A SURVEY OF STANLEY-REISNER THEORY

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Abstract. We survey the Stanley-Reisner correspondence in combinatorial commutative algebra, describing fundamental applications involving Alexander duality, associated primes, f- and h-vectors, and Betti numbers of monomial ideals.

1. Introduction

Stanley-Reisner theory provides the central link between combinatorics and commutative algebra. Pioneered in the 1970s, the correspondence between simplicial complexes and squarefree monomial ideals has been responsible for substantial progress in both fields. Among the most celebrated results are Reisner's criterion for Cohen-Macaulayness, Stanley's proof of the Upper Bound Conjecture for simplicial spheres, and Hochster's formula for computing multigraded Betti numbers of squarefree monomial ideals via simplicial homology. Moreover, techniques such as deformation and polarization can allow one to take a problem about homogeneous ideals and turn it into a question about squarefree monomial ideals.

This paper is written to give a self-contained introduction to Stanley-Reisner theory, directing it especially at relatively new graduate students in commutative algebra and combinatorics. Our goal is to provide enough background to enable readers to progress to more detailed treatments in one of the many excellent references (for example, [MS, P, St, V]).

In the following section, we define some important terms and explain the basics of the Stanley-Reisner correspondence. We introduce some running examples in Section 3. In Section 4, we discuss the important role Alexander duality plays in studying squarefree monomial ideals and simplicial complexes and its interaction with associated primes. We develop this topic further in Section 5, paying particular attention to shellability and the Cohen-Macaulay property. Section 6 is devoted to connections between f- and h-vectors on the combinatorial side and Hilbert series on the algebraic side. Finally, in Section 7, we explore Hochster's formula and the use of simplicial homology to compute multigraded Betti numbers of monomial ideals.

2. Background and notation

We begin by making explicit some familiar ideas and notation from combinatorics and algebra. Most readers should be comfortable skipping to §2.3, which defines the Stanley-Reisner correspondence.

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2.1. **Combinatorics.** Fix $n > 0$, and let $X = \{x_1, x_2, \ldots, x_n\}$. An *abstract simplicial complex* Δ on vertex set X is a collection of subsets of X (that is, $\Delta \subseteq 2^X$) such that $A \in \Delta$ whenever $A \subseteq B \in \Delta$. The members of Δ are called *simplices* or *faces*, and a simplex of Δ not properly contained in another simplex of Δ is called a *facet*. If all the facets of Δ have the same cardinality, then Δ is called pure.

If Δ contains no sets, it is called the *void* complex. Any simplicial complex other than the void complex contains the empty set as a face. This may seem like a minor point, but in the future it will be necessary to differentiate between the void complex, $\emptyset \subseteq 2^X$, and the complex whose only simplex is the empty simplex, $\{\emptyset\} \subseteq 2^X$.

Now fix some ℓ. If $k \leq \ell$, the convex hull of $k+1$ points in general position in $\mathbb{R}^\ell$ is a *geometric k-simplex*. In general, a collection Δ of geometric simplices in $\mathbb{R}^\ell$ is a *geometric simplicial complex* if $\sigma \cap \tau$ is a geometric simplex in Δ for any $\sigma, \tau \in \Delta$. It is a basic fact from combinatorial topology that every abstract simplicial complex has a geometric realization, and thus we often use the term "simplicial complex" to simultaneously refer to both these points of view. In Section 3, each figure is a geometric realization of the corresponding simplicial complex.

The *dimension* of a simplex $A \in \Delta$ is one less than the cardinality of A. This coincides with our usual geometric notion of a simplex, as it requires $k+1$ points in general position in Euclidean space to have a k-dimensional convex hull.

If $A \notin \Delta$ but $B \in \Delta$ for any $B \subsetneq A$, then A is called a *minimal non-face* of Δ. For example, the minimal non-faces of the complex in Example 3 are ad, ae, bd, and be.

2.2. **Algebra.** Let k be a field, and let $S = k[X] = k[x_1, x_2, \ldots, x_n]$.

A *monomial* of S is an element m which factors (uniquely, up to order) as a product of the variables in X; it is *squarefree* if no variable appears more than once in this factorization. The *degree* of m is the number of variables in its factorization; if m is squarefree, its degree is equal to the number of variables dividing m.

Given two monomials m and m', we say that m *divides* m' if every variable appears at least as many times in the factorization of m' as in the factorization of m. The monomials are partially ordered by divisibility, and this ordering refines degree: If m divides m', then $\deg m \leq \deg m'$, with equality only if $m = m'$.

A *monomial ideal* $I \subset S$ is an ideal which, when viewed as a k-vector space, has a (unique) basis consisting of monomials. Equivalently, a monomial ideal is one with a generating set consisting of monomials. Of course, I is generated by the set of all its monomials, but this generating set is highly redundant. We can remove the redundancy by restricting our attention to those monomials of I which are minimal under divisibility. This yields a generating set which is minimal in the sense that any monomial generating set of I contains it; this is called $\mathrm{gens}(I)$, or the *unique minimal monomial generating set* for I, and its elements are simply referred to as *generators* of I. A *squarefree monomial ideal* is a monomial ideal whose monomial generators are all squarefree.

2.3. **The Stanley-Reisner correspondence.** The Stanley-Reisner correspondence arises from two important observations connecting the information in simplices to that in squarefree monomials. The first observation is that these are in natural bijection.

Definition 2.1. Let $A \subset X$. Then the *monomial supported on* A is the squarefree monomial $m_A = \prod_{x_i \in A} x_i$. Conversely, if m is a squarefree monomial, then its *support* is $\operatorname{supp} m = \{x_i : x_i \text{ divides } m\}$.

Proposition 2.2. *For every squarefree monomial m, we have $m_{\operatorname{supp} m} = m$. For every subset $A \subset X$, we have $\operatorname{supp}(m_A) = A$. If m and m' are squarefree monomials, then m divides m' if and only if $\operatorname{supp} m \subseteq \operatorname{supp} m'$.*

Proof. The only difficulty is with the empty set. Observe that $m_\varnothing$ is the empty product, namely the monomial 1, and that the support of 1 is the empty set. □

Notation 2.3. Throughout the paper, we will abuse notation without comment by writing squarefree monomials in place of subsets of X. That is, we will refer to faces of a simplicial complex as m instead of as $\operatorname{supp} m$, and in the examples we will dispense with the set brackets and the commas.

The second observation of Stanley-Reisner theory is that simplicial complexes and monomial ideals have opposite behavior with respect to the partial orders of inclusion and divisibility. That is, simplicial complexes are closed under "shrinking" and monomial ideals are closed under "growing":

Proposition 2.4. *Let m be a squarefree monomial. If I is a squarefree monomial ideal and $m \in I$, then $m' \in I$ whenever m divides m'. On the other hand, if Δ is a simplicial complex and $m \in \Delta$, then $m' \in \Delta$ whenever $m' \subset m$.*

The corollary to this observation motivates the definitions at the heart of Stanley-Reisner theory.

Corollary 2.5. *If $I \subset S$ is a squarefree monomial ideal, then the set of monomials not contained in I forms a simplicial complex.*

Definition 2.6. For a squarefree monomial ideal I, the *Stanley-Reisner complex of I* is the simplicial complex consisting of the monomials not in I,

$$\Delta_I = \{m \subset X : m \notin I\}.$$

For a simplicial complex Δ, the *Stanley-Reisner ideal of* Δ is the squarefree monomial ideal generated by the non-faces of Δ,

$$I_\Delta = (m \subset X : m \notin \Delta).$$

The *face ring* or *Stanley-Reisner ring* of Δ is the quotient by the Stanley-Reisner ideal, $R_\Delta = S/I_\Delta$.

Observe that the minimal monomial generators of I_Δ are the minimal non-faces of Δ.
The following is immediate.

Proposition 2.7. *If I is a squarefree monomial ideal, then $I_{\Delta_I} = I$. If Δ is a simplicial complex, then $\Delta_{I_\Delta} = \Delta$.*

3. Running examples

In the examples, we set $X = \{a, b, c, \dots\}$ instead of $\{x_1, x_2, x_3, \dots\}$. Throughout the paper, we will refer to the examples in this section.

