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A CONVEX-EAR DECOMPOSITION FOR RANK-SELECTED SUBPOSETS OF SUPERSOLVABLE LATTICES*

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Abstract. Let L be a supersolvable lattice with nonzero Möbius function. We show that the order complex of any rank-selected subposet of L admits a convex-ear decomposition. This proves many new inequalities for the h-vectors of such complexes, and shows that their g-vectors are M-vectors.

Key words. supersolvable lattice, order complex, h-vector, convex-ear decomposition

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1. Introduction.

One of the most fundamental combinatorial invariants associated to a $(d-1)$-dimensional finite simplicial complex Δ is its *f-vector*, $\langle f_0, f_1, f_2, \ldots, f_d\rangle$, where f_i is the number of $(i-1)$-dimensional faces of Δ. By convention, $f_0 = 1$ whenever $\Delta \neq \emptyset$. Closely related to the f-vector of Δ is its *h-vector*, $\langle h_0, h_1, h_2, \ldots, h_d\rangle$, defined by the transformation $\sum_0^d f_i(x-1)^{d-i} = \sum_0^d h_i x^{d-i}$. Somewhat surprisingly, properties of a complex's f-vector are sometimes better expressed in the language of the h-vector. For instance, when Δ is the boundary complex of a simplicial d-polytope, $h_i = h_{d-i}$ for all i (these are the Dehn–Sommerville relations). The g-theorem, proven by Stanley [10] and Billera and Lee [1], says that an integral sequence $\langle h_0, h_1, h_2, \ldots, h_d\rangle$ is the h-vector of some simplicial polytope boundary if and only if the Dehn–Sommerville relations are satisfied and the associated *g-vector*, $\langle h_0, h_1 - h_0, h_2 - h_1, \ldots, h_{\lfloor \frac{d}{2}\rfloor} - h_{\lfloor \frac{d}{2}\rfloor -1}\rangle$, is an M-vector. An *M-vector* (called an *O-sequence* in some places) is the degree sequence of some order ideal of monomials.

Convex-ear decompositions, first introduced by Chari in [4], are an invaluable tool in proving several key inequalities of a complex's h-vector: when Δ admits a convex-ear decomposition, its h-vector satisfies $h_i \le h_{d-i}$ and $h_i \le h_{i+1}$ for all i with $0 \le i \le \lfloor \frac{d}{2}\rfloor$. Swartz has also proven an analogue of the g-theorem, meaning that the g-vector of a complex which admits a convex-ear decomposition is an M-vector [13].

The purpose of this paper is to prove the following theorem.

Theorem 1.1. *Let L be a rank r supersolvable lattice with nonzero Möbius function. Then for any $S \subseteq [r-1]$ the order complex of the rank-selected poset L_S admits a convex-ear decomposition.*

Here and in the remainder of this paper, we say that a poset P has a "nonzero Möbius function" if $\mu(x, y) \neq 0$ whenever $x, y \in P$ and $x < y$. Given the work of Chari and Swartz, the following is immediate.

Corollary 1.2. *Let L be as above, and let $S \subseteq [r-1]$. Then the h-vector of the order complex of L_S satisfies $h_i \le h_{r-i}$ and $h_i \le h_{i+1}$ whenever $0 \le i \le \lfloor \frac{r}{2}\rfloor$, and the associated g-vector is an M-vector.*

We start by finding a convex-ear decomposition for the order complex of a supersolvable lattice with nonzero Möbius function. This is by far the simplest convex-ear

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decomposition constructed in this paper, but the techniques used will help give the flavor of the decompositions to follow. Next we give a convex-ear decomposition for the order complex of a rank-selected subposet of a Boolean lattice. This decomposition is a good deal more complicated than the first, so it helps to have a feel for our techniques from the previous section. Our main theorem then follows from the first two decompositions. Although our first two decompositions are special cases of our main theorem, we have split our exposition into these three sections in hopes of better readability.

The results in this paper are part of a larger body of work and will be expanded upon in [8]. In this upcoming paper, we will give convex-ear decompositions for order complexes of rank-selected subposets of geometric lattices and certain rank-selected subposets of shellable complex face posets. We will also obtain enumerative results for the flag h-vectors of certain complexes.

2. Preliminaries.

Throughout this section, let Δ be a $(d-1)$-dimensional finite simplicial complex.

For $0 \le i \le d$, let f_i be the number of $(i-1)$-dimensional faces of Δ (by convention we set $f_0 = 1$). We should note that some authors use f_i to mean the number of i-dimensional simplices, but we deviate from that here. The *f-vector* of Δ is the sequence $\langle f_0, f_1, f_2, \dots, f_d\rangle$, and the *h-vector* of Δ is the sequence $\langle h_0, h_1, h_2, \dots, h_d\rangle$ defined by

$$\sum_{i=0}^{d} f_i(x-1)^{d-i} = \sum_{i=0}^{d} h_i x^{d-i}.$$

DEFINITION 2.1. *We say that Δ has a convex-ear decomposition if there exist pure $(d-1)$-dimensional subcomplexes $\Sigma_1, \dots \Sigma_n$ such that*

(i) $\bigcup_1^n \Sigma_i = \Delta$,

(ii) Σ_1 *is the boundary complex of a simplicial d-polytope, and for $i > 1$ there exists a simplicial d-polytope Δ_i so that Σ_i is a pure, full-dimensional subcomplex of* $\partial\Delta_i$,

(iii) *for $i > 1$, Σ_i is a simplicial ball, and*

(iv) *for $i > 1$, $(\bigcup_1^{i-1} \Sigma_j) \cap \Sigma_i = \partial\Sigma_i$.*

We refer to each Σ_i as an *ear* of the decomposition. Convex-ear decompositions were first introduced by Chari in [4], where they were used to prove the following.

THEOREM 2.2 (see [4]). *Let Δ be a $(d-1)$-dimensional simplicial complex that admits a convex-ear decomposition. Then for $i < d/2$ the h-vector of Δ satisfies*

(1) $h_i \le h_{d-i}$, *and*

(2) $h_i \le h_{i+1}$.

Swartz has also proven the following analogue of the g-theorem for complexes admitting such decompositions.

THEOREM 2.3 (see [13]). *Let Δ be as in the statement of the previous theorem. Then the g-vector of Δ, $\langle h_0, h_1 - h_0, h_2 - h_1, \dots, h_{\lfloor d/2\rfloor} - h_{\lfloor d/2\rfloor -1}\rangle$, is an M-vector.*

As an example, let Δ be the 2-dimensional simplicial complex with the vertex set $\{1, 2, 3, 4, 5, 6\}$ and facets $123, 124, 126, 134, 135, 145, 156, 234, 236, 345$, and 356, where we write "ijk" as shorthand for $\{i, j, k\}$. Let Σ_1 be the subcomplex with facets $123, 124, 134$, and 234, let Σ_2 be the subcomplex with facets $135, 145$, and 345, and let Σ_3 be the subcomplex with facets $126, 156, 236$, and 356. The sequence $\Sigma_1, \Sigma_2, \Sigma_3$ is a convex-ear decomposition of Δ. In Figures 1 and 2, we show Σ_2 being attached to Σ_1 and then Σ_3 being attached to $\Sigma_1 \cup \Sigma_2$.

