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Rim-finite, arc-free subsets of the plane

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Abstract

We investigate properties of rim-finite subsets of the plane (those which have topological bases whose elements have finite boundaries), which are also arc-free. Recently (see [K. Bouhjar, J.J. Dijkstra, Preprint], [K. Bouhjar, J.J. Dijkstra, J. van Mill, Topology Appl., to appear], [M.N. Charatonik, W.J. Charatonik, Comment. Math. Univ. Carolin., to appear], [D.L. Fearnley, J.W. Lamoreaux, Proc. Amer. Math. Soc., to appear] and [L.D. Loveland, S.M. Loveland, Houston J. Math. 23 (1997) 485–497]) there has been considerable research regarding n-point sets (sets which intersect each line in exactly n-points). These spaces are rim-finite (since the interior of a triangle has its boundary contained in a union of three lines, each of which has n points of the space), and our investigation provides a direction to generalize them. One of our main theorems seems to generalize all known results regarding the dimension of n-point sets (see, for example, [K. Bouhjar, J.J. Dijkstra, J. van Mill, Topology Appl., to appear], [D.L. Fearnley, J.W. Lamoreaux, Proc. Amer. Math. Soc., to appear] and [J. Kulesza, Proc. Amer. Math. Soc. 116 (1992) 551–553]), and beyond that has, as corollaries, the solutions to problems of Bouhjar and Dijkstra [Preprint], and L.D. Loveland and S.M. Loveland [Houston J. Math. 23 (1997) 485–497]. In Bouhjar and Dijkstra [Preprint] it is asked if all n-point sets which are arc-free must be zero-dimensional, and our result gives a positive answer. In [L.D. Loveland, S.M. Loveland, Houston J. Math. 23 (1997) 485–497] it is asked whether a connected 2-GM set must contain an arc, and again we give a positive answer.

Another main theorem states that if X is a subset of $\Re^2$ such that there is a nonnegative integer n so that every straight interval of length 1 has a local basis of open sets with boundaries which intersect X in a set of cardinality less than or equal to n, then either X is zero-dimensional or X contains an arc. We produce an example which demonstrates that, essentially, our theorem cannot be improved. The "straight interval of length 1" cannot be replaced by "point", because our example has a base of open sets whose boundaries have cardinality less than or equal to 72 and contains no arcs, yet has dimension 1. This example seems to be the first of a positive dimensional, rim-finite and arc-free separable metric space.

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1. Introduction

All spaces considered are separable and metrizable. Call a space *rim-finite* (*rim-n*) if it has a basis of open sets whose boundaries are finite (have at most n points). Rim-finite continua have been studied rather extensively; important to us is that rim-finite continua are arcwise connected, as shown by Ward in [9]. In this paper we study *arc-free* rim-finite subsets of $\Re^2$, that is, rim-finite subsets which do not contain homeomorphic images of $[0, 1]$. By the result of Ward, such sets cannot contain continua. In particular we are interested in the dimension properties of such sets, which must be at most one, since the finite boundaries have dimension at most 0.

Our motivation comes from the fact that these spaces are apparently not well understood, and yet simultaneously generalize the classes of *n-point sets*, sets which intersect each straight line in exactly n points, and n-GM sets, those with the property that if straight a lines separate two points of the set, then that line contains exactly n points from the set. There has been recent and interesting work involving these types of subsets of the plane in [1–4,6].

Regarding n-point sets, it is shown in [5], answering a question of Mauldin (in [7]), that each two-point set must be zero-dimensional. A key part of the proof was showing that a two-point set can contain no arc. More recently, in [2], it is shown that no three-point set could contain an arc, but that for $n \geqslant 4$ there are n-point sets which do contain arcs. This prompted the question, posed by Dijkstra, of whether a three-point set must be zero-dimensional. This question was recently answered in [4], where it is shown that three-point sets are, in fact, zero-dimensional. A natural generalization of this is posed in [1], where it is asked if an n-point set without any arcs must be zero-dimensional. Our main result in Section 2 has, as a corollary, that this is the case. Our result is much stronger that this though; it implies, for example, that if we only require that X intersects all lines from dense subsets of both the vertical and horizontal lines in at most n-points, and X is arc-free, then X is zero-dimensional. We point out that our method of proof expands, but relies on the main idea in [4].

We also show that the main theorem from Section 2 provides a solution to a problem from [6]: must a connected 2-GM set contain an arc? In fact, using a result from [6], we show that a positive-dimensional 2-GM set must be a simple closed curve, and also that a positive-dimensional n-GM set must contain an arc.

In Section 3, we prove a theorem based on the main result of Section 2, which states that if $X \subset \Re^2$ is arc-free and rim-n at each straight arc of length one, meaning that each arc of length one has a local base with at most n points of X on the boundaries of the base sets, then X is zero-dimensional. If the "straight arc" could be replaced by "point" in this theorem, then every arc-free, rim-n subset of $\Re^2$ would be zero-dimensional. However this is not the case, as is shown in Section 4, where an arc-free, rim-72 planar example with positive dimension is given. Thus the theorem of Section 3, in some sense, isolates the weakest set of requirements to guarantee an arc-free rim-finite planar set is zero-dimensional. We remark that, by Proposition 25 in [1], this example cannot be totally disconnected.