Example 3.1. Let K be the complex in Figure 1. The facets of K are abc, abe, ace, bcd, bce, and cde. The minimal nonfaces of K are ad and bce, so $I_K = (ad, bce)$. Both K and I_K are very well-behaved objects; for example, the geometric realization of K is homeomorphic to a sphere, and I_K is a complete intersection.

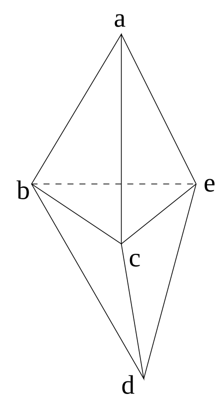

Figure 1. A simplicial complex homeomorphic to $\mathbb{S}^2$.

Example 3.2. Let Q be the standard triangulation of the real projective plane, as in Figure 2. The facets of Q are abd, abf, acd, ace, aef, bce, bcf, bde, cdf, and def, and I_Q is generated by the ten minimal nonfaces, all of which have cardinality three:

$$I_Q = (abc, abe, acf, ade, adf, bcd, bdf, bef, cde, cef).$$

The projective plane is globally strange (for example, its homology is characteristic-dependent) but locally well-behaved; we will see this in the ideal as well.

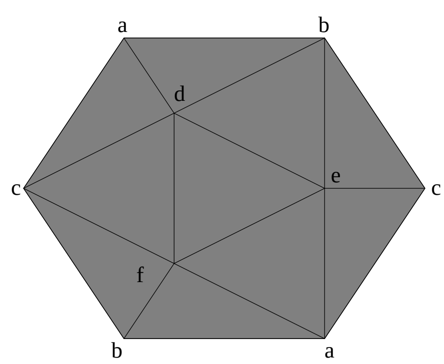

Figure 2. The standard minimal triangulation of the projective plane. (Note the identifications on the boundary.)

Example 3.3. Let B be the "bow-tie" complex shown in Figure 3. The facets of B are abc and cde, and the minimal non-faces of B are ad, ae, bd, and be, meaning $I_B = (ad, ae, bd, be)$.

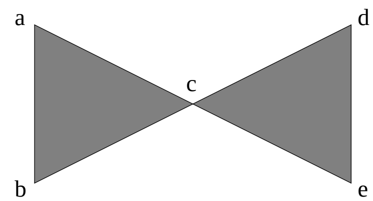

FIGURE 3. The bow-tie complex B.

4. Alexander duality and associated primes

The notion of Alexander duality comes from algebraic topology, where for a sufficiently "nice" subspace Γ of the n-dimensional sphere there is an isomorphism between $H_i(\Gamma)$ and $H^{n-i-1}(\Gamma^c)$. The combinatorial flavor of Alexander duality, which we discuss here, produces a dual complex $\Delta^\vee$ from a simplicial complex Δ, and relates this complex to the prime ideals associated to the Stanley-Reisner ideal I_Δ.

Definition 4.1. If Δ is a simplicial complex, the *Alexander dual* of Δ, denoted by $\Delta^\vee$, is the simplicial complex with faces $\{X \smallsetminus m : m \notin \Delta\}$. That is, faces of $\Delta^\vee$ are complements of non-faces of Δ.

Note that the facets of $\Delta^\vee$ are thus complements of *minimal* nonfaces of Δ.

Example 4.2. The complex K of Figure 6 has minimal non-faces ad and bce, and thus $K^\vee$ has facets bce and ad. Note that $K^\vee$ is the disjoint union of a line segment and solid triangle, meaning $K^\vee$ is homotopy equivalent to $\mathbb{S}^0$, whereas K is homotopy equivalent to $\mathbb{S}^2$.

Example 4.3. If Q is the triangulation of the projective plane shown in Figure 2, then Q is self-dual. Indeed, the minimal non-faces of Q are abc, abe, acf, ade, adf, bcd, bdf, bef, cde, and cef, and so facets of $Q^\vee$ are the complements of these non-faces: def, cdf, bde, bcf, bce, aef, ace, acd, abf, and abd. Note that these are exactly the facets of Q, and thus $Q^\vee = Q$.

Example 4.4. The complex B of Figure 3 has minimal non-faces ad, ae, bd, and be, and thus the facets of $B^\vee$ are bce, bcd, ace, and acd. The complex $B^\vee$ is shown below in Figure 4.

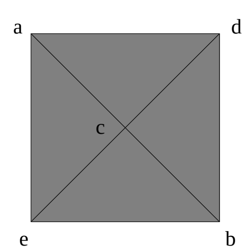

FIGURE 4. The dual of the complex B.

Observe that c is a cone point in both B and $B^\vee$. This illustrates the general fact that a cone point of a complex is also a cone point of the dual.

Remark 4.5. The name "Alexander dual" is justified by the fact that $\Delta^\vee$ is homotopic to the complement of Δ in the $(n-2)$-sphere. See [BCP, Theorem 2.1] or [MS, Remark 5.7].

The translation of Alexander duality to squarefree monomial ideals involves the set of associated primes, which is a natural generalization of the prime factorization of an integer.

Fact 4.6. *Let I be a squarefree monomial ideal. Then I decomposes as an intersection of prime monomial ideals,*

$$I = \bigcap_{P \supset I} P.$$

Here P ranges over the primes which contain I and are monomial ideals. Of course, this intersection is highly redundant. For example, the homogeneous maximal ideal appears in the intersection but can be deleted unless I is itself the maximal ideal. It is natural to remove the redundancy by restricting our attention to the monomial primes which are minimal among those containing I; these are called the *associated primes* of I.

Remark 4.7. The discussion above fails badly if I is not a squarefree monomial ideal. Monomial ideals cannot in general be written as intersections of primes, or even of powers of primes. Instead, we can find a *irredundant primary decomposition*: that is, we can write an arbitrary ideal as an irredundant intersection of primary ideals. This decomposition is usually not unique, but the set of primes which occur as radicals of the primary ideals is, and we call these the *associated primes*. The minimal primes containing an ideal are associated, but it is possible for non-minimal primes to be associated as well. Associated primes can be detected as the annihilators of elements as in the last condition of Theorem 4.8 below. For more details, see [E, Chapter 3].

Theorem 4.8. *Let I be a squarefree monomial ideal. Then the following are equivalent for a monomial prime ideal P.*

(1) *P contains I, and is minimal among the prime ideals that do so.*
(2) *I may be written as an irredundant intersection of primary ideals, $I = \bigcap Q_j$, and P is the radical of one of the Q_j in this intersection.*
(3) *There is a monomial $m \notin I$ with the property that $mx \in I$ if and only if $x \in P$.*

Definition 4.9. Let I be a squarefree monomial ideal. If P is a monomial prime ideal satisfying the equivalent conditions of Theorem 4.8, then we say that P is an *associated prime* of I. The set of all associated primes of I is written $\operatorname{Ass}(I)$.

To understand the associated primes of a monomial ideal, we need to study monomial primes. The chief observation is that every monomial prime is generated by a subset of the variables. Thus, prime ideals carry the same information that monomials do.

Notation 4.10. If $A \subset V$ is a subset of the variables, then write P_A for the prime ideal generated by the elements of A, $P_A = (x_i : x_i \in A)$. If m is a monomial, write P_m for $P_{\operatorname{supp} m}$.

This notation allows us to describe the facets of Δ in terms of the associated primes of I.

Proposition 4.11. *Let I be a squarefree monomial ideal, and let $\mathbf{X} = x_1 \dots x_n$ be the product of the variables. Let m be a squarefree monomial. Then P_m is associated to I if and only if $\frac{\mathbf{X}}{m}$ is a facet of Δ_I. More generally, P_m contains I if and only if $\frac{\mathbf{X}}{m} \in \Delta_I$.*

Proof. Observe that P_m contains I if and only if $\operatorname{supp} m$ shares a variable with $\operatorname{supp} \mu$ for every monomial $\mu \in I$. Equivalently, the complement of $\operatorname{supp} m$, $\frac{\mathbf{X}}{m}$, does not contain $\operatorname{supp} \mu$. This means that $\frac{\mathbf{X}}{m}$ is not divisible by any μ, so is not contained in I. In other words, $\frac{\mathbf{X}}{m} \in \Delta_I$. This proves the second claim.