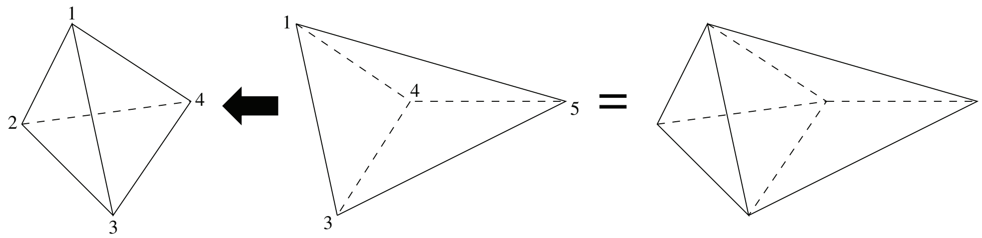

FIG. 1. *Attaching* Σ_2 *to* Σ_1.

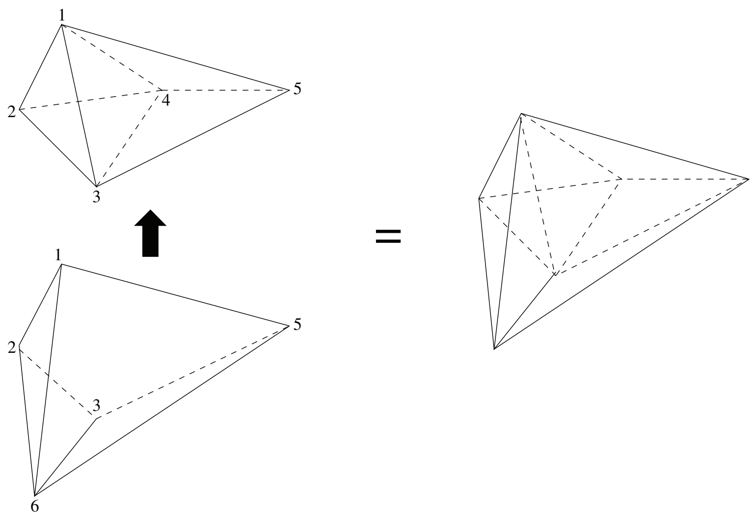

FIG. 2. *Attaching* Σ_3 *to* $\Sigma_1 \cup \Sigma_2$.

We leave it to the reader to verify that the above is a convex-ear decomposition. Note that $\Sigma_1, \Sigma_3, \Sigma_2$ is *not* a convex-ear decomposition, as $\Sigma_3 \cap \Sigma_1 \neq \partial\Sigma_3$.

Convex-ear decompositions can be viewed as a coarser counterpart to the following well-known concept of a *shelling*.

DEFINITION 2.4. *A pure* $(d-1)$*-dimensional finite simplicial complex* Δ *is shellable if there is an ordering of its facets* $F_1, F_2, \ldots, F_t$ *such that* $(\cup_{i=1}^{j-1} F_i) \cap F_j$ *is a nonempty union of facets of* ∂F_j *whenever* $1 < j \leq t$. *Such a facet ordering is called a shelling.*

We will employ shellings several times in this paper, but we will use this alternate definition, shown in [3].

PROPOSITION 2.5. *Let* Δ *be as in the previous definition. Then a facet ordering* $F_1, F_2, \ldots, F_t$ *is a shelling of* Δ *if and only if for all* i, j *with* $1 \leq i < j \leq t$ *there exists a* $k \leq j$ *such that* $F_i \cap F_j \subseteq F_k \cap F_j$ *and* $|F_k \cap F_j| = d - 1$.

We now give some necessary definitions from poset theory.

Let P be a rank r graded poset with a least element $\hat{0}$ and a greatest element $\hat{1}$, and let λ be a function that assigns an integer to each edge of the Hasse diagram of P. That is, $\lambda : \{\langle x, y\rangle \in P^2 : y \text{ covers } x\} \to \mathbb{Z}$. We call λ a labeling, and for some saturated chain $\mathbf{c} := x_i < x_{i+1} < x_{i+2} < \cdots < x_{i+j}$ in P (where each x_k has rank k) define the λ-label of $\mathbf{c}$ to be the word

$$\lambda(x_i, x_{i+1})\lambda(x_{i+1}, x_{i+2})\lambda(x_{i+2}, x_{i+3}) \ldots \lambda(x_{i+j-1}, x_{i+j}).$$

Definition 2.6. *We say λ is an EL-labeling of P if in each interval $x < y$ of P there is a unique saturated chain, starting with x and ending with y, with a strictly increasing λ-label and the label of this chain is lexicographically first among the labels of all saturated chains in this interval.*

Now let P be as above. The *order complex* of P is the simplicial complex whose faces are chains in $P \setminus \{\hat{0}, \hat{1}\}$. The main reason for introducing EL-labelings is the following result, shown in [3].

Theorem 2.7. *Let P be as above, and suppose P admits an EL-labeling λ. Then lexicographic order of the maximal chains of P (with respect to their λ-labels) gives a shelling of the order complex of P.*

Definition 2.8. *Let P be a graded poset with an EL-labeling λ, and let $\mathbf{c}$ be a nonmaximal chain in P. Let the completion of $\mathbf{c}$, written com($\mathbf{c}$), be the maximal chain that results from filling in each gap in $\mathbf{c}$ with the unique chain in that interval with an increasing label.*

Notice that *com*($\mathbf{c}$) depends on the labeling λ. The following helpful lemma follows easily from the definition of an EL-labeling.

Lemma 2.9. *Let P be as above, let P' be a full-rank subposet of P such that λ restricted to P' is an EL-labeling, and let $\mathbf{c}$ be a chain in P'. Then com($\mathbf{c}$) is a (maximal) chain in P'.*

Finally, if $\mathbf{c}$ is a chain containing an element of rank j, we write $\mathbf{c}_{-j}$ to denote the chain that results from removing that element.

We will refer several times to the Möbius function μ of a finite poset. For background on this topic, we refer the reader to [12]. The main property of the Möbius function that we use is the following.

Proposition 2.10 (see [12, Theorem 3.13.2]). *Let P be a poset admitting an EL-labeling λ, and let $x, y \in P$ with $x < y$. Then $|\mu(x, y)|$ is equal to the number of saturated chains in the interval $[x, y]$ whose λ-labels are weakly decreasing.*

3. The supersolvable case. We start by finding a convex-ear decomposition for order complexes of supersolvable lattices with nonzero Möbius function. This construction is motivated by Welker's result [15] that the order complex of a lattice of the above type is 2-Cohen–Macaulay. For a definition of this term, as well as the relevant background, see [11].

Let P be a poset. An *order completion* of P is a total ordering of its elements $x_1 < x_2 < \cdots < x_r$ such that if $x_i < x_j$ in P, then $i < j$. An *order ideal* of P is a subset $I \subseteq P$ such that if $y \in I$ and $x < y$, then $x \in I$. Let $\mathcal{I}(P)$ be the poset of order ideals of P ordered by inclusion.

The following definition is not the standard one but is equivalent by the fundamental theorem of finite distributive lattices (see, for instance, [12, Theorem 3.4.1]).

Definition 3.1. *A finite lattice L is distributive if there exists a poset P such that L is isomorphic to $\mathcal{I}(P)$.*

All distributive lattices admit EL-labelings. To see this, let I and J be two order ideals of some r-element poset P, and note that J covers I in $\mathcal{I}(P)$ if and only if $J = I \cup \{x\}$ for some $x \in P \setminus I$ that covers some $y \in I$. Thus there is a 1-1 correspondence between maximal chains in $\mathcal{I}(P)$ and order completions of P (and so $\mathcal{I}(P)$ is pure of rank r). Now let $x_1 < x_2 < \cdots < x_r$ be an order completion of P, and define the labeling λ by $\lambda(I, J) = n$, where $J = I \cup \{x_n\}$. It is an easy exercise to show that λ is in fact an EL-labeling.