2. Positive-dimensional n-point sets all contain arcs

We call $X \subseteq \Re^2$ an *n-point set* if for all lines $\ell \subseteq \Re^2$, $|\ell \cap X| = n$. We call X a *partial* n-point set if for all lines $\ell \subseteq \Re^2$, $|\ell \cap X| \leqslant n$. We call $X \subseteq \Re^2$ an n-GM-*set* if for all lines $\ell \subseteq \Re^2$ separating two points of X, $|\ell \cap X| = n$. It is easy to check that partial n-point sets (and thus n-point sets) are rim-$3n$; if X is a partial n-point set, the open sets whose boundaries are triangles have at most $3n$ boundary points in X. To see that a partial n-GM set X is rim-$3n$, fix $p \in X$, and consider that if there are points of X to the right of p, then there is a vertical line close to p and to its right with at most n points. Repeating this reasoning, using the half planes determined by the lines parallel to $y = x$ and $y = -x$ which contain p, the result is clear.

Let H and V be sets of horizontal and vertical lines, respectively, in $\Re^2$. Call $H' \subseteq H$ *dense* if for all $\ell_1, \ell_2 \in H$ there is $\ell \in H'$ such that ℓ is between ℓ_1 and ℓ_2. Similarly, call $V' \subseteq V$ *dense* if for all $\ell_1, \ell_2 \in V$ there is $\ell \in V'$ such that ℓ is between ℓ_1 and ℓ_2.

This is the main theorem of the section, which gives a large class of arc-free rim-finite sets which must be zero-dimensional.

Theorem 2.1. *Let H and V be countable dense sets of horizontal and vertical lines, respectively, and let $X \subseteq \Re^2$ such that X contains no arcs and for every compact, convex $D \subseteq \Re^2$ there is an n_D with $|X \cap D \cap \ell| \leqslant n_D$ for all lines $\ell \in H \cup V$. Then* $\dim(X) = 0$.

The method of proof we use makes extensive use of the technique used in [4]. We break the theorem down into several lemmas. For the remainder of the section, let A and B be two horizontal line segments with $\pi_1(A) = \pi_1(B)$, let $D \subseteq \Re^2$ be a compact, convex set whose interior contains the compact rectangular region specified by A and B, and let $n = n_D$ as defined in the theorem. For $x \in A$, let $S(x)$ denote the vertical line segment starting at A and ending at B, with $S(x) \cap A = \{x\}$.

Let $x \in A$ and let $y \in S(x)$. Generalizing the notion from [4], call y *avoidable* if there exist two horizontal line segments $\ell_1(y)$ and $\ell_2(y)$ such that the following properties hold:

(1) Each of $\ell_1(y)$ and $\ell_2(y)$ ends on $S(x)$;
(2) $\ell_1(y)$ is above y and $\ell_2(y)$ is below y;
(3) $\pi_1(\ell_1(y)) = \pi_1(\ell_2(y))$;
(4) $\ell_1(y) \cap X \subseteq S(x)$ and $\ell_2(y) \cap X \subseteq S(x)$;
(5) For all $\varepsilon > 0$ there is a continuum C such that $C \cap X = \emptyset$, $C \cap S(x) = \emptyset$, C connects $\ell_1(y)$ and $\ell_2(y)$, and $\pi_1(C) \subseteq (\pi_1(x) - \varepsilon, \pi_1(x) + \varepsilon)$.

More specifically, call y *left-avoidable* if $\ell_1(y)$ and $\ell_2(y)$ lie on the left side of $S(x)$, and call it *right-avoidable* if $\ell_1(y)$ and $\ell_2(y)$ lie on the right side of $S(x)$. Call y *unavoidable* if it is not avoidable. Note that y does not necessarily have to be in X to be unavoidable.

Lemma 2.2. *Let $x \in A$ and let $y \in S(x)$ be left-avoidable. Let $\ell \subseteq D$ be a horizontal line segment with $\pi_1(\ell) = [\pi_1(x) - \varepsilon, \pi_1(x)]$ for some $\varepsilon > 0$ and $y \notin \ell$ such that $\pi_2(\ell) \in [\pi_2(\ell_2(y)), \pi_2(\ell_1(y))]$. Then we can find a continuum C with $C \cap X \subseteq \{y\}$ such that C connects ℓ and y, $\pi_1(C) \subseteq \pi_1(\ell)$, and $C \cap S(x) = \{y\}$.*