For the first claim, observe that facets are maximal, associated primes are minimal, and the operation taking m to $\frac{\mathbf{X}}{m}$ is order-reversing. (Alternatively, $\frac{\mathbf{X}}{m}$ is a facet if and only if $x_i \frac{\mathbf{X}}{m} \in I$ for all x_i dividing m.) □

These observations inspire the definition of the (algebraic) Alexander dual of I as (abusing notation) the ideal generated by the associated primes of I.

Definition 4.12. Let I be a squarefree monomial ideal. Then the *Alexander dual* of I is

$$I^\vee = \big(m : P_m \in \operatorname{Ass}(I)\big).$$

We justify this name by showing that Alexander duality commutes with the Stanley-Reisner correspondence.

Theorem 4.13. *If I is a squarefree monomial ideal, then $\Delta_{(I^\vee)} = (\Delta_I)^\vee$. If Δ is a simplicial complex, then $I_{(\Delta^\vee)} = (I_\Delta)^\vee$. Thus, we can write $I_\Delta^\vee$ or $\Delta_I^\vee$ without confusion.*

Proof. The faces of $\Delta_{(I^\vee)}$ are the monomials m such that P_m does not contain I, i.e., the monomials whose complements are contained in I. The faces of $(\Delta_I)^\vee$ are the complements of the non-faces of Δ_I, i.e., the complements of the monomials in I. □

It immediately follows that algebraic Alexander duality is a duality operation, and that it gives us a somewhat more efficient way to compute the associated primes of a squarefree monomial ideal.

Corollary 4.14. *Let I be a squarefree monomial ideal. Then $(I^\vee)^\vee = I$.*

Corollary 4.15. *Let I be a squarefree monomial ideal. Then $I^\vee = \bigcap_{m \in \operatorname{gens}(I)} P_m$, and the associated primes of I correspond to the generators of $I^\vee$. That is, $\operatorname{Ass}(I) = \{P_\mu : \mu \in \operatorname{gens}(I^\vee)\}$.*

If we know the Stanley-Reisner ideal of a complex (or vice-versa) we can use it to compute the dual with less work.

Corollary 4.16. *Let $\mathbf{X} = x_1 \dots x_n$ be the product of the variables.*

1. *Let Δ be a simplicial complex. Then the facets of $\Delta^\vee$ are the monomials $\frac{\mathbf{X}}{m}$, where m ranges over the generators of I_Δ.*
2. *Let I be a squarefree monomial ideal. Then the generators of $I^\vee$ are the monomials $\frac{\mathbf{X}}{f}$, where f ranges over the facets of Δ_I.*

Corollary 4.17. *Let I be a squarefree monomial ideal. Then I is equidimensional (i.e., all its associated primes have the same height) if and only if $\Delta_I^\vee$ is pure (i.e., all its facets have the same dimension).*

Example 4.18. Let K be the complex in Example 3.1. Then $I_K = (ad, bce)$, so $I_K^\vee = (a,d) \cap (b,c,e) = (ab, ac, ae, bd, cd, de)$. Note that the Stanley-Reisner complex corresponding to $I_K^\vee$ has facets ad and bce. This is consistent with the computation in Example 4.2.

Example 4.19. Let Q be the triangulation of the projective plane from Example 3.2. As we saw in Example 4.3, Q is self-dual, and therefore I_Q is also self-dual.

Example 4.20. Let B be the bow-tie complex in Example 3.3. It has two facets, abc and cde. Therefore $I_B^\vee$ is generated by $\frac{\mathbf{x}}{abc} = de$ and $\frac{\mathbf{x}}{cde} = ab$; i.e., $I_B^\vee = (ab, de)$.

Alexander duality plays a vital role in linking the notions of Cohen-Macaulayness of simplicial complexes and linearity in free resolutions of squarefree monomial ideals, as we discuss in the next section.

5. Shellablity, Cohen-Macaulayness, and linear resolutions

Shellable complexes occur frequently throughout combinatorics. The shellability condition is particularly helpful because shellable simplicial complexes are homotopic to bouquets of spheres and, in particular, are Cohen-Macaulay over any field.

Definition 5.1. An ordering $F_1, F_2, \dots, F_t$ of the facets of a simplicial complex Δ is a *shelling* if, for each j with $1 < j \le t$, the intersection

$$\left(\bigcup_{i=1}^{j-1} F_i \right) \cap F_j$$

is a non-empty union of facets of ∂F_j. If there exists a shelling of Δ, then Δ is called *shellable.*

See [BW1, BW2] for more details, particularly in the nonpure case, and motivation.

Example 5.2. The complex in Figure 5 is shellable. (One shelling is cde, ad, ab, bc, bd.) Note, however, that any sequence of facets of this complex in which the sole 2-dimensional facet is not first cannot be a shelling.

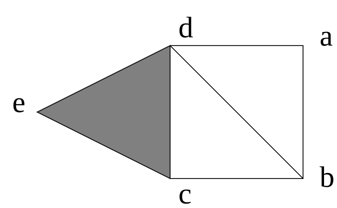

Figure 5. A shellable simplicial complex.

The complex K from Example 3.1 is shellable, as we discuss in detail in the next section. The other examples from Section 3 are not shellable.

We sketch the proof that shellable complexes are bouquets of spheres. Given a shelling $F_1, \dots, F_t$ of a simplicial complex Δ, call F_j a *full-restriction facet* if $\partial F_j \subseteq \displaystyle\bigcup_{k=1}^{j-1} F_k$.

Theorem 5.3. *Let Δ be a simplicial complex with shelling $F_1, F_2, \ldots, F_t$. Then Δ is homotopy equivalent to a wedge of spheres of the dimensions of the full restriction facets,*

$$\Delta \simeq \bigvee_{F_j} \mathbb{S}^{\dim F_j},$$

where the wedge is taken over the full restriction facets F_j.

The complex of Figure 5 is homotopy equivalent to a wedge of two 1-dimensional spheres.

Proof of Theorem 5.3 (sketch). Let Δ' denote the complex obtained from Δ by removing each full-restriction facet F_j (but keeping ∂F_j in Δ'). Then the given shelling order, after removing the full-restriction facets, is a shelling of Δ'.

A straightforward induction on the number of facets of Δ' shows that it is contractible. If F_j is an i-dimensional full-restriction facet of Δ, then contracting Δ' identifies the boundary of F_j, creating an i-dimensional sphere. The same holds for all full-restriction facets of Δ, proving the result. $\square$

Shellability of simplicial complex implies an important algebraic property known as sequential Cohen-Macaulayness.

Definition 5.4. Let M be a finitely-generated graded module. We say that M is *Cohen-Macaulay* if its depth is the same as its dimension; equivalently, if its projective dimension is the same as its codimension. The module M is *sequentially Cohen-Macaulay* if there exists a filtration

$$0 = M_0 \subset M_1 \subset \cdots M_r = M$$

such that each M_i/M_{i-1} is Cohen-Macaulay, and the Krull dimensions of the quotients are increasing:

$$\dim(M_1/M_0) < \dim(M_2/M_1) < \cdots < \dim(M_r/M_{r-1}).$$

See, e.g., [BH, E, Sc] for detailed discussions of Cohen-Macaulayness. For more details on sequential Cohen-Macaulayness, see [St], and see [HS, St] for equivalent definitions in terms of Ext modules.

We define a simplicial complex Δ to be (sequentially) Cohen-Macaulay if S/I_Δ is (sequentially) Cohen-Macaulay.

Theorem 5.5. *If Δ is shellable, then Δ is sequentially Cohen-Macaulay. If Δ is also pure, then Δ is Cohen-Macaulay.*

The result that a pure shellable complex is Cohen-Macaulay is [BH, Theorem 5.1.13], [MS, Theorem 13.45], or [St, Ch. 3, Theorem 2.5]. For the first statement, build the filtration from the Stanley-Reisner ideals of Δ_i, where Δ_i is the subcomplex of Δ generated by the facets of dimension at least i.