The EL-labeling constructed above is of a special type; each maximal chain in $\mathcal{I}(P)$ is labeled with a permutation of $[r]$. This leads to the following definition.

Definition 3.2. *Let P be a graded poset of rank r, and let λ be an EL-labeling of P. We say that λ is an $\mathcal{S}_r$-EL-labeling if every maximal chain of P is labeled by an element of S_r (when viewed as a word on the alphabet $[r]$).*

The fairly straightforward proof of the following, by induction on the rank of P, is left to the reader.

Lemma 3.3. *Let L be a distributive lattice of rank r, and let P be the poset for which L is the lattice of order ideals. Then every $\mathcal{S}_r$-EL-labeling λ of L is obtained from P in the fashion described above. That is, for every $\mathcal{S}_r$-EL-labeling λ, there exists a bijection $\nu : P \to [r]$ such that $\lambda(I, J) = n$ if and only if $J = I \cup \nu^{-1}(n)$, where I and J are order ideals of P.*

Supersolvable lattices were originally introduced by Stanley in [9] as a generalization of distributive lattices. They are so named because subgroup lattices of supersolvable groups are supersolvable lattices.

Definition 3.4. *Let L be a lattice. We say that L is supersolvable if there exists a maximal chain $\mathbf{c}_M$ of L, called the M-chain (not to be confused with an M-vector), such that the sublattice of L generated by $\mathbf{c}_M$ and any other (not necessarily maximal) chain of L is a distributive lattice.*

The next result gives an alternate characterization of supersolvability.

Theorem 3.5 (see McNamara [7]). *Let P be a poset of rank r. Then P is a supersolvable lattice if and only if it admits an S_r-EL-labeling.*

We will also need the following theorem of Stanley, implicitly shown in [9], for proving our theorem.

Theorem 3.6. *Let L be a rank r supersolvable lattice with S_r-EL-labeling λ and M-chain $\mathbf{c}_M$, let $\mathbf{d}$ be a chain in L, and let L' be the (distributive) sublattice of L generated by $\mathbf{c}_M$ and $\mathbf{d}$. Then λ restricted to L' is an S_r-EL-labeling.*

Also in [9], Stanley proves that, under an S_r-EL-labeling of a supersolvable lattice L, the unique maximal chain with increasing label is an M-chain.

The main result in this section is the following theorem.

Theorem 3.7. *Let L be a rank r supersolvable lattice such that $\mu(x, y) \neq 0$ whenever $x, y \in L$ and $x < y$. Then the order complex of L admits a convex-ear decomposition.*

For the remainder of this section, fix an S_r-EL-labeling of L. Call this labeling λ.

We now construct the ears of the decomposition. Let $\mathbf{d}_1, \mathbf{d}_2, \ldots, \mathbf{d}_t$ be all maximal chains of L with decreasing labels (the order of the list is arbitrary but fixed from here on). This list is nonempty, since $\mu(\hat{0}, \hat{1}) \neq 0$. For each i, let L_i be the sublattice of L generated by $\mathbf{d}_i$ and $\mathbf{c}_M$, and let Σ_i be the simplicial complex whose facets are given by maximal chains in $L_i \setminus \{\hat{0}, \hat{1}\}$ that are not chains in L_j for any $j < i$. We let the Σ_i's do double-duty, simultaneously representing the complex mentioned above and the set of (not necessarily maximal) chains in L that correspond to faces of that complex. Given the order below, it is sometimes helpful to think of maximal chains (i.e., facets) of Σ_i as "new" and maximal chains of $L_i \setminus \{\hat{0}, \hat{1}\}$ that are not in Σ_i as "old."

We claim that $\Sigma_1, \Sigma_2, \ldots, \Sigma_t$ is a convex-ear decomposition of $\Delta(L)$. We will show each part of the decomposition separately.

Proof of property (ii). By definition, each L_i is a distributive lattice. Fix i, and let P be the poset such that $\mathcal{I}(P) \simeq L_i$. By Theorem 3.6 and Lemma 3.3, the chain $\mathbf{c}_M$ in L_i gives us an order completion of P: $x_1 < x_2 < \cdots < x_r$. Similarly, the chain $\mathbf{d}_i$ gives another order completion of P: $x_r < x_{r-1} < \cdots < x_1$. So for any $x_j, x_k \in P$, one of the above order completions gives $x_j < x_k$, while the other gives $x_k < x_j$. Thus no two elements in P are comparable, and any subset of elements is an order ideal of P. So L_i is isomorphic to B_r, the Boolean lattice on r elements. Since the order complex of B_r is the first barycentric subdivision of the boundary of the r-simplex, and since $\Sigma_1 = L_1$ and $\Sigma_i \subsetneq L_i$ for $i > 1$ (because $\mathbf{c}_M$ is in every L_i), this completes our proof of property (ii) of the decomposition. ∎

Proof of property (i). Let $\mathbf{c} := \hat{0} = x_0 < x_1 < \cdots < x_r = \hat{1}$ be a maximal chain of L. We must show that $\mathbf{c}$ is a chain in L_i for some i, and we do this by induction on the number of ascents of the chain-label of $\mathbf{c}$. If the chain-label has no ascents, then $\mathbf{c} = \mathbf{d}_i$ for some i and is therefore in L_i. Otherwise, $\mathbf{c}$ has at least one ascent, say, at position j. Since L has nonzero Möbius function, the interval (x_{j-1}, x_{j+1}) has at least one element other than x_j. Let $\mathbf{c}'$ be the chain that results from replacing x_j in $\mathbf{c}$ with one of these other elements. Since $\mathbf{c}'$ has one fewer ascent than $\mathbf{c}$, it belongs to some L_i by induction. Since λ is an EL-labeling on L_i (Theorem 3.6), $com((\mathbf{c}')_{-j}) = \mathbf{c}$ is a chain in L_i by Lemma 2.9. ∎

Proof of property (iii). To prove that Σ_i is a ball for all $i > 2$, we show that the reverse lexicographic order of the maximal chains in Σ_i is a shelling. Invoking a result of Danaraj and Klee [5], which states that a shellable full-dimensional proper subcomplex of a sphere must be a ball, completes the proof. Let $\mathbf{c} := \hat{0} = x_0 < x_1 < \cdots < x_r = \hat{1}$ and $\mathbf{d}$ be two chains in Σ_i, with $\mathbf{d}$ lexicographically later (and therefore earlier in the shelling) than $\mathbf{c}$. By the argument given on pages 25–26 of [2], there must be some j such that $\mathbf{d}$ and $\mathbf{c}$ do not coincide at the jth rank and such that $\lambda(x_{j-1}, x_j) < \lambda(x_j, x_{j+1})$. Now let $\mathbf{c}'$ be the unique maximal chain of L_i that coincides with $\mathbf{c}$ everywhere but the jth position. Then, by definition of an EL-labeling, $\mathbf{c}'$ is lexicographically later than $\mathbf{c}$ (and thus earlier in the shelling), $|\mathbf{c} \setminus \mathbf{c}'| = 1$, and $\mathbf{c} \cap \mathbf{d} \subseteq \mathbf{c}'$. It remains to be shown that $\mathbf{c}'$ is in Σ_i. If $\mathbf{c}'$ were not a chain in Σ_i, it would be a chain in L_k for some $k < i$, meaning $(\mathbf{c}')_{-j}$ is a chain in L_k. But then, again by Lemma 2.9, we would have that $com((\mathbf{c}')_{-j}) = \mathbf{c}$ is a chain in L_k. This would mean that $\mathbf{c}$ is not a chain in Σ_i, which is a contradiction. ∎

We have yet to prove property (iv). Since we will use a very similar technique to prove this property in the coming sections, we outline the method here and refer back to this exposition later.

