

Balanced Non-Transitive Dice

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Abstract

We study triples of labeled dice in which the relation “is a better die than” is non-transitive. Focusing on such triples with an additional symmetry we call “balanced,” we prove that such triples of n -sided dice exist for all $n \geq 3$. We then examine the sums of the labels of such dice, and use these results to construct an $O(n^2)$ algorithm for verifying whether or not a triple of n -sided dice is balanced and non-transitive. Finally, we consider generalizations to larger sets of dice.

1 INTRODUCTION

Suppose we play the following game with the three six-sided dice in Figure 1: You choose a die, and then I choose a die (based on your choice). We roll our dice, and the player whose die shows a higher number wins.

A closer look at the dice in Figure 1 reveals that, in the long run, I will have an advantage in this game: Whichever die you choose, I will choose the one immediately to its left (and I will choose die C if you choose die A). In any case, the probability of my die beating yours is $19/36 > 1/2$.

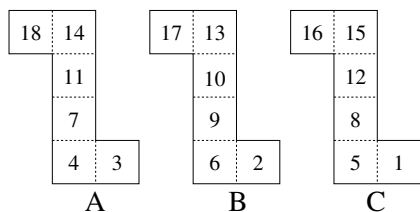


Figure 1: A set of balanced non-transitive 6-sided dice.

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This is a case of the phenomenon of *non-transitive* dice, first introduced by Martin Gardner in [2], and further explored in [3], [4], and [1].

We formally define a triple of dice as follows: Fix an integer $n > 0$. For our purposes, a set of n -sided *dice* is a collection of three pairwise-disjoint sets A, B , and C with $|A| = |B| = |C| = n$ and $A \cup B \cup C = [3n]$ (here and throughout, $[n] = \{1, 2, \dots, n\}$). We think of die A as being labeled with the elements of A , and so on. Each die is fair, in that the probability of rolling any one of its numbers is $1/n$. We also write $P(A \succ B)$ to indicate the probability that, upon rolling both A and B , the number rolled on A exceeds that on B .

Definition 1.1. *A set of dice is non-transitive if each of $P(A \succ B)$, $P(B \succ C)$, and $P(C \succ A)$ exceeds $1/2$. That is, the relation “is a better die than” is non-transitive.*

In this paper we (mostly) examine non-transitive sets of dice, but we introduce a new property as well.

Definition 1.2. *A set of dice is balanced if $P(A \succ B) = P(B \succ C) = P(C \succ A)$.*

Note that the set of dice in Figure 1 is balanced, as $P(A \succ B) = P(B \succ C) = P(C \succ A) = 19/36$.

In Theorem 2.1, we show that non-transitive balanced sets of n -sided dice exist for all $n \geq 3$. Surprisingly, this is also the first proof that non-transitive sets of n -sided dice exist for all $n \geq 3$. We then prove in Theorem 3.1 that a set of dice is balanced (but not necessarily non-transitive) if and only if the face-sums of the dice are equal (the *face-sum* of a die is simply the sum of the numbers with which it is labeled). This then yields an $O(n^2)$ algorithm for determining if a given triple of n -sided dice is non-transitive and balanced. Finally, we consider generalizations to sets of four dice.

2 BALANCED DICE

The main goal in this section is to prove the following.

Theorem 2.1. *For any $n \geq 3$, there exists a non-transitive set of balanced n -sided dice.*

First, we need some machinery. Fix $n > 0$. For our purposes, a *word* σ is a sequence of $3n$ letters where each letter is either an a, b , or c , and each of a, b , and c appears n times.

Definition 2.2. *If D is a set of n -sided dice, define a word $\sigma(D)$ by the following rule: the i^{th} letter of $\sigma(D)$ corresponds to the die on which the number i labels a side.*

Now let $\sigma = s_1 s_2 \cdots s_{3n}$ be a word. We define a function q_σ^+ on the letters of σ as follows.

$$q_\sigma^+(s_i) = \begin{cases} |\{j < i : s_j = b\}| & : s_i = a \\ |\{j < i : s_j = c\}| & : s_i = b \\ |\{j < i : s_j = a\}| & : s_i = c \end{cases}$$

Similarly, define a function q_σ^- by

$$q_\sigma^-(s_i) = \begin{cases} |\{j < i : s_j = c\}| & : s_i = a \\ |\{j < i : s_j = a\}| & : s_i = b \\ |\{j < i : s_j = b\}| & : s_i = c \end{cases}$$

For example, if $s_i = a$, then $q^+(s_i)$ is the number of sides of die B whose labels precede i . Similarly, $q^-(s_i)$ is the number of sides of die C whose labels precede i .