The characterization of when a simplicial complex is Cohen-Macaulay is known as *Reisner's criterion.* Recall that the *link* of a face F in a complex Δ, for which we write $\operatorname{link}_\Delta(F)$, is a simplicial complex whose faces are given as follows:

$$\operatorname{link}_\Delta(F) = \{G \in \Delta : F \cup G \in \Delta, F \cap G = \emptyset\}.$$

Theorem 5.6 (Reisner's criterion)**.** *A simplicial complex Δ is Cohen-Macaulay over k if and only if for any face F of Δ, $\dim_k(\tilde{H}_i(\mathrm{link}_\Delta(F);k)) = 0$ for $i < \dim(\mathrm{link}_\Delta(F))$.*

That is, Δ is Cohen-Macaulay over k if and only if the homology of each face's link vanishes below its top dimension.

The complex B of Example 3.3 is not Cohen-Macaulay over any field, as $\mathrm{link}_B(c)$ has facets ab and de, meaning $\dim_k(\tilde{H}_0(\mathrm{link}_B(c), k)) = 1$ for any k, whereas $\dim(\mathrm{link}_B(c)) = 1$.

The complex of Figure 5 is 2-dimensional, yet its first homology group is non-trivial. As the link of the empty face is the entire complex, this shows that this complex is not Cohen-Macaulay.

Finally, consider the complex K of Example 3.1. The link of every vertex of K is a 1-sphere (for example $\mathrm{link}_K(a)$ has facets bc, ce, and eb, while $\mathrm{link}_K(c)$ has facets ab, bd, de and ea), and the link of every edge of K is a 0-sphere (for example $\mathrm{link}_K(ab)$ has facets c and e). As $\mathrm{link}_K(\emptyset) = K$, which is a 2-sphere, it follows that no link of a face of K has nontrivial homology below its top dimension. Thus, K is Cohen-Macaulay (as is any simplicial sphere).

The following theorem of Eagon and Reiner [ER, Theorem 3] connects Cohen-Macaulayness and free resolutions. (For background on free resolutions, see [P].)

Theorem 5.7. *A simplicial complex Δ is Cohen-Macaulay over k if and only if $I_{\Delta^\vee}$ has linear free resolution over $S = k[x_1, \ldots, x_n]$.*

Example 5.8. Take Δ to be the simplicial complex of Figure 5 without the face cde. Then $I_\Delta = (ac, ae, be, bcd, abd, cde)$, and $I_{\Delta^\vee} = (abc, abd, abe, ade, bce, cde)$, which has a linear resolution. Therefore Δ is Cohen-Macaulay. One can verify this by, for example, noting that the codimension of I_Δ is three, as is the projective dimension of S/I_Δ.

Note the importance of the field k in Theorem 5.7. Reisner pointed out the minimal triangulation of the projective plane Q is Cohen-Macaulay if and only if $\mathrm{char}\, k \neq 2$. Thus $I_Q^\vee$ has linear resolution when $\mathrm{char}\, k \neq 2$ but nonlinear resolution when $\mathrm{char}\, k = 2$.

We can loosen the conditions in Theorem 5.7 to get another useful statement. Sequential Cohen-Macaulayness is a natural generalization of Cohen-Macaulayness in the case in which Δ is not pure. We define the appropriate homological analogue:

Definition 5.9. Let I be a homogeneous ideal, and write (I_d) for the ideal generated by the degree-d forms in I. The ideal I is *componentwise linear* if for all d, (I_d) has linear resolution.

Remark 5.10. The condition in Definition 5.9 is not as computationally difficult to check as it appears. One needs only check that (I_d) has a linear resolution for degrees d in which I has minimal generators.

Herzog and Hibi prove the following theorem [HH, Theorem 2.1].

Theorem 5.11. *A simplicial complex Δ is sequentially Cohen-Macaulay over k if and only if $I_{\Delta^\vee}$ is componentwise linear over $S = k[x_1, \ldots, x_n]$.*

One consequence of Theorem 5.11 is a partial converse of Theorem 5.5.

Proposition 5.12. *Fix a field k. Then S/I_Δ is Cohen-Macaulay over k if and only if Δ is pure and S/I_Δ is sequentially Cohen-Macaulay over k.*

Proof. One direction is clear. For the other, assume that Δ is pure and that S/I_Δ is sequentially Cohen-Macaulay over k. By Theorem 5.11, because S/I_Δ is sequentially Cohen-Macaulay over k, $I_{\Delta^\vee}$ is componentwise linear. Moreover, the fact that Δ is pure means that $I_{\Delta^\vee}$ is generated in a single degree. Thus $I_{\Delta^\vee}$ has a linear resolution, and by Theorem 5.7, Δ is Cohen-Macaulay. □

One can also detect shellability with similar algebraic methods. Suppose I is a monomial ideal, minimally generated by monomials $m_1, \dots, m_r$, where $\deg m_i \le \deg m_{i+1}$ for all i. We say that I has *linear quotients* if for each $2 \le i \le r$, $(m_1, \dots, m_{i-1}) : (m_i)$ is generated by a subset of the variables. Herzog and Takayama note in [HT] that the statements that I_Δ has linear quotients and $\Delta^\vee$ is nonpure shellable are "almost tautologically equivalent." See also Proposition 6.13. Moreover, if I_Δ has linear quotients and is generated in a single degree, then I_Δ has a linear resolution, and therefore $\Delta^\vee$ is Cohen-Macaulay.

Remark 5.13. We conclude with a brief discussion of the use of Alexander duality in the study of edge and cover ideals of graphs. Given a graph on vertices $\{x_1, \dots, x_n\}$, the edge ideal I_G of G is generated by all monomials $x_i x_j$ such that $\{x_i, x_j\}$ is an edge of G. The cover ideal J_G is generated by monomials m such that each edge of G contains at least one element of $\operatorname{supp} m$. Therefore $J_G = I_G^\vee$, and one can investigate properties of the graph G by studying either the edge or the cover ideal of G, often passing back and forth between the two. See [MV] for a survey of recent work on edge and cover ideals.

6. Hilbert functions and f-vectors

Probably the most important numerical invariant of a graded ideal is its Hilbert function, which associates to each degree d the dimension of the ideal's degree-d piece. This measures the size of the ideal, and contains a lot of other important information, such as its dimension and multiplicity.

Definition 6.1. Let M be a graded module. Then the *Hilbert function* of M is

$$\begin{aligned} \mathrm{HF}_M : \mathbf{Z} &\to \mathbf{Z} \\ d &\mapsto \dim_k(M_d). \end{aligned}$$

In order to use generating function techniques, we also define the *Hilbert series* as the generating function on the Hilbert function, $\mathrm{HS}_M(t) = \sum_d \mathrm{HF}_M(d) t^d$.

Given an ideal I, we study both the Hilbert function of I, HF_I, and the Hilbert function of the quotient of S by I, $\mathrm{HF}_{S/I}$. Observe that $\mathrm{HF}_I + \mathrm{HF}_{S/I} = \mathrm{HF}_S$.

We begin by computing the Hilbert function of S.

Proposition 6.2. *Let $S = k[x_1, \dots, x_n]$. Then $\mathrm{HS}_S(t) = \frac{1}{(1-t)^n}$.*

Proof. We induct on n. If $n = 1$, then $S = k[x]$ has basis $\{1, x, x^2, \dots\}$ and Hilbert series $t^0 + t^1 + t^2 + \cdots = \frac{1}{1-t}$.