Proof of property (iv). Fix $i > 1$, and note that a chain $\mathbf{c}$ in Σ_i is in $\partial\Sigma_i$ if and only if there exist two maximal chains containing it, $\mathbf{c}_{old}$ and $\mathbf{c}_{new}$, such that $\mathbf{c}_{old}$ is a maximal chain of L_i but not Σ_i, and $\mathbf{c}_{new}$ is a maximal chain in Σ_i.

From the above description of chains in the boundary of Σ_i, $\partial\Sigma_i \subseteq (\bigcup_1^{i-1} \Sigma_j) \cap \Sigma_i$. To see the reverse inclusion, let $\mathbf{c}$ be a chain in $(\bigcup_1^{i-1} \Sigma_j) \cap \Sigma_i$. Then $\mathbf{c}$ is, by definition, a subchain of some facet of Σ_i. This chain is the required $\mathbf{c}_{new}$. To complete the proof, we must find a suitable $\mathbf{c}_{old}$. However, since $\mathbf{c}$ is a chain in $\bigcup_1^{i-1} \Sigma_j$, it must be a chain in some L_j for $j < i$. Then Lemma 2.9 guarantees that $com(\mathbf{c})$ is in L_j, so set $\mathbf{c}_{old} = com(\mathbf{c})$. ∎

4. The rank-selected Boolean case.

Definition 4.1. *Let P be a graded poset of rank r, and let $S \subseteq [r-1]$. The rank-selected subposet P_S is defined to be the poset with elements $\{x \in P : \mathrm{rank}(x) \in S \cup \{\hat{0}, \hat{1}\}\}$ and order inherited from P.*

Recall that B_r denotes the rank r Boolean lattice. This section is devoted to proving the following theorem.

Theorem 4.2. *For any $S \subseteq [r-1]$, the order complex of the rank-selected subposet $(B_r)_S$ admits a convex-ear decomposition.*

Throughout this section, we fix an $\mathcal{S}_r$-EL-labeling λ of B_r defined as follows: view the elements of B_r as subsets of $[r]$, and note that y covers x if and only if $y = x \cup \{n\}$ for some $n \in [r] \setminus x$. To define the labeling λ, set $\lambda(x, y) = n$. It is easy to see that λ is an $\mathcal{S}_r$-EL-labeling.

For any subset $S \subseteq [r-1]$ and any maximal chain $\mathbf{c}$ of B_r, let $\mathbf{c}_S$ denote the subchain of $\mathbf{c}$ consisting of all elements in $\mathbf{c}$ whose ranks are in $S \cup \{0, r\}$. In particular, we write $\mathbf{c}_j$ as shorthand for $\mathbf{c}_{\{j\}}$, the element of $\mathbf{c}$ of rank j with $\hat{0}$ and $\hat{1}$ adjoined. Note that $\mathbf{c}_S$ is a maximal chain in $(B_r)_S$.

Now fix a subset $S \subseteq [r-1]$ for the remainder of this section, and write S as a disjoint union of intervals, where $a_1 < a_2 < \cdots < a_s$:

$$S = [a_1, b_1] \cup [a_2, b_2] \cup \cdots \cup [a_s, b_s]$$

and no $a_i - 1$ or $b_i + 1$ is a member of S and $b_i < a_{i+1}$ for all i. Where appropriate, we also set $b_0 = 0$ and $a_{s+1} = r$.

Because maximal chains in B_r, under their λ-labels, are in bijection with permutations of $[r]$, we do much of our work in the context of $\mathcal{S}_r$, where we write permutations in word form: $\sigma = \sigma(1)\sigma(2)\ldots\sigma(r)$. When $1 \le m < n \le r$, we write $\sigma(m, n)$ to mean the set $\{\sigma(m), \sigma(m+1), \ldots, \sigma(n)\}$.

Let $\mathbf{c}$ be a maximal chain in B_r with $\lambda(\mathbf{c}) = \sigma \in \mathcal{S}_r$. We wish to characterize the labels of all chains that coincide with $\mathbf{c}$ at ranks in S. This will turn out to be the coincidence set $C(\sigma)$ described below. Similarly, the set $Sp(\sigma)$ defined below is the set of labels of chains that coincide with $\mathbf{c}$ at ranks *not* in S.

First, for a permutation $\sigma \in \mathcal{S}_r$, define the *coincidence set* of σ, written $C(\sigma)$, as the set of all $\tau \in \mathcal{S}_r$ such that $\tau(m) = \sigma(m)$ for all $m \in S \setminus \{a_1, a_2, \ldots, a_s\}$ and $\sigma(b_i+1, a_{i+1}) = \tau(b_i+1, a_{i+1})$ for all i. To visualize the set $C(\sigma)$, define the bracketed word σ^C to be the word of σ with a left bracket inserted before each $\sigma(b_i+1)$ and a right bracket inserted after each $\sigma(a_i)$ (as usual, we let $b_0 = 0$ and $a_{s+1} = r$). Then $C(\sigma)$ is the set of permutations that can be obtained by permuting the elements between the brackets of σ^C.

For example, suppose $r = 7$, $S = \{2, 3, 4, 6\}$, and $\sigma = 5\,3\,7\,4\,1\,6\,2$. Then $S = [2, 4] \cup [6, 6]$, and the bracketed word defined above is

$$\sigma^C = [\,5\,3\,]\,7\,4\,[\,1\,6\,]\,[\,2\,].$$

Thus the set $C(\sigma)$ consists of four permutations: $3\,5\,7\,4\,1\,6\,2$, $3\,5\,7\,4\,6\,1\,2$, $5\,3\,7\,4\,1\,6\,2 = \sigma$, and $5\,3\,7\,4\,6\,1\,2$.

Now define the *span* of σ, written $Sp(\sigma)$, to be the set of all permutations $\tau \in \mathcal{S}_r$ such that $\tau(m) = \sigma(m)$ whenever $b_i + 1 < m < a_i$ for some i, and $\tau(a_i, b_i + 1) = \sigma(a_i, b_i + 1)$ for all i. Here, we do not follow our convention that $b_0 = 0$ and $a_{s+1} = r$. As before, define a bracketed word σ^{Sp} as follows: insert a left bracket before each $\sigma(a_i)$ and a right bracket after each $\sigma(b_i+1)$. Then $Sp(\sigma)$ consists of all permutations obtained from σ by permuting the elements between the brackets of σ^{Sp}.

Continuing with our example,

$$\sigma^{Sp} = 5\,[\,3\,7\,4\,1\,]\,[\,6\,2\,].$$

Thus a permutation in $Sp(\sigma)$ is given by permuting the set $\{1, 3, 4, 7\}$ within the first bracket and the set $\{2, 6\}$ within the second. (When no confusion can result, we use "bracket" to mean the word specified by a pair of brackets.)

Note that our above definitions depend on our choice of the set $S \subseteq [r-1]$. However, as we have fixed one choice of S for the entire section, we suppress "S"

from our notation. Given the bracket interpretations of the sets $C(\sigma)$ and $Sp(\sigma)$, the following lemma is obvious.