Proof. Without loss of generality, assume that $\pi_2(\ell) < \pi_2(y)$. Choose a sequence $\{y_i\colon i \in N\} \subseteq S(x)$ between $\ell \cap S(x)$ and y converging to y such that $\{\pi_2(y_i)\colon i \in N\}$ is strictly increasing, and such that for all positive integers m, ℓ_m is a horizontal line segment with $y_m \in \ell_m$, $\pi_1(\ell_m) \subseteq [\pi_1(x) - \varepsilon/m, \pi_1(x)]$, and $\ell_m \cap X \subseteq \{y_m\}$. Let $\ell_0 = \ell$. Notice that the construction of such a sequence is possible by choosing line segments that are subsets of lines in H, the countable dense set of horizontal lines. The last property is given by the fact that for all $\ell' \in H \cup V$, $|X \cap \ell' \cap D| \leqslant n$, so we may find some interval on these lines disjoint from X. Now we use the avoidability of y to construct a continuum with the desired properties. For each ℓ_m, find a continuum C_m connecting ℓ_m to ℓ_{m+1}, with $\pi_1(C_m) \subseteq \pi_1(\ell_m)$, and with $C_m \cap X = \emptyset$. Notice that this is possible, since ℓ_m and ℓ_{m+1} lie between $\ell_1(y)$ and $\ell_2(y)$, so we may find continua arbitrarily close to $S(x)$. Furthermore, for each nonnegative integer m, let $K_m \subseteq \ell_m$ be a closed sub-arc containing $C_m \cap \ell_m$ and $C_{m-1} \cap \ell_m$ with $K_m \cap S(x) = \emptyset$. Therefore, each K_m is disjoint from X. Let $C' = (\bigcup K_m) \cup (\bigcup C_m)$. Now, by construction, C' intersects ℓ. If we let $C = \mathrm{Cl}(C')$, then C intersects ℓ as well. Furthermore, it is clear that $C \cap C' \cap S(x) = \{y\}$, since any other point on $S(x)$ different from y cannot be captured by the closure, as it cannot be a limit point of the sequence $\{y_i\colon i \in N\}$, and therefore cannot be a limit of the continua connecting them. Similarly, it is clear that $y \in C$ by the same reasoning. Therefore, C is our desired continuum.

Notice that Lemma 2.2 is symmetric, and also works with points that are right-avoidable.

Lemma 2.3. *Let $x \in A$, let ℓ_1 and ℓ_2 be horizontal line segments strictly in between A and B with $\pi_1(\ell_1) = \pi_1(\ell_2)$ such that $\pi_1(x) \in \mathrm{Int}(\pi_1(\ell_1))$, and suppose that all $y \in S(x)$ between the two line segments are avoidable. Then there is a continuum C with $C \cap X = \emptyset$ such that C connects ℓ_1 and ℓ_2 and $\pi_1(C) \subseteq \pi_1(\ell_1)$.*

Proof. Let $S \subseteq S(x)$ be the closed interval in between ℓ_1 and ℓ_2. For each $y \in S$, let $U_y \subseteq S(x)$ be the open (in $S(x)$) interval indicated by the intersections of $\ell_1(y)$ and $\ell_2(y)$ with $S(x)$. Now since the set $\{U_y\colon y \in S\}$ covers S, which is compact, we can find a minimal finite set $\{y_1, y_2, \ldots, y_t\}$ such that $\{U_{y_r}\colon 1 \leqslant r \leqslant t\}$ covers S. For each y_r, let δ_r be the length of the line segment $\ell_1(y)$, and let $\delta = \min\{\delta_r\colon 1 \leqslant r \leqslant t\}$. Notice that $\delta > 0$. Assume that $(\pi_1(x) - \delta, \pi_1(x) + \delta) \subseteq \pi_1(\ell_1) = \pi_1(\ell_2)$. If not we can pick a new, smaller, δ such that this holds. Now for each y_r, make the rectangular region $B(r)$ as follows: Trim each of $\ell_1(y_r)$ and $\ell_2(y_r)$ so that they each still end on $S(x)$ but now each has length δ. Next, union each of the two line segments with its reflection across $S(x)$. These two line segments we have just obtained make up one pair of sides of $B(r)$. The other sides are obtained from connecting the line segments to each other vertically. Now it is clear that the set $\{\mathrm{Int}(B(r))\colon 1 \leqslant r \leqslant t\}$ covers S. For each $B(r)$, we can find a continuum $C(r)$ connecting the top and bottom of $B(r)$, and in the side of $B(r)$ that the line segments for y_r started on, such that $C(r) \cap X = \emptyset$ and $C(r) \subseteq B(r)$. Notice that no more than two of these rectangles can overlap in one place, since that would contradict our choice of a minimal covering set. Now for each r and s where $r \neq s$ and $B(r) \cap B(s) \neq \emptyset$, find a point $i(r,s) \in \mathrm{Int}(B(r) \cap B(s)) \cap S$ such that $i(r,s) \notin X$. This is possible because otherwise $\mathrm{Int}(B(r) \cap B(s)) \cap S \subseteq X$, contradicting our assumption that X does not contain an arc. Suppose that $B(s)$ is above $B(r)$. Now suppose that $C(r)$ lies in the left side of $B(r)$. Using Lemma 2.2, connect the left half of the line segment that is the top of $B(r)$ to $i(r,s)$ with a continuum $C(r,s)$ such that $C(r,s) \cap X = \emptyset$, $C(r,s) \subseteq B(r)$. Do the obvious parallel construction for the bottom of $B(r)$, connecting the left half of the line segment that is the bottom of $B(r)$ to $i(r,t)$ with a continuum $C(r,t)$ such that $C(r,t) \cap X = \emptyset$, $C(r,t) \subseteq B(r)$, and where $B(t)$ is the box that overlaps $B(r)$ on the bottom. Repeat this process for all $B(r)$, remembering to find continua on the same side of $B(r)$ that $C(r)$ is on. Once this process is done, simply connect the different continua to each other using segments from the top and bottom sides of each $B(r)$ but disjoint from the intersection of the sides with S. Doing this clearly gives us a continuum C that connects ℓ_1 and ℓ_2, by taking the union of all the constructed continua. To see that this continuum is close enough to S, notice that $\pi_1(C) \subseteq \pi_1(B(r))$ for any p, and that for any r, $\pi_1(B(r)) \subseteq (\pi_1(\ell_1))$. To see that $C \cap X = \emptyset$, notice that each continuum we found was disjoint from X, and that the only parts used from the horizontal line segments were used to connect $C(r)$ to $C(r,s)$ for some r and s, and so were disjoint from $S(x)$. However, this means the pieces were also disjoint from X, and so $C \cap X = \emptyset$.