In general, let $\mathcal{B}$ be a graded k-basis for $k[x_1, \dots, x_{n-1}]$. Then S decomposes (as a vector space) as

$$S = \bigoplus_{b \in \mathcal{B}} b \cdot k[x_n].$$

If b has degree d, the Hilbert series of the corresponding summand $b \cdot k[x_n]$ is $t^d + t^{d+1} + t^{d+2} + \cdots = \frac{t^d}{1-t}$. We have

$$\begin{aligned}
\mathrm{HS}_S(t) &= \sum_{b\in\mathcal{B}} \mathrm{HS}_{b\cdot k[x_n]}(t) \\
&= \sum_{b\in\mathcal{B}} \frac{t^{\deg b}}{1-t} \\
&= \frac{1}{1-t}\sum_{b\in\mathcal{B}} t^{\deg b} \\
&= \left(\frac{1}{1-t}\right) \mathrm{HS}_{k[x_1,\dots,x_{n-1}]}(t) \\
&= \left(\frac{1}{1-t}\right) \frac{1}{(1-t)^{n-1}}. \qquad \square
\end{aligned}$$

It turns out that the Hilbert series is always a rational function.

Fact 6.3. *Let M be a finitely generated S-module. Then*

$$\mathrm{HS}_M(t) = \frac{p(t)}{(1-t)^n}$$

for some polynomial $p(t)$. If we write this in lowest terms, it becomes

$$\mathrm{HS}_M(t) = \frac{h(t)}{(1-t)^d},$$

*where d is the Krull dimension of M. Here, $h(t)$ is called the h-*polynomial *of M and $h(1)$ is the* multiplicity *of M.*

If I is a monomial ideal, then the Hilbert function of I counts the monomials appearing in I, and the Hilbert function of S/I counts the monomials not appearing in I. If I is a squarefree monomial ideal, this is still true, but it's much easier to count the smaller number of squarefree monomials in I. To this end, we define another object, the *squarefree Hilbert function.*

Definition 6.4. Let I be a squarefree monomial ideal. Then the *squarefree Hilbert function* of I is the function

$$\begin{aligned}
\mathrm{HF}_I^{\mathrm{sqfree}} &: \mathbb{Z} \to \mathbb{Z} \\
&d \mapsto \#\{m \in I : m \text{ is a squarefree monomial of degree } d\}
\end{aligned}$$

We define $\mathrm{HF}_{S/I}^{\mathrm{sqfree}}$ similarly, and define the squarefree Hilbert series $\mathrm{HS}_I^{\mathrm{sqfree}}$ and $\mathrm{HS}_{S/I}^{\mathrm{sqfree}}$ to be the generating functions on the squarefree Hilbert functions. Note that the squarefree Hilbert series is actually a polynomial, since there are no squarefree monomials of degree greater than n.

In order to understand the relationship between the Hilbert function and squarefree Hilbert function of a squarefree monomial ideal, we need a little more machinery.

Definition 6.5. Let $m = \prod x_i^{e_i}$ be a monomial. Then the *squarefree part* of m is the result of "deleting the exponents" from m,

$$\operatorname{sqfree}(m) = \prod_{e_i \geq 1} x_i = \prod_{x_i \text{ divides } m} x_i.$$

The idea is that a monomial's presence in or absence from a squarefree ideal depends only on its squarefree part.

Lemma 6.6. *Let I be a squarefree monomial ideal and m a monomial. Then $m \in I$ if and only if* $\operatorname{sqfree}(m) \in I$.

This observation allows us to compute so-called "Stanley decompositions" of I and S/I.

Corollary 6.7. *Let I be a squarefree monomial ideal. Then, viewed as vector spaces, I and S/I have the decompositions*

$$I = \bigoplus_{m \in I} m \cdot k[\operatorname{supp} m] \qquad \text{and} \qquad S/I = \bigoplus_{m \notin I} m \cdot k[\operatorname{supp} m],$$

both sums being taken over the set of squarefree monomials.

These decompositions will enable us to compute the Hilbert functions in terms of the corresponding squarefree Hilbert functions.

Lemma 6.8. *Let m be a squarefree monomial. Then the Hilbert series of $m \cdot k[\operatorname{supp} m]$ is* $\left(\frac{t}{1-t}\right)^{\deg m}$.

Proof. By Proposition 6.2, the Hilbert series of $k[\operatorname{supp} m]$ is $\frac{1}{(1-t)^{\deg(m)}}$. Multiplying by m increases the degree of everything by $\deg m$, so multiplies the Hilbert series by $t^{\deg m}$. □

Theorem 6.9. *Let I be a squarefree monomial ideal. Then $\mathrm{HS}_I(t) = \mathrm{HS}_I^{\mathrm{sqfree}}\left(\frac{t}{1-t}\right)$ and* $\mathrm{HS}_{S/I}(t) = \mathrm{HS}_{S/I}^{\mathrm{sqfree}}\left(\frac{t}{1-t}\right)$.

Proof. By Corollary 6.7 and Lemma 6.8 we have

$$\begin{aligned}
\mathrm{HS}_I(t) &= \sum_{m \in I, \text{ squarefree}} \mathrm{HS}_{m \cdot k[\operatorname{supp} m]}(t) \\
&= \sum_{m \in I, \text{ squarefree}} \left(\frac{t}{1-t}\right)^{\deg m} \\
&= \mathrm{HS}_I^{\mathrm{sqfree}}\left(\frac{t}{1-t}\right).
\end{aligned}$$

The computation for the quotient is identical. □

The Stanley-Reisner analogue of the squarefree Hilbert Function, which counts squarefree monomials, is the f-vector, which counts faces of a simplicial complex.

Definition 6.10. The *f-vector* of a $(d-1)$-dimensional complex Δ is the sequence $f(\Delta) = (f_{-1}, f_0, \ldots, f_{d-1})$ where f_i is the number of i-dimensional faces of Δ (and $f_0 = 1$ whenever Δ is not the void complex). The f-polynomial is the generating function of the f-vector,

$$f_\Delta(t) = f_{-1}t^d + f_0 t^{d-1} + \cdots + f_{d-2}t + f_{d-1}.$$

Note that the coefficient of t^i is the number of *co*dimension-i faces (reversing the convention of the Hilbert series).

The f-vector is perhaps the most natural combinatorial invariant of simplicial complexes. In a sense, study of the f-vector dates back to Euler: The *reduced Euler characteristic* of Δ, which is a topological invariant, is given by

$$\tilde{\chi}(\Delta) = \sum_{i=-1}^{d-1} (-1)^i f_i.$$

The *h-vector* of a complex Δ is the result of an invertible transformation applied to the f-vector of Δ.

Definition 6.11. Let Δ be a $(d-1)$-dimensional simplicial complex. The *h-polynomial* of Δ, written $h_\Delta(t)$, is the polynomial given by

$$h_\Delta(t) = f_\Delta(t-1).$$

The *h-vector* of Δ is the sequence $h(\Delta) = (h_0, h_1, \ldots, h_d)$ of coefficients of $h_\Delta(t)$:

$$h_\Delta(t) = h_0 t^d + h_1 t^{d-1} + \cdots + h_{d-1}t + h_d.$$

Remark 6.12. The h-vector of a complex Δ is typically studied only when Δ is pure.

Although the f-vector and h-vector contain the same information, properties of some simplicial complexes are often easier to express in terms of the h-vector. One example of this are the *Dehn-Sommerville relations*, which state that the h-vector of a simplicial polytope boundary is palindromic.

Note that the entries of the h-vector are not guaranteed to be nonnegative. Indeed, the complex B in Example 3.3 has facets abc and cde, meaning it has f-vector $(1, 5, 6, 2)$ and h-vector $(1, 2, -1, 0)$. However, when Δ is a pure shellable complex its h-vector counts something, and thus has nonnegative entries. First, we examine a shelling in greater detail.

A shelling of a complex Δ can be thought of as a recipe for building Δ one facet at a time. In the shelling shown in Figure 6 of the complex K from Example 1, we begin with the void complex (the complex with no faces). Adding the facet abc then adds the faces $\emptyset, a, b, c, ab, ac, bc$, and abc to the complex. With respect to inclusion, this set of faces has a unique minimal face: $\emptyset$. Next, we add the facet bcd, which adds the faces d, bd, cd, and bcd to the complex. Again, this set has a unique minimal element: d. Continuing on, the minimal new faces obtained by adding the facets abe, bde, cde, and ace are, respectively, e, de, ce, and ace.