LEMMA 4.3. *Fix two permutations $\sigma, \tau \in \mathcal{S}_r$. Then $\sigma \in C(\tau)$ if and only if $C(\sigma) = C(\tau)$, and $\sigma \in Sp(\tau)$ if and only if $Sp(\sigma) = Sp(\tau)$.*

For a permutation $\sigma \in \mathcal{S}_r$, let $\mathbf{c}^\sigma$ denote the unique maximal chain in B_r with σ as its λ-label. That is,

$$\mathbf{c}^\sigma := \hat{0} = x_0 < x_1 < \cdots < x_{r-1} < x_r = \hat{1}$$

and $\sigma(m) = \lambda(x_{m-1}, x_m)$ for all m. For a subset $T \subseteq [r-1]$, we write $\mathbf{c}^\sigma_T$ as shorthand for $(\mathbf{c}^\sigma)_T$. The following is our reason for introducing the sets $C(\sigma)$ and $Sp(\sigma)$.

PROPOSITION 4.4. *Let $\sigma, \tau \in \mathcal{S}_r$. Then $C(\sigma) = C(\tau)$ if and only if $\mathbf{c}^\sigma_S = \mathbf{c}^\tau_S$, and $Sp(\sigma) = Sp(\tau)$ if and only if $\mathbf{c}^\sigma_{[r-1]\setminus S} = \mathbf{c}^\tau_{[r-1]\setminus S}$.*

Proof. Suppose $C(\sigma) = C(\tau)$, and let $m \in S$. Then there are two possible cases: either $\sigma(j)$ is in no bracket of σ^C, or it is the rightmost element in some bracket. In either case, $\tau(1, m) = \sigma(1, m)$, since rearranging elements in a bracket of σ^C cannot remove an element from, or add an element to, the set $\sigma(1, m)$. Viewing elements of B_r as subsets of $[r]$, we have $\mathbf{c}^\sigma_m = \sigma(1, m) = \tau(1, m) = \mathbf{c}^\tau_m$, and so $\mathbf{c}^\sigma_S = \mathbf{c}^\tau_S$.

For the reverse implication, suppose that $\mathbf{c}^\sigma_S = \mathbf{c}^\tau_S$, and fix some $m \in S \setminus \{a_1, a_2, \ldots, a_s\}$. Then $m - 1 \in S$, meaning $\mathbf{c}^\sigma_{m-1} = \mathbf{c}^\tau_{m-1}$. Since $\mathbf{c}^\sigma_m = \mathbf{c}^\tau_m$,

$$\sigma(m) = \lambda(\mathbf{c}^\sigma_{m-1}, \mathbf{c}^\sigma_m) = \lambda(\mathbf{c}^\tau_{m-1}, \mathbf{c}^\tau_m) = \tau(m).$$

Now fix some i with $0 \le i \le s$. Then $\mathbf{c}^\sigma_{b_i} = \mathbf{c}^\tau_{b_i}$ and $\mathbf{c}^\sigma_{a_{i+1}} = \mathbf{c}^\tau_{a_{i+1}}$. It follows that the sets $\sigma(b_i + 1, a_{i+1})$ and $\tau(b_i + 1, a_{i+1})$ are equal, since each is equal to $\mathbf{c}^\sigma_{a_{i+1}} \setminus \mathbf{c}^\sigma_{b_i}$ where again elements of B_r are viewed as subsets of $[r]$. Thus $\tau \in C(\sigma)$, or equivalently $C(\sigma) = C(\tau)$.

The proof of the lemma's second statement is completely analogous to the proof of the first. ◻

In Figure 3, we show (between the chain with increasing label and the chain with decreasing label) the four maximal chains in B_7 whose labels are permutations in $C(\sigma)$, where σ and S are as in our running example.

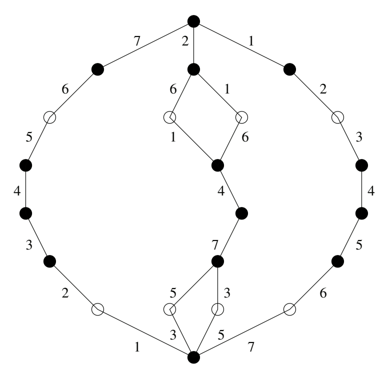

FIG. 3. *Maximal chains whose labels are in $C(\sigma)$. Elements whose ranks are in $S \cup \{0, 7\}$ are filled in.*

Let P be any graded poset of rank r that admits an EL-labeling. Then the order complex of P_S is shellable and homotopy equivalent to t-many spheres (see [3]), where t is the number of maximal chains of P whose labels have descent set S. Recall that the descent set of a permutation $\sigma \in \mathcal{S}_r$ is $des(\sigma) = \{m \in [r-1] : \sigma(m) > \sigma(m+1)\}$. In the case we treat, where $P = B_r$, t is the number of permutations in $\mathcal{S}_r$ with descent set S. It makes sense, then, that our convex-ear decomposition is constructed from the set $D = \{\delta \in \mathcal{S}_r : des(\delta) = S\}$.

For any $\sigma \in \mathcal{S}_r$, define a permutation δ_σ as follows: first, let π_σ be the permutation obtained by replacing each bracket in σ^C with the increasing word in those letters. In keeping with our running example,

$$\pi_\sigma^C = [\,3\,5\,]\,7\,4\,[\,1\,6\,]\,[\,2\,],$$

where we have written π_σ^C rather than just π_σ in hopes of better readability. Next, obtain δ_σ by replacing the contents of each bracket in π_σ^{Sp} with the decreasing word in those letters. Continuing with our example,

$$\pi_\sigma^{Sp} = 3\,[\,5\,7\,4\,1\,]\,[\,6\,2\,], \text{ and so } \delta_\sigma^{Sp} = 3\,[\,7\,5\,4\,1\,]\,[\,6\,2\,].$$

Note that, by construction, π_σ is in both $C(\sigma)$ and $Sp(\delta_\sigma)$, and so $C(\sigma) \cap Sp(\delta_\sigma) \neq \emptyset$.

Proposition 4.5. *For any $\sigma \in \mathcal{S}_r$, $\delta_\sigma \in D$.*

Proof. Let $n \in S$. Then $\delta_\sigma(n)$ and $\delta_\sigma(n+1)$ are in the same bracket of δ_σ^{Sp}. Because δ_σ is obtained from π_σ by putting the contents of each bracket of π_σ^{Sp} in decreasing order, it must be the case that $\delta_\sigma(n) > \delta_\sigma(n+1)$. Thus $S \subseteq des(\delta_\sigma)$. Suppose $S \neq des(\delta_\sigma)$, and choose some $m \in des(\delta_\sigma) \setminus S$. Then $m = a_j - 1$ or $m = b_j + 1$ for some j. Suppose $m = a_j - 1$. $\pi_\sigma(a_j - 1)$ is in the same bracket of π_σ^C as $\pi_\sigma(a_j)$, so $\pi_\sigma(a_j - 1) < \pi_\sigma(a_j)$. Furthermore, $\pi_\sigma(a_j)$ is the leftmost element of some bracket of π_σ^{Sp}, and so by construction $\delta_\sigma(a_j) \geq \pi_\sigma(a_j)$. Similarly, $\pi_\sigma(a_j - 1)$ either is not in any bracket of π_σ^{Sp} or is the rightmost element in some bracket, so $\delta_\sigma(a_j - 1) \leq \pi_\sigma(a_j - 1)$. Stringing these inequalities together,

$$\delta_\sigma(m) = \delta_\sigma(a_j - 1) \leq \pi_\sigma(a_j - 1) < \pi_\sigma(a_j) \leq \delta_\sigma(a_j) = \delta_\sigma(m+1),$$

which is a contradiction. The proof for the case in which $m = b_j + 1$ for some j is symmetric. Thus $des(\delta_\sigma) = S$, and so $\delta_\sigma \in D$. ∎

Now choose $\sigma, \delta, \tau \in \mathcal{S}_r$, with $\tau \in C(\sigma) \cap Sp(\delta)$. By Proposition 4.4, $\mathbf{c}_S^\tau = \mathbf{c}_S^\sigma$ and $\mathbf{c}_{[r-1]\setminus S}^\tau = \mathbf{c}_{[r-1]\setminus S}^\delta$. Because only one maximal chain in B_r can satisfy both these constraints, it follows that the permutation τ is uniquely determined. Thus for any $\sigma, \delta \in \mathcal{S}_r$, $|C(\sigma) \cap Sp(\delta)| \leq 1$.