Lemma 2.4. *Let $x \in A$, let $y \in S(x)$, and let $\{x_i\colon i \in N\} \subseteq A \setminus \{x\}$ be a sequence converging to x such that $\{\pi_1(x_i)\colon i \in N\}$ is a strictly increasing sequence. Let $\gamma > 0$ and suppose that, for all positive integers i, $S(x_i) \cap N_\gamma(y)$ contains only avoidable points. Then y itself is avoidable.*

Proof. Let $\varepsilon > 0$. Choose horizontal line segments ℓ_1 and ℓ_2 as in the definition of left avoidability for y so that each lies within $N_\gamma(y)$. Once again, this is possible by picking line segments that are subsets of lines in H. Now find subsets of ℓ_1 and ℓ_2, ℓ_1' and ℓ_2' such that each is still a horizontal line segment ending at $S(x)$ with $\pi_1(\ell_1') = \pi_1(\ell_2')$, but now each has length less than ε. Now choose $x_t \in \{x_i\colon i \in N\}$ so that $d(x_t, x) < \varepsilon$. Now since $S(x_t) \cap N_\gamma(y)$ contains only avoidable points, we may find, by Lemma 2.3, a continuum C connecting ℓ_1' and ℓ_2' such that $\pi_1(C) \subseteq \pi_1(\ell_1')$ and $C \cap X = \emptyset$. But then C is our required continuum for ε, and so y is avoidable.

Since this proof is perfectly symmetric, it also works with sequences that are strictly decreasing. Since from any sequence $\{x_i\colon i \in N\} \subseteq A$ which both converges to x and has infinitely many distinct terms we can pick out either a strictly increasing or strictly decreasing subsequence converging to x, we have the following:

Corollary 2.5. *Let $x \in A$ and let $y \in S(x)$. Let $\{x_i\colon\ i \in N\} \subseteq A$ be a sequence, with infinitely many distinct points, converging to x, and let $\gamma > 0$ be such that $S(x_i) \cap N_\gamma(y)$ contains only avoidable points for all positive integers i. Then y is avoidable.*

Lemma 2.6. *There is a continuum C connecting A and B such that $C \cap X = \emptyset$ and $\pi_1(C) \subseteq \pi_1(A)$.*

Proof. If there is some $x \in \mathrm{Int}(A)$ with $S(x)$ containing only avoidable points, then by Lemma 2.3 we can find a continuum with the desired properties by choosing horizontal line segments A' and B' with $\pi_1(A') = \pi_1(B') = \pi_1(A) = \pi_1(B)$ strictly above and below A and B, respectively. The result is obtained by taking $A = h_1$ and $B = h_2$ as in the hypotheses of the lemma. So assume that for all $x \in \mathrm{Int}(A)$, $S(x)$ contains at least one unavoidable point. Let $V' = \{x \in A\colon$ there is a line $\ell \in V$ with $x \in \ell\}$. Notice that, for all $x \in V'$, $|S(x) \cap X| \leqslant n$. For each integer t with $1 \leqslant t \leqslant n$, let $W_t = \{x \in V'\colon\ S(x)$ has exactly t unavoidable points$\}$. Note that