It turns out that any shelling has this property: with the addition of each facet F_i, the corresponding set of all "new" faces has a unique minimal element with respect to inclusion. Conversely, any ordering of the facets satisfying this property is a shelling. We prove this in the next proposition.

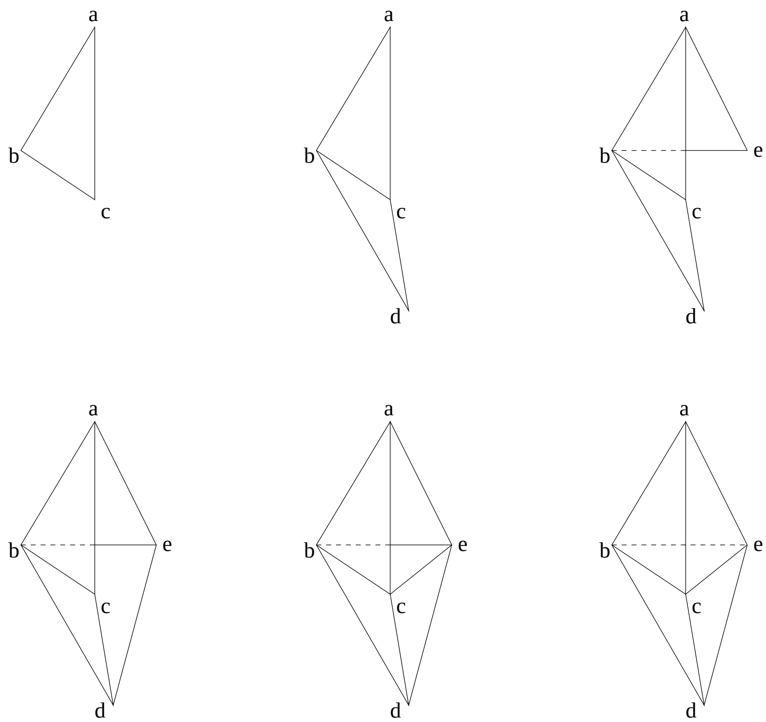

FIGURE 6. Shelling the pure simplicial complex K from Example 1.

Proposition 6.13. *Let Δ be a simplicial complex and $F_1, F_2, \ldots, F_t$ an ordering of its facets. Then this ordering is a shelling of Δ if and only if the following property holds:*

- *For each j with $1 \leq j \leq t$, the set of faces contained in $\bigcup_{i=1}^{j} F_i$ but not in $\bigcup_{i=1}^{j-1} F_i$ has a unique minimal element with respect to inclusion.*

We often call this unique minimal element the minimal face *associated to F_j.*

Proof. First assume that the property is satisfied, and fix $j > 1$. We need to show that $F_j \cap \left(\bigcup_{i=1}^{j-1} F_i\right)$ is a non-empty union of facets of ∂F_j. Let A be the minimal face associated to F_j, and note that A must be nonempty, as $\bigcup_{i=1}^{j-1} F_i$ already contains the empty set as a face. For a face B of F_j, we have that $B \subseteq F_j \cap \left(\bigcup_{i=1}^{j-1} F_i\right)$ if and only if there is an $a \in A$ with $a \notin B$. As $F_j \smallsetminus a \subseteq \bigcup_{i=1}^{j-1} F_i$ for all $a \in A$, it follows that

$$\bigcup_{a \in A} F_j \smallsetminus a = F_j \cap \left(\bigcup_{i=1}^{j-1} F_i\right),$$

which means the ordering is a shelling.

The converse is proven analogously: Fix $j > 1$, and let $A = \{a \in F_j : F_j \smallsetminus a \subseteq \bigcup_{i=1}^{j-1} F_i\}$. Then a face $B \subseteq F_j$ is contained in $\bigcup_{i=1}^{j-1} F_i$ if and only if it is contained in $F_j \smallsetminus a$ for some $a \in A$, as $F_j \cap \left(\bigcup_{i=1}^{j-1} F_i\right)$ is a union of facets of ∂F_j. Thus, a face of F_j is not contained in $\bigcup_{i=1}^{j-1} F_i$ if and only if it contains A, meaning the property is satisfied. $\square$

Looking back at the shelling depicted in Figure 6, we found that the minimal faces associated to the facets in the shelling were $\emptyset, d, e, de, ce$, and ace. Clearly, a different shelling of K would not necessarily produce the same set of minimal faces. For example, if we use the shelling $abc, ace, abe, bcd, bde, cde$, the associated minimal faces are, in order, $\emptyset, e, be, c, de, cde$.

While these two sets of minimal faces are distinct, they each contain one (-1)-dimensional face (namely, $\emptyset$), two 1-dimensional faces, two 2-dimensional faces, and one 2-dimensional face. The next theorem asserts that this information is recorded by the h-vector of K. Indeed, the f-vector of K is $(1,5,9,6)$, and thus the h-vector of Δ is $(1,2,2,1)$.

Theorem 6.14. *Let Δ be a pure $(d-1)$-dimensional complex with shelling $F_1, F_2, \ldots, F_t$, and define Γ_j as above. Then the h-vector $(h_0, h_1, \ldots, h_d)$ is given as follows: For each i,*

$$h_i = |\{j : \text{the minimal face corresponding to } F_j \text{ is } (i-1)\text{-dimensional}\}|$$

Proof. Consider a facet F_j in a shelling of Δ, and let A be the minimal face associated to F_i. Then every new face obtained by adding F_j to the union $\bigcup_{i=1}^{j-1} F_i$ must contain A, meaning these faces add $(t+1)^{d-|A|}$ to $f_\Delta(t)$, and so

$$h_\Delta(t+1) = f_\Delta(t).$$

□

The next theorem expresses the relationship between the Hilbert series of a squarefree ideal and the f- and h-vectors of its Stanley-Reisner complex. It is often paraphrased by saying that the relationship between the Hilbert function and the squarefree Hilbert function is the same as the relationship between the f-vector and the h-vector.

Theorem 6.15. *Let Δ be a $(d-1)$-dimensional simplicial complex with f-vector $(f_{-1}, f_0, \ldots, f_{d-1})$. Recall that we write R_Δ for the quotient S/I_Δ. Then:*

(1) $HS^{\text{sqfree}}_{R_\Delta}(t) = \displaystyle\sum_{i=0}^{d} f_{i-1}t^i.$

(2) $HS_{R_\Delta}(t) = \displaystyle\sum_{i=0}^{d} \frac{f_{i-1}t^i}{(1-t)^i} = \frac{1}{(1-t)^d}\sum_{i=0}^{d} h_i t^i.$

Proof. In part (1), for each i, the squarefree monomials in S/I_Δ of degree i are exactly those monomials which correspond to $(i-1)$-dimensional faces of Δ.

For part (2), the first equality follows from part (1) and Theorem 6.9. For the second equality, we first write out the relation $h_\Delta(t) = f_\Delta(t-1)$:

$$\sum_{i=0}^{d} h_i t^{d-i} = \sum_{i=0}^{d} f_{i-1}(t-1)^{d-i}.$$

If we substitute $1/t$ for t, this becomes

$$\sum_{i=0}^{d} \frac{h_i}{t^{d-i}} = \sum_{i=0}^{d} f_{i-1}\left(\frac{1-t}{t}\right)^{d-i},$$

and multiplying through by t^d yields

$$\sum_{i=0}^{d} h_i t^i = \sum_{i=0}^{d} f_{i-1} t^i (1-t)^{d-i}.$$

Now we have

$$\begin{aligned} HS_{R_\Delta}(t) &= \sum_{i=0}^{d} \frac{f_{i-1}t^i}{(1-t)^i} \\ &= \frac{1}{(1-t)^d} \sum_{i=0}^{d} f_{i-1} t^i (1-t)^{d-i} \\ &= \frac{1}{(1-t)^d} \sum_{i=o}^{d} h_i t^i. \end{aligned}$$ □

We illustrate these ideas using the complex K of Example 4.2. The minimal non-faces of Δ are ae and bcd, and thus $I_\Delta = (ae, bcd)$ and $R_\Delta = k[a,b,c,d,e]/(ae,bcd)$. The f-vector and h-vector of Δ are $(1,5,9,6)$ and $(1,2,2,1)$, respectively. By Theorem 6.15,

$$HS_{R_\Delta}(t) = 1 + \frac{5t}{(1-t)} + \frac{9t^2}{(1-t)^2} + \frac{6t^3}{(1-t)^3} = \frac{1+2t+2t^2+t^3}{(1-t)^3}.$$

7. Hochster's formula and Betti numbers

We conclude with some connections of Stanley-Reisner theory to computing Betti numbers of monomial ideals. Many of the results we discuss in this section are for *multigraded* Betti numbers; that is, the degree of the syzygy is a monomial or, equivalently, a vector in $\mathbb{N}^n$.

