Notes on the Subposet Lattice

All our posets are on the ground set $[n]$, and are naturally labeled (meaning $1 < 2 < \cdots < n$ is always an order completion). For a poset P, let $R(P) = \{(i,j) \in [n]^2 : i <_P j\}$. Note that, since P is naturally labeled, $(i,j) \in R(P) \Rightarrow i < j$. The subposet lattice of P, which we denote $N(P)$, has ground set $\{Q : R(Q) \subseteq R(P)\}$ with order relation $Q <_{N(P)} Q'$ whenever $R(Q) \subseteq R(Q')$.

Definition 0.1. *Let P be a graded poset of rank r. An EL-labeling of the Hasse diagram of P is an* S_r-EL-labeling *if every maximal chain of P is labeled with a permutation of the set $[r]$.*

Theorem 0.2. *(MacNamara) If P has an S_r-EL-labeling, it is a supersolvable lattice.*

Let C be the n-element chain (note that $R(C) = \{(i,j) : i < j\}$). We construct an S_n-EL-labeling of $N(C)$, which is simply the lattice of naturally labeled posets on $[n]$. Order the relations in $R(C)$ by $(i,j) < (k,l)$ if either $i < k$ or $i = k$ and $j > l$. For example, the ordering in the case $n = 4$ is:

$$(1,4) < (1,3) < (1,2) < (2,4) < (2,3) < (3,4)$$

Lemma 0.3. *Let $P < Q$ in $N(C)$, and let (i,j) be the largest relation in $R(Q) \setminus R(P)$. Then there is a poset Q' such that $R(Q') = R(Q) \setminus (i,j)$.*

Proof. Let Q' be the smallest poset containing the relations $R(Q) \setminus (i,j)$. Failure of the lemma would imply that $Q' = Q$, meaning that the relation (i,j) is implied by two relations $(i,k),(k,j) \in R(Q)$. If both these relations are in $R(P)$, then $(i,j) \in R(P)$, contradicting our assumption that $(i,j) \in R(Q) \setminus R(P)$. Otherwise, one of (i,k) or (k,j) belongs to $R(Q) \setminus R(P)$. Since $i < k < j$, each of these relations is later in our ordering than (i,j), which is a contradiction. □

Corollary 0.4. *(to a lemma?!) If P and Q are posets with $R(P) \subseteq R(Q)$, then Q covers P if and only if $|R(P)| = |R(Q)| - 1$. Thus $N(C)$ (and hence $N(P)$ for any P) is ranked, where the rank of P is $|R(P)|$.*

Theorem 0.5. *Labeling each cover $P \prec Q$ with the unique relation in $R(Q) \setminus R(P)$ gives an $S_{\binom{n}{2}}$-EL-labeling on $N(C)$.*

Proof. Let $P < Q$ be an interval in $N(C)$. Let $P \prec P_1 \prec P_2 \prec \cdots \prec P_t = Q$ by the saturated chain defined as follows. If (i,j) is the largest relation in $R(P_k) \setminus R(P)$, let P_{k-1} be the poset such that $R(P_{k-1}) = R(P_k) \setminus (i,j)$ (the existence of P_{k-1} is guaranteed by Lemma 0.3). The label of this saturated chain is clearly increasing. Since any saturated chain between P and Q must be labeled with a permutation of the set $R(Q) \setminus R(P)$, it follows that this increasing chain is unique. □

Corollary 0.6. *For any P, the labeling constructed above is an $S_{|R(P)|}$-EL-labeling of $N(P)$.*

Corollary 0.7. *The lattice $N(P)$ is supersolvable.*

We write $\Delta(N(P))$ to denote the order complex of the proper part of $N(P)$, for any poset P.

Proposition 0.8. *For a poset P, $\Delta(N(P))$ is contractible if and only if $ht(P) > 1$.*

Proof. Let P be a poset with $ht(P) \leq 1$. Then no two relations in $R(P)$ imply a third, meaning every subset $X \subseteq R(P)$ is the set of relations of some poset, and so $N(P)$ is the Boolean lattice $B_{|R(P)|}$, meaning $\Delta(N(P))$ is the boundary of the $(|R(P)| - 1)$-dimensional simplex.

Now let P be a poset with $ht(P) > 1$. Then there exist relations $(i,j),(j,k),(i,k) \in R(P)$. Suppose $\hat{0} \prec P_1 \prec P_2 \prec \cdots \prec P_{|R(P)|} = P$ is a maximal chain of $N(P)$ with decreasing label. The relations (j,k) and (i,j) must precede (i,k) in the label of this chain, meaning some P_t contains the relations (i,j) and (j,k) but not (i,k), which is a contradiction. Thus $N(P)$ has no decreasing chains, and $\Delta(N(P))$ is contractible. □

Corollary 0.9. *For a poset P,*

$$\mu(\hat{0}, P) := \begin{cases} (-1)^{|R(P)|} & \text{if } ht(P) \leq 1 \\ 0 & \text{if } ht(P) > 1 \end{cases}$$

Definition 0.10. *For $n > 0$, let $\mathcal{P}_n$ denote the lattice of* all *posets on the ground set $[n]$, ordered by $P <_{\mathcal{P}_n} Q$ whenever $R(P) \subseteq R(Q)$. We affix a unique maximal element $\hat{1}$ to $\mathcal{P}_n$.*

In (Björner, Edelman, and Welker), a nerve construction is employed to show that the order complex $\Delta(\mathcal{P}_n)$ is homotopy equivalent to the $(n-2)$-sphere, meaning that Möbius function μ of $\mathcal{P}_n$ satisfies $\mu(\hat{0}, \hat{1}) = (-1)^{n-2}$. This can also be proven combinatorially, since Corollary 0.9 implies the following.

$$\mu(\hat{0}, \hat{1}) = -\sum_{P} (-1)^{|R(P)|}$$

where the sum is taken over all posets on $[n]$ of height ≤ 1.