$$V' = \bigcup_{1 \leqslant t \leqslant n} W_t.$$

Now since V' is dense in A, there is some minimal m such that W_m is somewhere dense. Let $J' \subseteq A$ be an interval in which W_m is dense. Now since for each $t < m$ W_t is nowhere dense, we can find a subinterval $J \subseteq J'$ such that, for all $t < m$, $W_t \cap J = \emptyset$. We now show that for all $x \in J$, $S(x)$ cannot have more than m unavoidable points. Suppose that $x \in J$ and $S(x)$ has $m + 1$ unavoidable points. Choose a sequence $\{x_i\colon\ i \in N\} \subseteq J \cap W_m$ converging to x such that $\{\pi_1(x_i)\colon\ i \in N\}$ is strictly increasing. Now for each x_i in the sequence, $S(x_i)$ has only m unavoidable points. Therefore, there is some unavoidable point $y \in S(x)$ and some $\gamma > 0$ such that there are infinitely many points x_i in the sequence with $S(x_i) \cap N_\gamma(y)$ containing only avoidable points. However, these infinitely many points can be formed into a subsequence satisfying the hypotheses of Lemma 2.4, and so y is avoidable, a contradiction. Therefore, unless $t = m$, $W_t \cap J = \emptyset$.

Now for each integer t with $1 \leqslant t \leqslant m$, let $Y_t = \{x \in J \setminus W_m\colon x$ has exactly t unavoidable points$\}$. Let $Y = \bigcup_{1 \leqslant t \leqslant m} Y_t$. Now since V' is countable and $W_m \subseteq V'$, there is some least k such that Y_k is somewhere dense in J. Let $I' \subseteq J$ be an interval in which Y_k is dense. Now since each Y_t, with $t < k$, is nowhere dense, we can find an interval $I \subseteq I' \subseteq J$ disjoint from all Y_t where $t < k$. Now we already know that $k \leqslant m$. We want to show that $k = m$. If $k < m$, pick an $x \in W_m \cap I$, and choose a strictly increasing sequence in $I \cap Y_k$ converging to x. But duplicating the argument used earlier, we see that this would mean that one of the unavoidable points in $S(x)$ is actually avoidable, a contradiction. Therefore, $k = m$. Now since $I = W_m \cup Y_k$ and $k = m$, we have that for all $x \in I$ $S(x)$ has exactly m unavoidable points.

We now define a function $f : I \mapsto X$. For all $x \in I$, let $f(x)$ be the highest unavoidable point on $S(x)$. Clearly, f is 1–1. Now, to show f is continuous, fix $x \in I$ and $\varepsilon > 0$; we may assume that ε is less than half the distance between each pair of unavoidable points on $S(x)$. Since $f(x)$ is unavoidable, there is a $\delta > 0$ such that if $z \in I$ and $d(x,z) < \delta$, then there is an unavoidable point of $S(z)$ within ε of each unavoidable point of $S(x)$, including $f(x)$. It is easy to see that this point must be $f(z)$. Thus f is continuous, and X contains an arc, a contradiction.

Note that everything we have done up to Lemma 2.6 could have been done to obtain horizontal continua instead of vertical continua, by doing everything with the plane rotated by $\pi/2$.

Proof of Theorem 2.1. Let $p \in X$, let ℓ be the vertical line through p, and let $\varepsilon > 0$. Let $D \subset \Re^2$ be compact and convex, with $N_\varepsilon(p) \subseteq D$, and let $n = n_D$. Choose two horizontal line segments ℓ_1 and ℓ_2 above and below p, respectively, such that each is contained in $N_\varepsilon(p)$, $\pi_1(\ell_1) = \pi_1(\ell_2)$, $\ell_1 \cap X \subseteq \ell$, $\ell_2 \cap X \subseteq \ell$, and $\pi_1(\ell) \in \mathrm{Int}(\pi_1(\ell_1))$. This choice is possible, once again, by choosing line segments that are subsets of lines in H. Now, by Lemma 2.6, we can connect each side of the line segments with continua that are disjoint from X. Taking the union of the two horizontal line segments, we have a set whose intersection with X is at most the two points where ℓ_1 and ℓ_2 intersect ℓ. However, by the nature of Lemma 2.6, we can find two continua above and below p, respectively, that connect the first two continua. This is by the fact that Lemma 2.6 can be applied to obtain continua connecting vertical lines rather than horizontal lines. Therefore, the union of the four continua (two from the first application of Lemma 2.6 and two from the second application) gives us the boundary of a set, closed and open in X, around p, and inside $N_\varepsilon(p)$, proving the theorem.

Corollary 2.7. *If X is an n-point set, partial n-point set, or n-GM set containing no arcs, then* $\dim(X) = 0$.

Proof. The proof for the n-point set and partial n-point set is immediate. Now suppose X is an n-GM set. There are at most two horizontal lines that hit X but do not separate two points of it. Therefore, the set of horizontal lines hitting X in n or fewer places is dense. Similarly, the set of vertical lines hitting X in n or fewer places is dense, and we can now apply the theorem.

This also gives us the answer to a question of L.D. Loveland and S.M. Loveland in [6], namely must a connected 2-GM set contain an arc? (If a 2-GM set contains an arc, it must be a simple closed curve, by a result in [6].) We therefore have a result, that any positive-dimensional 2-GM set is a simple closed curve.