Notation 7.1. Throughout the section, let $\mathbf{b}$ be a monomial. Abusing notation, we identify $\mathbf{b}$ with its exponent vector $\mathbf{b} \in \mathbb{N}^n$, and we write $x_i \in \mathbf{b}$ to indicate that x_i divides $\mathbf{b}$. We adopt the common shorthand $|\mathbf{b}| = \deg \mathbf{b}$.

Hochster's formula, which appeared in [H], has been a central tool in combinatorial commutative algebra for over 30 years. We begin by defining induced subcomplexes of a simplicial complex.

Definition 7.2. Let Δ be a simplicial complex on X, and let $Y \subseteq X$. The *induced subcomplex* $\Delta[Y]$ is the simplicial complex consisting of all faces of Δ whose vertices lie in Y.

Example 7.3. Let B be the bowtie complex from Example 3.3. The induced subcomplex $B[abde]$, shown in Figure 7, is the union of two disjoint line segments. Note that while B is contractible, $B[abde]$ has nontrivial homology. This illustrates that the collection of induced subcomplexes contains more information about the complex than its homotopy type.

Hochster's formula shows that the multigraded Betti numbers of a squarefree monomial ideal I are encoded in the homology of induced subcomplexes of Δ_I. When $\mathbf{b}$ is a multidegree and Δ a simplicial complex, we often write $\Delta[\mathbf{b}]$ to mean the induced subcomplex on the associated subset of X.

FIGURE 7. The induced subcomplex $B[acde]$.

Before stating Hochster's formula, we need a quick introduction to (multigraded) Betti numbers. We have done our best to minimize the machinery required in this section. However, the material is necessarily more involved than what comes before. Readers who are interested only in the statements or applications of Hochster's formula should skip ahead to Theorem 7.11.

There are many ways to define Betti numbers, and it is usually most convenient to think of them in terms of a free resolution or as the ranks of certain Tor modules. However, for our purposes, it is easiest to define Betti numbers in terms of the homology of an object called the *(algebraic) upper Koszul complex*, which we construct below.

Notation 7.4. For each variable x_j, define an object e_j called the *differential* of x_j. Given a monomial g of degree d, write $g = x_{j_1}x_{j_2}\dots x_{j_d}$, and set the *differential of* g equal to

$$Dg = e_{j_1} \wedge e_{j_2} \wedge \cdots \wedge e_{j_d},$$

where $\wedge$ (pronounced "wedge" and called the *exterior product*) is an associative operation satisfying the anticommutativity relation $e_r \wedge e_s = -e_s \wedge e_r$. If k has characteristic other than 2, it follows that $Dg \neq 0$ if and only if g is squarefree. (If $\operatorname{char} k = 2$, we take this as an additional axiom.) We endow the differential Dg with multidegree g.

Definition 7.5. The *Koszul complex* is the algebraic chain complex

$$K_\bullet : 0 \to F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to k \to 0,$$

where F_i is the free S-module with basis given by $\{Dg : \deg g = i\}$. We formally set the empty wedge $D(1)$ equal to 1, so $F_0 = S$. The boundary maps are given by the formula $\phi(Dg) = \sum_{x_j \in g} \pm x_j D\left(\frac{g}{x_j}\right)$, where the sign convention is the standard simplicial boundary convention: if $g = \prod x_{j_s}$, then $Dg = \sum (-1)^{s+1} x_{j_s} D\left(\frac{g}{x_{j_s}}\right)$. Note that ϕ preserves multidegree.

Remark 7.6. If we stipulate that $\wedge$ is also distributive over addition, and set $E = \bigoplus F_i$, then E is an anticommutative k-algebra, called the *exterior algebra on* $\{x_1, \dots, x_n\}$.

Remark 7.7. The Koszul complex is essentially the usual simplicial chain complex arising from the $(n-1)$-simplex $\{x_1, \dots, x_n\}$, but viewed with coefficients in S. The coefficients in the boundary maps give it multigraded structure. It is a standard exercise in commutative algebra that the Koszul complex is exact and hence a free resolution of k.

Definition 7.8. Let I be a monomial ideal. Then the *(algebraic) upper Koszul complex* of I is the tensor product of I with the Koszul complex,

$$K_\bullet(I) : 0 \to IF_n \to IF_{n-1} \to \cdots \to IF_1 \to IF_0 \to I/(x_1, \ldots x_n)I \to 0.$$

The i^{th} Betti number of I is the dimension of the i^{th} homology of this complex,

$$b_i(I) = \dim_k H_i(K_\bullet(I)),$$

and the $(i, \mathbf{b})^{\text{th}}$ multigraded Betti number of I is the dimension of the degree $\mathbf{b}$ part of this homology,

$$b_{i,\mathbf{b}}(I) = \dim_k \left(H_i(K_\bullet(I))\right)_{\mathbf{b}},$$

The module IF_i has k-basis consisting of symbols of the form fDg, where $f \in I$ is a monomial and g is a squarefree monomial of degree i.

The following standard theorems about Betti numbers are considerably easier to derive from one of the many standard treatments (see, for example, [P]) than from the definitions above.

Theorem 7.9. *Let I be a squarefree monomial ideal and $\mathbf{b}$ a multidegree. Then*

(1) $b_{i,\mathbf{b}}(I) = b_{i+1,\mathbf{b}}(S/I)$.
(2) *If $\mathbf{b}$ is not squarefree, then $b_{i,\mathbf{b}}(I) = 0$.*

The upper Koszul complex $K^\bullet(I)$ has k-basis consisting of symbols of the form fDg, for monomials $f \in I$ and squarefree monomials g. The symbol fDg has multidegree fg and homological degree $\deg g$, and its differential is $\phi(fDg) = \sum_{x_j \in g} \pm (fx_j)D(\frac{g}{x_j})$, where the signs alternate according to the standard convention on the order of the x_j.

If we restrict to a squarefree multidegree $\mathbf{b}$, $K^\bullet(I)$ becomes a complex $K^{\mathbf{b}}(I)$ of vector spaces, with basis $\mathcal{B} = \left\{B_g = \frac{\mathbf{b}}{g}Dg\right\}_{g \in \mathcal{G}}$, where the index set is

$$\begin{aligned} \mathcal{G} &= \left\{ g : g \text{ divides } \mathbf{b} \text{ and } \frac{\mathbf{b}}{g} \in I \right\} \\ &= \left\{ g : g \text{ divides } \mathbf{b} \text{ and } \frac{\mathbf{b}}{g} \notin \Delta_I \right\} \\ &= \left\{ g : g \text{ divides } \mathbf{b} \text{ and } \frac{\mathbf{b}}{g} \notin (\Delta_I)[\mathbf{b}] \right\} \\ &= ((\Delta_I)[\mathbf{b}])^\vee . \end{aligned}$$

Thus, $K^{\mathbf{b}}(I)$ has basis $\left\{B_g : g \in ((\Delta_I)[\mathbf{b}])^\vee\right\}$, and the differential is $\phi(B_g) = \sum_{x_j \in g} \pm B_{\frac{g}{x_j}}$.

Remark 7.10. Note that $\mathcal{G} = ((\Delta_I)[\mathbf{b}])^\vee$ is a simplicial complex. Bayer, Charalambous, and Popescu call it the *upper Koszul simplicial complex* in multidegree $\mathbf{b}$. See [BCP, MS]. We will revisit this object later in the section.