Example 2.3. *Let D be the following set of dice.*

$$\begin{aligned} A &= 9 \ 5 \ 1 \\ B &= 8 \ 4 \ 3 \\ C &= 7 \ 6 \ 2 \end{aligned}$$

Then $\sigma(D) = acbbacba$. Note that this set of dice is balanced non-transitive, as $P(A \succ B) = P(B \succ C) = P(C \succ A) = \frac{5}{9}$.

Conversely, given a word σ , let $D(\sigma)$ denote the unique set of dice corresponding to the word σ . As this is a one-to-one correspondence, we often speak of a set of dice and the associated word interchangeably. For instance, if $\sigma = s_1 s_2 \cdots s_{3n}$ is a $3n$ -letter word, the probability of die A beating die B is given by

$$P(A \succ B) = \frac{1}{n^2} \sum_{s_i=a} q^+(s_i),$$

and the other probabilities may be computed analogously. Thus, the property of a set D of dice being balanced is equivalent to $\sigma(D)$ satisfying

$$\sum_{s_i=a} q^+(s_i) = \sum_{s_i=b} q^+(s_i) = \sum_{s_i=c} q^+(s_i).$$

Furthermore, if D is a set of n -sided dice, then D is non-transitive if and only if each of

$$\sum_{s_i=a} q_{\sigma(D)}^+(s_i), \sum_{s_i=b} q_{\sigma(D)}^+(s_i), \text{ and } \sum_{s_i=c} q_{\sigma(D)}^+(s_i)$$

exceeds $n^2/2$.

Although a set of dice D and its associated word $\sigma(D)$ hold the same information, this alternate interpretation will prove invaluable in showing Theorem 2.1. First, we need some lemmas. Recall that the *concatenation* of two words σ and τ , for which we write $\sigma\tau$, is simply the word σ followed by τ .

Lemma 2.4. *Let σ and τ be balanced words. Then the concatenation $\sigma\tau$ is balanced.*

Proof. Let $|\sigma| = 3m$, $|\tau| = 3n$. If $i \leq 3m$, then $q_{\sigma\tau}^+(s_i) = q_\sigma^+(s_i)$ (q^+ is defined as a subset of the s_j with $j < i$, so concatenating τ after σ contributes nothing to these). Otherwise (for $3m < i \leq 3m + 3n$), $q_{\sigma\tau}^+(s_i) = q_\tau^+(s_i) + m$, because every letter from τ beats all m letters from the appropriate die in σ , in addition to whichever letters it beats from the structure of τ itself. Then

$$\sum_{s_i=a} q_{\sigma\tau}^+(s_i) = \sum_{s_i=a} q_\sigma^+(s_i) + \sum_{s_i=a} q_\tau^+(s_i) + mn. \quad (1)$$

We may repeat the argument for $s_i = b, c$, and then we are done as σ and τ are balanced. \square

While Lemma 2.4 is only useful for balanced words (or sets of dice), the next result can be applied to arbitrary sets of non-transitive dice.

Lemma 2.5. *Let σ and τ be non-transitive words. Then the concatenation $\sigma\tau$ is non-transitive.*

Proof. Let σ be a word of length $3m$. Because $m^2 P_\sigma(A \succ B)$ counts the number of rolls of dice A and B in which die A beats die B , we note that

$$m^2 P_\sigma(A \succ B) = \sum_{s_i=a} q^+(s_i),$$

and an analogous statement holds for $m^2 P_\sigma(B \succ C)$ and $m^2 P_\sigma(C \succ A)$. Define a quantity V_σ by

$$V_\sigma = m^2 \cdot \min\{P_\sigma(A \succ B), P_\sigma(B \succ C), P_\sigma(C \succ A)\}.$$

Now let τ be a word of length $3n$, and define quantities V_τ and $V_{\sigma\tau}$ as above. Note that

$$V_\sigma > \frac{m^2}{2}, \quad V_\tau > \frac{n^2}{2},$$

because σ and τ are non-transitive. By Equation 1, we have

$$\begin{aligned} V_{\sigma\tau} &= V_\sigma + V_\tau + mn \\ &> \frac{m^2}{2} + \frac{n^2}{2} + mn \\ &= \frac{(m+n)^2}{2}, \end{aligned}$$

and so $\sigma\tau$ is non-transitive. \square

With the two lemmas above in place, we are now able to provide a quick proof of Theorem 2.1, the main result of this section.