3. A similar result for sets forming finite bases around arcs of length 1

Notice that our hypotheses for Theorem 2.1 could be changed to: Suppose $X \subseteq \Re^2$ and H and V are dense sets of horizontal and vertical lines, respectively. Suppose further that for any compact, convex set $D \subseteq \Re^2$ there is a positive integer n_D such that, for any line

$\ell \in H \cup V$, $\ell \cap X \cap D$ contains at most n_D unavoidable points. The proof of the theorem would then be virtually unchanged, as the reader can verify.

Let $U \subseteq P(\Re^2)$ be a set of open sets. Call U an *arc-basis* if for any $A \subseteq \Re^2$ where A is an arc of length 1 and for any closed set $C \subseteq \Re^2$ with $C \cap A = \emptyset$ there is a $V \in U$ such that $A \subseteq V$ and $\mathrm{Cl}(V) \cap C = \emptyset$.

For any positive integer n, call $X \subseteq \Re^2$ an *n-arc-finite set* if there is some arc-basis U such that for all $V \in U$ $|\mathrm{Bd}(V) \cap X| \leqslant n$.

Theorem 3.1. *Let n be a positive integer, and let $X \subseteq \Re^2$ be an n-arc-finite set with the associated arc-basis U. Then either X contains an arc, or* $\dim(X) = 0$.

Proof. We give only an outline, as this proof mirrors the proof of Theorem 2.1. First, pick two horizontal line segments A and B with $\pi_1(A) = \pi_1(B)$ and $\pi_2(A) - \pi_2(B) = 1$. Now look at all the vertical line segments of length 1 connecting A to B. Let C be one of these segments. Let $\{V_i \in U\colon i \in N\}$ be a sequence of open sets around C, such that for all $i \in N$, $\mathrm{Cl}(V_{i+1}) \subseteq V_i$. This is possible because of the basis property. Now this means that, since $|\mathrm{Bd}(V_i) \cap X| \leqslant n$, for all i, C cannot have more than n unavoidable points, since if it has $n+1$, one of these points must have a sequence of continua disjoint from X, each a part of the boundary for some V_i, converging to it, meaning it is avoidable. Therefore, each vertical line segment connecting A and B has at most n unavoidable points, and we can apply the proof of Theorem 2.1.

From [1], a subset X of $\Re^2$ is called *almost an n-point set* if there is a nowhere dense subset Z of the angles in $[0, \pi]$ such that every line with angle of inclination not in Z meets X in exactly n points. It is clear that an almost n-point set is $3n$-arc-finite, using triangles with sides whose angles of inclination are not in Z. Therefore we have:

Corollary 3.2. *Each arc-free, almost n-point set is zero-dimensional.*

4. A positive-dimensional rim-72 set with no arcs

With Theorem 3.1 in place, one may conjecture a positive answer to the following question:

If $X \subseteq \Re^2$, n is a positive integer, and U is a basis for the plane (in the usual sense) such that for all $V \in U$, $|\mathrm{Bd}(V) \cap X| \leqslant n$, does $\dim(X) > 0$ imply that X contains an arc?

It turns out that the answer to this question is “no”, and we now turn to giving an example which illustrates this.

Example. There is a rim-72 subset X of $\Re^2$ which contains no arcs but which satisfies $\dim(X) = 1$.

Construction. We call a number in $[0, 1]$ *triadic* if it can be written in the form $n/3^k$ for some integer n satisfying $0 \leqslant n \leqslant 3^k$, and *triadic of order k* if the denominator is 3^k

when it is written in lowest terms. We let $V(k)$ and $H(k)$ denote the vertical and horizontal lines, respectively, whose fixed coordinate is triadic of order k, and let $V = \bigcup_{k\in N} V(k)$, $H = \bigcup_{k\in N} H(k)$, and $L = V \cup H$, with $L(k)$ defined in the obvious fashion. The example X will be a subset of $[0,1]^2$. Let $F = [0,1]^2 \cap (V \cup H)$; when we refer to a line, we will actually mean its intersection with $[0,1]^2$. By a *k-square* we mean the square whose sides come from four lines, two lines in H determined by fixed coordinates $n/3^k$ and $(n+1)/3^k$ for some $0 \leqslant n < 3^k$, and two lines in V determined by fixed coordinates $m/3^k$ and $(m+1)/3^k$ for some $0 \leqslant m < 3^k$. It is clear that a basis for X can be obtained by using the sets which are, for some k, either interiors of k-squares, or are squares formed by starting with four k-squares which intersect at a common point, and taking the interior of their union (this needs to be adjusted slightly in an obvious way at points of the boundary of $[0,1]^2$). The boundaries of these basis elements are clearly contained in F. Our goal is to construct X so that $X \cap \mathrm{bd}(S) \subset F$, for every k-square S, is a boundedly finite set. To this end, let P denote the set of points in F which are the intersections of lines of the form $l_v \in V(k)$ and $l_h \in H(m)$ where $|k-m| \leqslant 2$. P is a countable set. Notice that if S is a k-square for any k, then P contains at most 10 points of any side of S, hence $|\mathrm{bd}(S) \cap P| \leqslant 40$ (in fact one can see that $|\mathrm{bd}(S) \cap P| \leqslant 36$ since either a corner of a square is in P and thus counted twice in this, or there are at most 9 points on the sides containing that corner). We will, in fact, construct X so that $X \cap F = P$; thus, from the above discussion, we will be sure that the basis elements which are interiors of squares are at most rim-36, and the basis elements which are interiors of unions of four intersecting squares are at most rim-72.