Lemma 4.6. *Let $\sigma \in \mathcal{S}_r$ and $\delta \in D$, and suppose that $C(\sigma) \cap Sp(\delta) = \{\tau\}$. Then $\delta = \delta_\sigma$ if and only if the contents of each bracket of τ^C are increasing.*

Proof. Suppose each bracket of τ^C is increasing. $\tau \in C(\sigma)$, so it follows that $\tau = \pi_\sigma$, as defined in the proof of Proposition 4.5. Since δ_σ is obtained by permuting elements in the brackets of $\pi_\sigma^{Sp} = \tau^{Sp}$, $\tau \in Sp(\delta_\sigma)$. By assumption, $\tau \in Sp(\delta)$, and so by Lemma 4.3 $Sp(\delta_\sigma) = Sp(\delta)$. Because both δ and δ_σ are members of D, each bracket of δ^{Sp} and δ_σ^{Sp} must be decreasing, so $\delta = \delta_\sigma$.

Now suppose some bracket of τ^C is nonincreasing. Put another way, the word $\tau(b_j+1)\tau(b_j+2)\ldots\tau(a_{j+1})$ is nonincreasing for some j. Choose an m with $b_j + 1 \leq m \leq a_{j+1} - 1$ and $\tau(m) > \tau(m+1)$. If it were the case that $b_j + 1 < m < a_{j+1} - 1$, then we would necessarily have $\delta(m) = \tau(m)$ and $\delta(m+1) = \tau(m+1)$, since both entries are outside the brackets of δ^{Sp} and $\tau \in Sp(\delta)$. But then $m \in des(\delta) = S$, which is a contradiction. Therefore, either $m = b_j + 1$ or $m = a_{j+1} - 1$. We treat only the first case, the proof of the second being similar.

Note that $\tau \in C(\sigma) = C(\pi_\sigma)$, and so $\pi_\sigma = \pi_\tau$. Because π_τ is obtained by putting the brackets of τ^C in increasing order, $\tau(b_j+1) > \tau(b_j+2)$, and so $\pi_\tau(b_j+1) < \tau(b_j+1)$. It follows that $Sp(\pi_\tau) \neq Sp(\tau)$. Putting this together,

$$Sp(\delta_\sigma) = Sp(\pi_\sigma) = Sp(\pi_\tau) \neq Sp(\tau) = Sp(\delta),$$

and so $\delta \neq \delta_\sigma$. ◻

PROPOSITION 4.7. *Let $\sigma \in \mathcal{S}_r$. Then δ_σ is the lexicographically least permutation in the set $\{\delta \in D : C(\sigma) \cap Sp(\delta) \neq \emptyset\}$.*

Proof. Fix $\delta \in D \setminus \{\delta_\sigma\}$ such that $C(\sigma) \cap Sp(\delta) = \{\tau\}$ for some $\tau \in \mathcal{S}_r$. By the previous proposition, some bracket of τ^C is nonincreasing, meaning the word $\tau(b_j+1)\tau(b_j+2)\ldots\tau(a_{j+1})$ is nonincreasing for some j. So, in forming the permutation π_τ, this bracket is put in increasing order. It follows that $\delta_\tau = \delta_\sigma$ is lexicographically less than δ. ◻

We now use our work in $\mathcal{S}_r$ to construct a convex-ear decomposition for the order complex of $(B_r)_S$. Let $\delta_1, \delta_2, \ldots, \delta_t$ be all permutations in D, listed in lexicographic order of their labels. For each i let $\mathbf{d}_i = \mathbf{c}^{\delta_i}$ (in other words, $\mathbf{d}_i$ is the unique maximal chain in B_r with δ_i as its λ-label). Also let L_i be the poset generated by all maximal chains in $(B_r)_S$ of the form $\mathbf{c}_S$, where $\mathbf{c}$ is a maximal chain in B_r such that $\mathbf{c}_{[r-1]\setminus S} = (\mathbf{d}_i)_{[r-1]\setminus S}$. Finally, let Σ_i be the simplicial complex whose facets are given by maximal chains in $L_i \setminus \{\hat{0}, \hat{1}\}$ that are not chains in L_j for any $j < i$. As in the previous section, we use Σ_i to refer to both the simplicial complex above and the poset whose chains correspond to (not necessarily maximal) chains in $(B_r)_S$.

PROPOSITION 4.8. *$\Sigma_1, \Sigma_2, \ldots, \Sigma_t$ is a convex-ear decomposition of the order complex of $(B_r)_S$.*

To every maximal chain $\mathbf{e}$ in $(B_r)_S$, associate an equivalence class of maximal chains in B_r, namely, all maximal chains $\mathbf{c}$ such that $\mathbf{c}_S = \mathbf{e}$. By Proposition 4.4, this equivalence class can be viewed as the set $\{\mathbf{c}^\tau : \tau \in C(\sigma)\}$ for some $\sigma \in \mathcal{S}_r$. We refer to $C(\sigma)$ as the class corresponding to $\mathbf{e}$.

Next let $\mathbf{c}$ be a maximal chain in B_r such that $\mathbf{c}_S$ is a maximal chain in L_i. $\mathbf{c}_{[r-1]\setminus S} = (\mathbf{d}_i)_{[r-1]\setminus S}$, and so, by Proposition 4.4, $\lambda(\mathbf{c}) \in Sp(\delta_i)$. Let $\sigma = \lambda(\mathbf{c})$. The chain $\mathbf{c}_S$ then corresponds to the equivalence class $C(\sigma)$, and we have proven half of the following lemma.

LEMMA 4.9. *Let $\sigma \in \mathcal{S}_r$, and let $\mathbf{e}$ be a maximal chain in $(B_r)_S$ corresponding to the equivalence class $C(\sigma)$. Then $\mathbf{e}$ is a maximal chain in L_i if and only if $C(\sigma) \cap Sp(\delta_i) \neq \emptyset$.*

Proof. We have already proven the "only if" direction above. For the other direction, suppose $C(\sigma) \cap Sp(\delta_i) \neq \emptyset$. Choose the unique τ in this intersection. By Proposition 4.4, $\mathbf{c}^\tau_S = \mathbf{e}$ and $\mathbf{c}^\tau_{[r-1]\setminus S} = (\mathbf{d}_i)_{[r-1]\setminus S}$, and so $\mathbf{e}$ is a maximal chain in L_i. ◻

Now let $\mathbf{e}$ and σ be as in the statement of the above lemma, and suppose $\mathbf{e}$ is a facet in Σ_i. Then δ_i is the lexicographically first permutation δ in D such that $C(\sigma) \cap Sp(\delta) \neq \emptyset$, and so, by Proposition 4.7, $\delta_i = \delta_\sigma$. Summarizing, we have the following lemma.