Proof of Theorem 2.1. Example 2.3, along with

Example 2.6.

$$\begin{aligned} A &= 12 \ 10 \ 3 \ 1 \\ B &= 9 \ 8 \ 7 \ 2 \\ C &= 11 \ 6 \ 5 \ 4 \end{aligned}$$

and

Example 2.7.

$$\begin{aligned} A &= 15 \ 11 \ 7 \ 4 \ 3 \\ B &= 14 \ 10 \ 9 \ 5 \ 2 \\ C &= 13 \ 12 \ 8 \ 6 \ 1 \end{aligned}$$

provide balanced, non-transitive sets of dice for $n = 3, 4, 5$, which give rise to balanced words for these n , the smallest representatives (in the context of the theorem) for each congruence class modulo 3. Lemmas 2.4 and 2.5 then give that the concatenation of two balanced non-transitive words is a balanced non-transitive word, and the correspondence between words and sets of dice completes the proof. \square

3 FACE-SUMS

After taking a closer look at Example 2.3 as well as the sets of balanced, non-transitive dice given in the proof of Theorem 2.1, one may notice the following phenomenon: In any one of these sets of dice, the sum of the labels of any two dice are equal. Formally, if D is a set of n -sided dice and $\sigma(D) = s_1 s_2 \cdots s_{3n}$, we define the *face-sums* of D to be

$$\sum_{s_i=a} i, \sum_{s_i=b} i, \text{ and } \sum_{s_i=c} i.$$

Theorem 3.1. *A set of dice D (or the corresponding word) is balanced if and only if the face-sums of its dice are all equal.*

Proof. (*only if*) Let D be a set of balanced dice, and $\sigma(D)$ the word associated with it. Recall our definition for balanced words:

$$\sum_{s_i=a} q_{\sigma}^{+}(s_i) = \sum_{s_i=b} q_{\sigma}^{+}(s_i) = \sum_{s_i=c} q_{\sigma}^{+}(s_i),$$

which is obviously equivalent to

$$\sum_{s_i=a} q_{\sigma}^{-}(s_i) = \sum_{s_i=b} q_{\sigma}^{-}(s_i) = \sum_{s_i=c} q_{\sigma}^{-}(s_i).$$

Further define

$$q_\sigma(s_i) = |\{j < i : s_j = s_i\}|.$$

We focus on die A (with face-sum $\sum_{s_i=a} i$), and make two observations: First, for a face of A , its label i may be written as

$$i = q_\sigma^+(s_i) + q_\sigma^-(s_i) + q_\sigma(s_i) + 1.$$

Second, since A has n sides,

$$\sum_{s_i=a} q_\sigma(s_i) = \frac{n(n-1)}{2}.$$

Then,

$$\sum_{s_i=a} i = \sum_{s_i=a} (q_\sigma^+(s_i) + q_\sigma^-(s_i) + q_\sigma(s_i) + 1) = \sum_{s_i=a} q_\sigma^+(s_i) + \sum_{s_i=a} q_\sigma^-(s_i) + \frac{n(n-1)}{2} + n.$$

However, this computation was independent of our choice of A , so the other two are analogous, and every term in sight is equal as $\sigma(D)$ is balanced.

(if) Let D be a set of n -sided dice (with word $\sigma(D)$), and assume that

$$\sum_{s_i=a} i = \sum_{s_i=b} i = \sum_{s_i=c} i.$$

By the above, this is equivalent to

$$\sum_{s_i=a} q_\sigma^+(s_i) + \sum_{s_i=a} q_\sigma^-(s_i) = \sum_{s_i=b} q_\sigma^+(s_i) + \sum_{s_i=b} q_\sigma^-(s_i) = \sum_{s_i=c} q_\sigma^+(s_i) + \sum_{s_i=c} q_\sigma^-(s_i).$$

Write

$$a^+ = \sum_{s_i=a} q_\sigma^+(s_i), \quad a^- = \sum_{s_i=a} q_\sigma^-(s_i)$$

and analogously define

$$b^+, b^-, c^+, \text{ and } c^-.$$

Then, we have

$$a^+ + a^- = b^+ + b^- = c^+ + c^-,$$

and

$$a^+ + b^- = b^+ + c^- = c^+ + a^- = n^2,$$

giving six equations in six unknowns. Some straightforward linear algebra gives

$$a^+ = b^+ = c^+,$$

whence we also have

$$a^- = b^- = c^-.$$

□

Applying the result of Theorem 3.1, we obtain the following algorithm for checking if a given partition of $[3n]$ into 3 size- n subsets determines a set of balanced non-transitive dice.

