linear system of parameters on A (WLOG one can assume k is infinite). That is $h_i = \dim_k B_i$. We can write $B = S'/J$, where S' is a polynomial ring in $n-d = s$ variables. But since I_{Δ^\vee} is generated in degree c or higher, the same holds for J , so for $i < c$, the i -th degree part of B is the same as that of S' . Thus $h_i = \binom{s+i-1}{i}$ for $i < c$. Thus

$$b_j(S/I_\Delta) \geq \sum_{i \geq j-1}^{c-1} \binom{i}{j-1} \binom{s+i-1}{i} = \binom{s+j-2}{j-1} \sum_{i=j-1}^{c-1} \binom{s+i-1}{s+j-2}$$

We now prove the last claim. From the proof, it is clear that any equality occurs if and only if $h_i = 0$ for $i \geq c$. But since J is generated in degree c or higher, this is equivalent to $h_c = 0$. Which is, in turn, equivalent to $b_{c+1}(S/I) = 0$. But this happens if and only if $\text{pd}_S S/I = c$, which is equivalent to S/I being Cohen-Macaulay.

For (3), note that $h_c = 0$ is equivalent to $J_c = S'_c$, or the minimal number of generators of J equals to $\dim_k S'_c = \binom{s+c-1}{c}$. But the minimal number of generators of J is the same as the minimal number of generators of I_{Δ^\vee} , and the latter is simply the number of facets of Δ . Lastly, the equivalence of (4) to the other one can be shown by simply noting that (4) is equivalent to the equality for $b_1(S/I_\Delta)$, which is simply the number of minimal generators of I_Δ , and this is the number of minimal non-faces. \square

Remark 4.2. Theorem 4.1 gives precise conditions for when Δ is both linear resolution and Cohen-Macaulay. These have been studied in the literature, for example [9, 11]. For example, the Bi-Cohen-Macaulay bipartite graphs describe in [11] satisfy the conditions on the number of facets or minimal non-faces given above.

Corollary 4.3. *Let Δ be a simplicial complex such that $I = I_\Delta$ has a linear resolution. Let c be the codimension of I . Then for each j such that $1 \leq j \leq c$ we have*

$$b_j(S/I_\Delta) \geq j \binom{c+1}{j+1}.$$

In particular $\text{td}(\Delta) \geq 0$. Furthermore, the equality occurs for some j if and only if Δ is the clique complex of a chordal graph with exactly $c+1$ maximal cliques of size $n-c$.

Proof. Since s is the minimal size of a non-face of Δ , it is obvious that $s \geq 2$. Plug in $s = 2$ in the inequality of the last Theorem, we have for each j such that $1 \leq j \leq c$:

$$b_j(S/I_\Delta) \geq j \sum_{i=j}^c \binom{i}{j} = j \binom{c+1}{j+1}.$$

The equality occurs for some i if and only if $s = 2$ and the complex Δ has $c+1$ facets. As $s = 2$, we know that $I = I_\Delta$ is an edge ideal, so Δ must be the clique complex of a chordal graph G by Froberg's Theorem. \square

One can classify the situation of equality in the previous corollary precisely. Call a $d-1$ complex *facet constructible* if either Δ is a simplex or $\Delta = \Delta_1 \sqcup_{E=E'} \Delta_2$ where Δ_1 is facet constructible, Δ_2 is a simplex and E is a facet of its boundary.

Theorem 4.4. *Let G be chordal graph on n vertices and Δ be its clique complex of dimension $d-1$. The following are equivalent:*

- (1) Δ has with exactly $n-d+1$ maximal facets of size d .
- (2) G has $\frac{(d-1)(2n-d)}{2}$ edges.
- (3) Δ is Cohen-Macaulay.
- (4) Δ is facet constructible.
- (5) $h_2(\Delta) \geq 0$.
- (6) Δ is (S_2) in the sense of [13].
- (7) Δ is shellable.

Proof. The equivalence of (1), (2), (3) is just Theorem 4.1. To prove (4) implies (1), we use induction on the number of simplices to show that the skeleton of Δ is chordal with the correct numbers of facets. To prove that (1) implies (4) we can use induction again. Let G be a chordal graph with exactly $n-d+1$ maximal cliques of size d . Then there is a simplicial vertex v , which has to belong to some maximal clique. Remove v we get a chordal graph with $n-d$ maximal cliques, so induction hypothesis applies. For the last two items just note that (6) implies (5) by the main result of [13], (5) implies (3) by Corollary 4.5, and (3) implies (6) by definition. Finally, to see that (7) is equivalent to the others, it is clear that any facet constructible complex is shellable, and it is a standard result that any shellable complex is Cohen-Macaulay. \square

It is not hard to rewrite the conditions of Theorem 4.1 in terms of h -vector or f -vectors of Δ . We explain how to do it for h -vectors. The linear resolution of I necessarily looks like:

$$0 \rightarrow S(-s-l+1)^{b_l} \rightarrow \cdots \rightarrow S(-s-1)^{b_2} \rightarrow S(-s)^{b_1} \rightarrow S \rightarrow S/I \rightarrow 0$$

so the alternating sum of Hilbert series must be 0, which gives:

$$\frac{1 - b_1 t^s + b_2 t^{s+1} - \cdots + (-1)^l t^{s+l-1}}{(1-t)^n} = \frac{\sum h_i(\Delta) t^i}{(1-t)^{n-c}}$$

From this one gets $h_i(\Delta) = \binom{c+i-1}{i}$ for $i < s$. After that one gets $h_s = \binom{c+s-1}{s} - b_1$, etc. As a consequence, we have, for instance:

Corollary 4.5. *Let Δ be a simplicial complex such that $I = I_\Delta$ has linear resolution. Let s be the generating degree of I . Then $h_s(\Delta) \leq 0$, and equality occurs if and only if Δ is Cohen-Macaulay.*

Finally, we give our main application, showing that the inequalities in Theorem 4.1 extend to all homogenous ideals.

Theorem 4.6. *Let k be a field of any characteristic and $I \subset S = k[x_1, \dots, x_n]$ a homogenous ideal with a linear resolution. Let c be the codimension of I and s be its generating degree (the degree of any minimal generator of I). Then for each j such that $1 \leq j \leq c$ we have*

$$b_j(S/I_\Delta) \geq \binom{s+j-2}{j-1} \sum_{i=j-1}^{c-1} \binom{s+i-1}{s+j-2}.$$

Proof. We follow the proof of [3, Theorem 3.3]. Starting with a homogenous ideal with linear resolution, one can arrive at a square-free monomial one with the same Betti sequence via taking generic initial ideals and stretching. Note that this works over any field as explained in *loc. cit.* Also, it is easy to see that the process does not impact the numbers c and s . So the result follows from Theorem 4.1. \square

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