LEMMA 4.10. *Let $\mathbf{e}$ be a maximal chain in $(B_r)_S$ corresponding to the class $C(\sigma)$ for some $\sigma \in \mathcal{S}_r$. Then $\mathbf{e}$ represents a facet in Σ_i if and only if $\delta_i = \delta_\sigma$.*

We are now ready to prove the properties of our convex-ear decomposition.

Proof of property (i). We must show that any maximal chain $\mathbf{e}$ in $(B_r)_S$ is a maximal chain in some L_i. By Lemma 4.9, we must find some $\delta \in D$ such that

$C(\sigma) \cap Sp(\delta) \neq \emptyset$, where $C(\sigma)$ is the class corresponding to $\mathbf{e}$. But Lemma 4.5 guarantees such a permutation, namely, δ_σ. □

Proof of property (ii). Fix $\mathbf{d}_i$, and write $\mathbf{d}_i := \hat{0} = x_0 < x_1 < \cdots < x_r = \hat{1}$. A maximal chain in L_i is determined by a choice of maximal chain in each open interval (x_{a_j-1}, x_{b_j+1}). Each of these intervals is isomorphic to $B_{b_j-a_j+2} \setminus \{\hat{0}, \hat{1}\}$. As noted before, the order complex of $B_n \setminus \{\hat{0}, \hat{1}\}$ is $b(\partial\Delta_{n-1})$, where b denotes the first barycentric subdivision and Δ_{n-1} denotes the $(n-1)$-dimensional simplex. Thus the order complex of L_i is the product

$$b(\partial\Delta_{b_1-a_1+1}) * b(\partial\Delta_{b_2-a_2+1}) * \cdots * b(\partial\Delta_{b_s-a_s+1}),$$

where $*$ denotes simplicial join (see [6] for background on this operation and [16] for its application to polytopes). It follows that the order complex of each L_i is the boundary complex of a simplicial polytope. Since Σ_1 is the order complex of L_1, it remains to be shown that Σ_i is a *proper* subcomplex of the order complex of L_i when $i > 1$.

Fix δ_i with $i > 1$, and define a permutation $\sigma \in Sp(\delta_i)$ by putting each bracket of δ_i^{Sp} in increasing order. There are two cases to consider: first, suppose that $\sigma = 1\,2 \ldots r$. In this case, we leave it to the reader to show that $\delta_i = \delta_1$, the lexicographically first permutation in $\mathcal{S}_r$ with descent set S, contradicting our assumptions. Now suppose otherwise. Since each bracket of σ^{Sp} is increasing, it must be the case that some bracket of σ^C is nonincreasing. Then, by Lemma 4.6, $\delta_i \neq \delta_\sigma$, since $C(\sigma) \cap Sp(\delta_i) = \{\sigma\}$. Finally, by Proposition 4.7, δ_σ precedes δ_i lexicographically, and so $\delta_\sigma = \delta_j$ for some $j < i$. □

Proof of property (iii). Fix $i > 1$, and let $\mathbf{e}$ be a maximal chain representing a facet in Σ_i. Pick a $\sigma \in \mathcal{S}_r$ such that $\mathbf{e}$ corresponds to the equivalence class $C(\sigma)$. Define $\pi_\mathbf{e}$ to be the permutation π_σ. Let $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ be the maximal chains of $(B_r)_S$ corresponding to facets of Σ_i. Writing π_j as shorthand for $\pi_{\mathbf{e}_j}$, let the above order be so that π_j is lexicographically greater than π_k whenever $j < k$. In particular, $\pi_1 = \delta_i$. We claim that this ordering is a shelling of Σ_i.

Let $j < k$. Since $Sp(\pi_j) = Sp(\delta_i) = Sp(\pi_k)$, π_j^{Sp} and π_k^{Sp} coincide outside of their brackets. Because π_k lexicographically precedes π_j, there must be some ascent, $\pi_k(m) < \pi_k(m+1)$, such that $\pi_k(1,m) \neq \pi_j(1,m)$ and so that $\pi_k(m)$ and $\pi_k(m+1)$ are in the same bracket of π_k^{Sp}. We claim that the proof of this assertion is, as in the proof of property (iii) in the previous section, analogous to the discussion on pages 25–26 of [2]. This is because $\pi_k(1,m) \neq \pi_j(1,m)$ if and only if $\mathbf{c}_m^{\pi_k} \neq \mathbf{c}_m^{\pi_j}$, by Proposition 4.4. Let π_k' be the permutation obtained from π_k by switching $\pi_k(m)$ and $\pi_k(m+1)$.

Note that π_k' is lexicographically greater than π_k. It is clear that $C(\pi_k') \cap Sp(\delta_i) = \{\pi_k'\}$. Now fix some p, and consider the following bracket in π_k^C:

$$\pi_k(b_p+1)\pi_k(b_p+2)\ldots\pi_k(a_{p+1}).$$

$\pi_k(m)$ and $\pi_k(m+1)$ are in the same bracket of π_k^{Sp}, so there are only three possibilities for the placement of $\pi_k(m)$ within the above bracket: $m+1 = b_p+1$, $m = a_{p+1}$, or $\{m, m+1\} \cap [b_p+1, a_{p+1}] = \emptyset$. In the first case, $m = b_p$, and the corresponding bracket in $(\pi_k')^C$ is

$$\pi_k'(m+1)\pi_k'(b_p+2)\ldots\pi_k'(a_{p+1}) = \pi_k(m)\pi_k(b_p+2)\ldots\pi_k(a_{p+1}).$$

Because this bracket is increasing in π_k^C (by Lemmas 4.10 and 4.6) and $\pi_k(m) < \pi_k(m+1)$, it must be increasing in $(\pi_k')^C$ as well, meaning $\delta_i = \delta_{\pi_k'}$ (by Lemma 4.6).

The proof for the second case is again symmetric to the case we have proven, and the proof for the third case is trivial (since the bracket's contents are unchanged). Thus $\pi'_k = \pi_\ell$ for some $\ell < k$, since π'_k is lexicographically later than π_k.

To complete the proof, we have to show that $\mathbf{e}_j \cap \mathbf{e}_k \subseteq \mathbf{e}_j \cap \mathbf{e}'_k$. Since $\mathbf{e}_k$ coincides with $\mathbf{e}'_k$ everywhere except at rank m, it is enough to show that $\mathbf{e}_j$ and $\mathbf{e}_k$ do not intersect at that rank. But this follows immediately, since $\mathbf{c}_m^{\pi_k} \neq \mathbf{c}_m^{\pi_j}$. ∎

Proof of property (iv). We take our cue from the proof of property (iv) from the first section, since the Σ_i are defined analogously. That is, let $\mathbf{e}_i$ and $\mathbf{e}_j$ be facets of Σ_i and Σ_j, where $i < j$, and let $\mathbf{e} = \mathbf{e}_i \cap \mathbf{e}_j$. By the discussion in the proof of property (iv) in the previous section, it suffices to find a facet $\mathbf{e}'$ of some Σ_k with $k < j$ such that $\mathbf{e}'$ contains $\mathbf{e}$.

Define the maximal chain $\mathbf{e}'$ by $\mathbf{e}' = (com(\mathbf{e}))_S$, and let σ be the λ-label of $com(\mathbf{e})$. By construction, $\pi_\sigma = \sigma$. Now let τ be the λ-label of some maximal chain $\mathbf{c}$ in B_r with $\mathbf{c}_S = \mathbf{e}_i$. It is clear that π_τ is independent of the choice of maximal chain $\mathbf{c}$ and that π_σ is lexicographically less than or equal to π_τ. It follows that δ_σ is lexicographically less than or equal to δ_τ, which means that $\mathbf{e}'$ is a facet of Σ_k for some $k \leq i < j$. ∎

5. The rank-selected supersolvable case. It is implicit in our earlier work that supersolvable lattices are composed of Boolean lattices that are pieced together in an orderly fashion. Using the previous sections, we can prove the following theorem.