Algorithm 3.2. Suppose we are given a partition of $[3n]$ into 3 size n subsets A, B , and C . First, check the sums of the elements of these subsets. These sums are equal if and only if the set of dice is balanced (by Theorem 3.1). If this condition is met, check $P(A \succ B)$. If $P(A \succ B) = 1/2$, the set of dice is balanced but fair. If $P(A \succ B) > 1/2$, the set is balanced and non-transitive. If $P(A \succ B) < 1/2$, switching the labels of sets B and C produces a balanced non-transitive set of dice. Since this algorithm must check each pair of sides from dice A and B , it clearly runs in $O(n^2)$ time.

4 OTHER CONSTRUCTIONS

4.1 *Non-transitive dice and Fibonacci numbers*

In [4], Savage forms sets of non-transitive dice from consecutive terms of the Fibonacci Sequence. We briefly explain his construction. (We let f_i denote the i^{th} Fibonacci number, so that $f_1 = f_2 = 1, f_3 = 2$, et cetera.)

Algorithm 4.1 ([4]). Given a Fibonacci number f_k , consider the sequence

$$f_{k-2}, f_{k-1}, f_k, f_{k-1}, f_{k-2}.$$

Beginning with the number $3f_k$, label die A with f_{k-2} consecutive descending integers. Then label die B with the next f_{k-1} values, die C with the next f_k values, A with the next f_{k-1} values, and B with the last (ending in 1) f_{k-2} values. This produces a set of non-transitive dice (which is never balanced).

In the case where f_k is an *odd* Fibonacci number, a simple addition to this construction actually yields a balanced set.

Algorithm 4.2. Perform Algorithm 4.1 to obtain a set of non-transitive dice. Then, swap the last element of the first set of values ($3f_k - f_{k-2} + 1$, to be precise), which is on die A , with the first element of the second set of values ($3f_k - f_{k-2}$), which is the largest number on die B . The resulting set of dice is non-transitive and balanced.

4.2 *Sets of four dice*

Consider a modification of *set of dice* to mean four dice, labeled A, B, C, D . Then

Example 4.3.

$$\begin{array}{l} A: 12 \ 5 \ 2 \\ B: 11 \ 8 \ 1 \\ C: 10 \ 7 \ 3 \\ D: 9 \ 6 \ 4 \end{array}$$

Example 4.4.

A : 16 10 7 1
 B : 15 9 6 4
 C : 14 12 5 3
 D : 13 11 8 2

Example 4.5.

A : 20 13 10 6 4
 B : 19 15 9 8 3
 C : 18 16 12 5 1
 D : 17 14 11 7 2

give minimal examples for balanced non-transitive sets of dice. Modifying to $4n$ -letter words (with n each of a, b, c, d), the proof of Theorem 2.1 generalizes, which gives us the following.

Theorem 4.6. *For any $n \geq 3$, there exists a set of four balanced, non-transitive, n -sided dice.*

However, Example 4.3 has unequal face-sums, proving that Theorem 3.1 does not generalize.

5 FURTHER QUESTIONS

Given the proof of Theorem 2.1, it seems natural to define the following.

Definition 5.1. *Let σ be a balanced non-transitive word. If there do not exist balanced non-transitive words τ_1 and τ_2 (both nonempty) such that $\sigma = \tau_1\tau_2$, we say that σ (and its associated set of dice) is irreducible.*

Question 5.2. *For any $n \geq 3$, does there necessarily exist an irreducible balanced non-transitive set of n -sided dice?*

The notion of non-transitive triples of dice also suggests the following broad generalization.

Definition 5.3. *Let G be an orientation of K_m , the complete graph on the vertex set $\{v_1, v_2, \dots, v_m\}$. Define an realization of G to be an m -tuple of n -sided dice A_1, A_2, \dots, A_m for some n (where, as before, the A_i 's are pairwise disjoint) satisfying the following property:*

$$P(A_i \succ A_j) > \frac{1}{2} \Leftrightarrow (v_i \rightarrow v_j) \text{ is an edge of } G.$$

Theorem 2.1 gives us the following as a corollary.

Corollary 5.4. *Let G be an orientation of K_3 . Then there exists a realization of G using n -sided dice for any $n \geq 3$.*

Proof. If G is a directed cycle, Theorem 2.1 gives the result. Otherwise, G is acyclic, meaning the orientation corresponds to a total ordering of the vertices. Then the dice $A = \{1, 2, \dots, n\}$, $B = \{n + 1, n + 2, \dots, 2n\}$, and $C = \{2n + 1, 2n + 2, \dots, 3n\}$, appropriately placed, will provide a realization. \square

References

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