The procedure will now be described. First, we need:

Lemma 4.1. *If K is a continuum in $[0,1]^2$ from $[0,1]\times\{0\}$ to $[0,1]\times\{1\}$, then either $K \cap P \neq \emptyset$ or $|K \cap [0,1]^2 \backslash F| = \mathbf{c}$.*

This lemma and transfinite induction will be used to find a subset X^* of $[0,1]^2\backslash F$ which intersects every continuum of the type mentioned in the lemma. This guarantees that $X = X^* \cup P$ satisfies $\dim(X) \geqslant 1$. Also, the construction is done in such a way that $[0,1]^2\backslash X$ intersects each arc of $[0,1]^2$. It is then clear that X is as desired.

Assuming the lemma, we construct X^*. Enumerate by $\{A_\alpha\colon \alpha < \mathbf{c}\}$ the collection of arcs in $[0,1]^2$, and by $\{K_\alpha\colon \alpha < \mathbf{c}\}$ the collection of continua in $[0,1]^2$ from the top to the bottom. By transfinite induction, for each $\alpha < \mathbf{c}$, we find points x_α and y_α (with x_α's not necessarily distinct). Defining $X_\gamma = \{x_\beta\colon \beta \leqslant \gamma\}$ and $Y_\gamma = \{y_\beta\colon \beta \leqslant \gamma\}$, we will require that $y_\alpha \in A_\alpha \backslash ((\bigcup_{\beta<\alpha} X_\alpha) \cup P)$ and $x_\alpha \in K_\alpha \backslash (F \cup Y_\alpha)$. It follows that $X^* = \bigcup_{\alpha<\mathbf{c}} X_\alpha$ intersects each member of $\{K_\alpha\colon \alpha < \mathbf{c}\}$ and does not intersect $Y^* = \bigcup_{\alpha<\mathbf{c}} Y_\alpha$, which intersects each member of $\{A_\alpha\colon \alpha < \mathbf{c}\}$. Thus X^* will be as required above.

Assume, for all $\beta < \alpha$ points x_β and y_β have been found appropriately. Observe that $P \cup (\bigcup_{\beta<\alpha} X_\beta)$ has fewer that $\mathbf{c}$ points and so we can choose $y_\alpha \in A_\alpha \backslash (P \cup (\bigcup_{\beta<\alpha} X_\beta))$. Then if $K_\alpha \cap P = \emptyset$, by Lemma 4.1, there are enough points of $K_\alpha \backslash F$ to choose $x_\alpha \in K_\alpha \backslash (F \cup Y_\alpha)$.

Before proving Lemma 4.1, we introduce some more notation and concepts. For each line $l \in L$, $l \backslash P$ is a collection of open intervals. Let $S(k)$ denote the set of such intervals arising from $L(k)$ lines, and let $S = \bigcup_{k \in N} S(k)$. Most elements of $S(k)$ can be seen to have length $1/3^{k+2}$, except where they have length $1/3^{k+1}$ due to crossings with lines in $L(r)$ for some $r < k-2$. For $s \in S$, by the *chain component of s* we mean $\bigcup\{t \in S\colon$ there is a sequence $s = s_0, s_1, \ldots, s_r = t$ of elements of S with $s_i \cap s_{i+1} \neq \emptyset$, for $0 \leqslant i < r\}$. The collection of chain components partitions $F \backslash P$ into countably many sets, $\{C_i\colon i \in N\}$, since there are only countably many intervals in S, and the chaining process gives an equivalence relation. The intervals from S which make up a chain component will be called its *links*.

The next lemma illustrates the critical relationship between pairs of chain components:

Lemma 4.2. *For any two distinct chain components C_i and C_j, $\mathrm{cl}(C_i) \cap \mathrm{cl}(C_j) = \emptyset$, or else the points of intersection are endpoints of links of both C_i and C_j (and are thus in P).*

If s is a closed side of a k-square B, the *quadrant of s in B* is the union of s with the open triangular region which has s for one side, and the other two sides, whose union we call $D(s, B)$, are the segments from the endpoints of s to the center of B.