THEOREM 5.1. *Let L be a rank r supersolvable lattice such that $\mu(x, y) \neq 0$ whenever $x, y \in L$ and $x < y$, and let $S \subseteq [r-1]$. Then the order complex of L_S admits a convex-ear decomposition.*

Fix an S_r-EL-labeling λ of L. Let $\mathbf{d}_1, \mathbf{d}_2, \ldots, \mathbf{d}_t$ be a fixed ordering of the maximal chains in L with decreasing λ-label. For each i, let L_i be the sublattice of L generated by $\mathbf{d}_i$ and the unique maximal chain in L with increasing λ-label. From our convex-ear decomposition for supersolvable lattices, we know that each L_i is isomorphic to B_r. For a fixed i, let $\mathbf{d}_i^1, \mathbf{d}_i^2, \ldots, \mathbf{d}_i^t$ be a list of the maximal chains in L_i whose labels have descent set S, where the chains are listed in lexicographic order of their labels. For each j, let L_i^j be the poset generated by all maximal chains in $\mathbf{c}$ in L_i such that $\mathbf{c}_{[r-1]\setminus S} = (\mathbf{d}_i^j)_{[r-1]\setminus S}$. In other words, L_i^j is just the poset L_j as defined in our convex-ear decomposition for $(B_r)_S$, when L_i is viewed as the Boolean lattice B_r. Finally, let Σ_i^j be the simplicial complex whose facets are given by the maximal proper chains in L_i^j that are not maximal chains in any L_i^k for some $k < j$ or any L_m^n for some $m < i$.

PROPOSITION 5.2. *Once we eliminate all $\Sigma_i^j = \emptyset$, the sequence $\langle \Sigma_i^j \rangle$, ordered lexicographically with respect to the tuples $\langle i, j \rangle$, is a convex-ear decomposition of the order complex of L_S.*

Property (i) is immediately verified by our earlier decompositions. Property (ii) is almost verified as well; we know from the previous section that the order complex of each L_i^j is the boundary complex of some simplicial r-polytope, and it follows from the definitions that Σ_1^1 is the order complex of L_1^1. However, we still need to know that Σ_i^j is a *proper* subcomplex of the order complex of L_i^j whenever $j > 1$ or $i > 1$.

Let $j > 1$. Then, by our decomposition of the rank-selected Boolean lattice, some maximal chain in L_i^j is a maximal chain in L_i^k for some $k < j$. Now suppose $j = 1$. Then the label of $\mathbf{d}_i^1$ is the lexicographically first permutation in $\mathcal{S}_r$ with descent set S. It follows that $\mathbf{c}_S$ is a maximal chain in L_i^1, where $\mathbf{c}$ is the unique chain in L with increasing λ-label. Thus $\mathbf{c}_S$ is a maximal chain in L_1^1, proving the remainder of property (ii).

Proof of property (iii). We claim that, as in the previous section, reverse lexicographic order of the facets of Σ_i^j is a shelling. In fact, let $\mathbf{e}_j, \mathbf{e}_k$, and $\mathbf{e}'_k$ be as in the proof of property (iii) given there. The only way in which this proof could fail to work in this case is if $\mathbf{e}'_k$ is a chain in L_m^n for some $m < i$. Suppose this is the case. Let p be the unique rank level at which $\mathbf{e}_k$ and $\mathbf{e}'_k$ do not coincide, let $\mathbf{c}$ be the unique maximal chain in L such that $\mathbf{c}_S = (\mathbf{e}_k)_S$ and $\mathbf{c}_{[r-1]\setminus S} = (\mathbf{d}_i^j)_{[r-1]\setminus S}$, and define $\mathbf{c}'$ analogously. Then $\mathbf{c}' = com(\mathbf{c}_{-p})$. λ restricts to an EL-labeling on L_m, and thus, by Lemma 2.9, $\mathbf{c}$ is a maximal chain in L_m, which means that $\mathbf{e}_k = \mathbf{c}_S$ is a maximal chain in L_m^k for some k, which is a contradiction. ◻

Proof of property (iv). As above, we refer to the proof of property (iv) in the previous section and show that the same technique works here. Indeed, let $\mathbf{e}_i^j$ and $\mathbf{e}_m^n$ be facets of Σ_i^j and Σ_m^n, respectively, where $\langle i, j\rangle$ lexicographically precedes $\langle m, n\rangle$. Let $\mathbf{e} = \mathbf{e}_i^j \cap \mathbf{e}_m^n$. As discussed earlier, we need only find a maximal chain $\mathbf{e}'$ in L_m^n that is old (i.e., that is not a facet of Σ_m^n) such that $\mathbf{e}'$ contains $\mathbf{e}$ as a subchain. If $i = m$, our previous proof guarantees such a chain. Otherwise $i < m$, so let $\mathbf{c}'$ be the maximal chain $com(\mathbf{e})$. Then Lemma 2.9 guarantees that $\mathbf{c}'$ is a maximal chain in L_i. ◻

Suppose that $\Sigma_i^j \neq \emptyset$. Since reverse lexicographic order is a shelling of Σ_i^j, $(\mathbf{d}_i^j)_S$ is a facet of Σ_i^j. Because $|\mu((B_r)_S)|$ is the number of maximal chains of B_r whose labels have descent set S and $(\mathbf{d}_i^j)_S$ is not a maximal chain in any Σ_k^l for $\langle i, j\rangle \neq \langle k, l\rangle$, we obtain the following as a corollary.

COROLLARY 5.3. *For any i and j, let Δ_i^j denote the order complex of L_i^j. Then $\{\Delta_i^j : \Sigma_i^j \neq \emptyset\}$ is a homology basis for the order complex of $(B_r)_S$.*

6. Final remarks.

Recall that a simplicial complex Δ is *Cohen–Macaulay* if the reduced homology of the link of any face (including the empty set) vanishes in all but the top dimension. Δ is 2-*Cohen–Macaulay* if Δ is Cohen–Macaulay and, for any vertex v of Δ, $\Delta - v$ is Cohen–Macaulay and of the same dimension as Δ.

THEOREM 6.1 (see [13]). *If Δ admits a convex-ear decomposition, then Δ is 2-Cohen–Macaulay.*

Theorem 3.7 was originally motivated by Welker's result [15] that the order complex of a supersolvable lattice with nonzero Möbius function is 2-Cohen–Macaulay. Since rank-selected subposets of 2-Cohen–Macaulay posets are 2-Cohen–Macaulay (see [11] for background), we obtain the following as a corollary of Welker's result.

COROLLARY 6.2. *Let L be a rank r supersolvable lattice with nonzero Möbius function, and let $S \subseteq [r-1]$. Then the order complex of L_S is 2-Cohen–Macaulay.*

The above can also be obtained as a corollary of Theorems 1.1 and 6.1.

It is not hard to construct 2-Cohen–Macaulay complexes that have no convex-ear decomposition (for instance, any nonpolytopal triangulation of a sphere). However, Björner and Swartz have conjectured the following partial converse.

CONJECTURE 6.3 (see Swartz [13]). *Let Δ be a 2-Cohen–Macaulay simplicial complex. Then the g-vector of Δ is an M-vector.*

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