Meanwhile, the chain complex associated to $((\Delta_I)[\mathbf{b}])^\vee$ has basis consisting of symbols $\left\{C_g : g \in ((\Delta_I)[\mathbf{b}])^\vee\right\}$. The homological degree of C_g is $\dim g = \deg g - 1$, and the differential is $\partial(C_g) = \sum_{x_j \in g} \pm C_{\frac{g}{x_j}}$.

Thus, the map sending B_g to C_g is an isomorphism of chain complexes, shifting homological degree by one. It induces an isomorphism between the i^{th} homology of $K^{\mathbf{b}}(I)$ and the $(i-1)^{\text{th}}$ homology of $(\Delta_I)[\mathbf{b}]$. This gives us the first form of Hochster's formula:

Theorem 7.11 (Hochster's formula, dual form)**.** *Let I be a squarefree monomial ideal and* $\mathbf{b}$ *be a squarefree multidegree. Then*

$$b_{i,\mathbf{b}}(I) = \dim_k \tilde{H}_{i-1}\left(((\Delta_I)[\mathbf{b}])^\vee\right)$$

and

$$b_{i,\mathbf{b}}(S/I) = \dim_k \tilde{H}_{i-2}\left(((\Delta_I)[\mathbf{b}])^\vee\right).$$

We can remove the dual from this formula at the price of passing to cohomology. Recall classical Alexander duality.

Theorem 7.12 (Alexander duality)**.** *Let Δ be a simplicial complex on ℓ vertices. Then $\tilde{H}_i(\Delta^\vee; k)$ is isomorphic to $\tilde{H}^{\ell-i-3}(\Delta; k)$.*

Applying this to Hochster's formula immediately gives us a second form of Hochster's formula.

Corollary 7.13 (Hochster's formula, cohomology form)**.** *Let I be a squarefree monomial ideal and* $\mathbf{b}$ *a squarefree multidegree. Then*

$$b_{i,\mathbf{b}}(S/I) = \dim_k \tilde{H}^{|\mathbf{b}|-i-1}(\Delta_I) \qquad \text{and} \qquad b_{i,\mathbf{b}}(I) = \dim_k \tilde{H}^{|\mathbf{b}|-i-2}(\Delta_I).$$

Remark 7.14. We can also obtain the cohomological form directly from the upper Koszul complex of S/I. The computation is similar in spirit to the development of the dual form of Hochster's formula, but the details are much messier, and thus we omit it.

Remark 7.15. If k is characteristic zero, then the ranks of corresponding homology and cohomology groups are equal, and we may thus rephrase Corollary 7.13 as:

$$b_{i,\mathbf{b}}(I) = \dim_k \tilde{H}_{|\mathbf{b}|-i-2}(\Delta_I[\mathbf{b}]; k).$$

Example 7.16. Let Δ be the complex from Figure 5. Then $\Delta[acde]$ is contractible, so $\tilde{H}^i(\Delta[acde], k)$ is trivial for any i, and $b_{i,acde}(I_\Delta) = 0$ for all i.

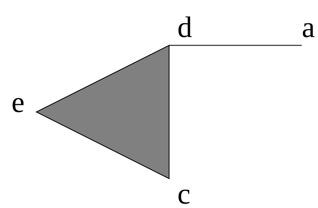

FIGURE 8. The induced subcomplex $\Delta[acde]$.

However, $\Delta[abcd]$ is homotopy equivalent to the wedge product of two circles (see Figure 9), and so

$$b_{1,abcd} = \dim_k \tilde{H}^{|abcd|-1-2}(\Delta[abcd], k) = \dim_k \tilde{H}^1(\Delta[abcd], k) = 2.$$

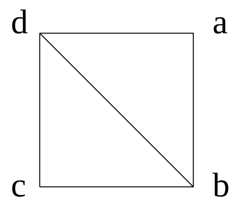

FIGURE 9. The induced subcomplex $\Delta[abcd]$.

We turn now to the computation of multigraded Betti numbers of arbitrary monomial ideals via simplicial complexes. When I is not squarefree, there is no Stanley-Reisner complex to work with, and thus Hochster's formula cannot be applied directly. However, the *upper Koszul simplicial complex*, introduced by Bayer, Charalambous, and Popescu [BCP], continues to make sense. (See also [MS].)

Definition 7.17. The *upper Koszul simplicial complex* of a monomial ideal I in multidegree $\mathbf{b}$ is

$$K^{\mathbf{b}}(I) = \{g \in 2^X : \frac{\mathbf{b}}{g} \in I\}$$

(Note: This is the simplicial complex $K_{\mathbf{b}}(I)$ in [BCP]; we have adopted the notation of [MS] to avoid confusion with another simplicial complex defined in [MS].)

Bayer, Charalambous, and Popescu prove the following theorem.

Theorem 7.18 ([BCP])**.** *Given a monomial ideal I, the multigraded Betti numbers of I are*

$$b_{i,\mathbf{b}}(I) = \dim \tilde{H}_{i-1}(K^{\mathbf{b}}(I); k).$$

Example 7.19. We compute some multigraded Betti numbers of a monomial ideal that is not squarefree to illustrate Theorem 7.18. Let $I = (a^3, b^3, c^4, abc, ac^2, bc^2) \subset S = k[a, b, c]$. We compute $K^{abc^2}(I)$. Note that $abc^2 \in I$, so $\emptyset \in K^{abc^2}(I)$. Moreover, dividing abc^2 by any of a, b, or c yields a monomial in I, meaning $K^{abc^2}(I)$ contains vertices a, b, and c. Because I is generated in degree three and higher, we cannot divide abc^2 by a degree two or higher monomial and stay in I, meaning that the facets of $K^{abc^2}(I)$ are exactly the three isolated vertices a, b, and c. From this, we can compute $b_{i,abc^2}(I)$ for any i. When $i = 0$, we compute $\dim \tilde{H}_{-1}(K^{abc^2}(I); k)$, which is zero. When $i = 1$, we are counting the number of connected components of $K^{abc^2}(I)$ minus one, giving us two. Hence $b_{1,abc^2}(I) = 2$. There is no higher homology, meaning there are two minimal first syzygies of multidegree abc^2, and $b_{i,abc^2}(I) = 0$ for $i \neq 1$.

In another direction, Terai proved a beautiful result in [T, Corollary 0.3] showing that the projective dimension of a squarefree monomial ideal and the regularity are dual notions. Suppose M is a finitely graded S-module with minimal free resolution

$$0 \to \bigoplus_j S(-b_{r,j}) \to \cdots \to \bigoplus_j S(-b_{1,j}) \to \bigoplus_j S(-b_{0,j}) \to M \to 0.$$

The *regularity* of M is the maximum value of $b_{i,j} - i$, and it is the label on the bottom row of the Macaulay 2 Betti diagram of the resolution of M.

Theorem 7.20 (Terai)**.** *The regularity of I_Δ equals the projective dimension of $S/I_{\Delta^\vee}$.*

Terai actually proves something a bit stronger, namely that the difference between the regularity of I_Δ and the smallest degree of a generator of I_Δ is the same as the difference between the projective dimension of $S/I_{\Delta^\vee}$ and $\operatorname{codim} I_{\Delta^\vee}$. (This is [T, Theorem 2.1], rephrased using the Auslander-Buchsbaum Theorem.) Theorem 5.7 is an immediate consequence because $\Delta^\vee$ is Cohen-Macaulay if and only if $\operatorname{pd} S/I_{\Delta^\vee} = \operatorname{codim} I_{\Delta^\vee}$, and an ideal has a linear resolution if and only if its regularity equals the smallest degree of a minimal generator.

Bayer, Charalambous, and Popescu generalized these ideas further in [BCP] with the notion of *extremal Betti numbers*, which are Betti numbers that are nonzero but occupy the upper left corner of a block of Betti numbers that are otherwise zero in a Macaulay 2 Betti diagram. See [BCP, Section 3] for some illustrative examples.

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