This lemma which is needed to prove Lemma 4.2, indicates how limited chains are, in terms of length:

Lemma 4.3. *If s is a closed side of a k-square B, the intersection, with B, of the closure of the union of all chain components which intersect s is contained in the quadrant of s in B.*

Proof. If an element s_1 of $S(r_1)$ intersects s at a point p_1, then $r_1 \geqslant k+1$ and s_1 is perpendicular to s. The distance from p_1 to any other point of $s_1 \cap B$ is less than $1/3^{r_1+2}$, while the distance from p_1 to $D(s, B)$ is at least $(\sqrt{2}/2)(1/3^{r_1})$. Now, if an element s_2 of $S(r_2)$ intersects s_1 at a point p_2 of B, then $r_2 \geqslant r_1 + 3$ and the distance of p_2 to any other point of s_2 is at most $1/3^{r_2+2} \leqslant 1/3^{r_1+5}$. Continuing in this fashion, we see that if $s = s_0, s_1, s_2, \ldots, s_m$ is a chain from p_1 into B, then the maximum distance from p_1 to a point on the chain is bounded by the sum $1/3^{r_1+2}(1 + 1/27 + 1/27^2 + \cdots) = (1/3^{r_1+2})(27/26) < 1/2(\sqrt{2}/2)(1/3^{r_1})$. Thus the distance from the point of intersection of a chain component into B with s to any point of that chain component in B is less than half the distance of that point of intersection to the set $D(s, B)$. The lemma follows.

Proof of Lemma 4.2. Suppose C_1 and C_2 are distinct chain components, and p is not an endpoint of a link of each C_i. We must show that p is not in $\bigcap_{i \in \{1,2\}} \mathrm{cl}(C_i)$. First, consider the case where p is in the interior of some k-square B such that the complement of B contains points of both C_i's. Then both C_i's intersect the boundary of B, or else p is not in the closure of $C_i \cap B$, for both $i \in \{1, 2\}$. But p is in only one quadrant of B, (by Lemma 4.3, p is not on a diagonal; otherwise p cannot be in the closure of either C_i which must intersect sides of B). If the C_i's do not intersect a common side of B then by

Lemma 4.3 we are done. But even if they do, we can subdivide along that side and apply Lemma 4.3 to the smaller boxes to see that p can only be in the closure of one of the C_i's. The case that p is not in the interior of some k-square requires p being either on one or two lines in L. Then we can take B to be, for small enough k, the union of all k-squares containing p (there are either two or four of these), so that both C_1 and C_2 contain points outside the union of these boxes, and also small enough so that for at least one $i \in \{1, 2\}$, the L lines through p when intersected with B contain no points of C_i. Then applying Lemma 4.3 to each of the sides of the k-squares which are in the boundary of B, it is clear that p cannot be in the closure of C_i, since C_i's chains can only enter through these sides.

Proof of Lemma 4.1. Fix K a continuum from the top to the bottom of $[0, 1]^2$. By the methods from the proof of Lemma 4.3, one can easily check that no chain component has diameter greater than $1/3$. Thus K is not contained in a chain component. Either the sets of the form $K \cap \mathrm{cl}(C_i)$ are pairwise disjoint, or else K contains a point of P, by Lemma 4.2. Assuming $K \cap P = \emptyset$, then K is partitioned into closed sets by the collection $\{K \cap \mathrm{cl}(C_i)\colon\ i \in N\} \cup \{\{x\}\colon\ x \in K \setminus \bigcup_{i \in N} \mathrm{cl}(C_i)\}$. By the Sierpinski Theorem [8, Exercise 4.6.1] no continuum can be partitioned by countably many closed sets, so $\{\{x\}\colon\ x \in K \setminus \bigcup_{i \in N} \mathrm{cl}(C_i)\}$ must be uncountable. Its cardinality is $\mathbf{c}$ because this set is G_δ in K, and cannot be countable.

Remark. Since both V and H are dense in the sets of vertical and horizontal lines which intersect X, and X intersects each such line in a finite set, this means the boundedness conditions in Theorems 2.1 and 3.1 cannot be relaxed to just finite. Thus the condition $|X \cap D \cap \ell| \leqslant n_D$ in Theorem 2.1, and the n-arc-finite condition of Theorem 3.1 seem to be necessary.

References

[1] K. Bouhjar, J.J. Dijkstra, On the structure of n-point sets, Preprint.

[2] K. Bouhjar, J.J. Dijkstra, J. van Mill, Three-point sets, Topology Appl., to appear.

[3] M.N. Charatonik, W.J. Charatonik, On Mazurkiewicz sets, Comment. Math. Univ. Carolin., to appear.

[4] D.L. Fearnley, J.W. Lamoreaux, Every three-point set is zero-dimensional, Proc. Amer. Math. Soc., to appear.

[5] J. Kulesza, A two-point set must be zero-dimensional, Proc. Amer. Math. Soc. 116 (1992) 551–553.

[6] L.D. Loveland, S.M. Loveland, Planar sets that lines hit twice, Houston J. Math. 23 (1997) 485–497.

[7] R.D. Mauldin, Problems in topology, in: J. van Mill, G.M. Reed (Eds.), Open Problems in Topology, North-Holland, Amsterdam, 1990.

[8] J. van Mill, Infinite Dimensional Topology, North-Holland, Amsterdam, 1989.

[9] L.E. Ward, The Hahn–Mazurkiewicz theorem for rim-finite continua, Gen. Topology Appl. 6 (1976) 183–190.