Honors Add-on – Counting & Enumeration

Jay Schweig

February 2, 2018

Lecture One

1 Introduction

At first glance, the idea of "counting things" seems pretty elementary. After all, to count a set of objects, just line them up and begin counting: "$1, 2, 3, \ldots$"

However, even some fairly simple "real world" setups can lead to sets of objects that are difficult to count. For example, suppose there two different types of donuts, and that donuts of the same type are indistinguishable. How many ways are there to order five donuts? Well, if the types are, for example, glazed and chocolate, then our order is uniquely determined by the number of glazed donuts ordered. So there are six possibilities: 0 glazed, 1 glazed, . . . , 5 glazed. But now suppose there are seven types of donuts! How many ways are there to order five? Even in a small case such as this, it's simply far too messy to write out all the possibilities.

This is where basic enumeration comes in. By thinking about the problem in the "right way," often a seemingly complicated problem can be reduced to a straightforward computation. However, as with most math (and indeed, most worthwhile things), you've got to practice!

2 Factorials and Permutations

One of the basic techniques in simple counting problems is what's sometimes called *The Multiplication Principle*. Loosely speaking, this principle says that if there are m ways to do one task and n ways to do another, then there are mn ways to do both tasks.

For example, suppose I want to count all 2-digit sequences XY where X is a number between 1 and 3 (inclusive) and Y is a letter between A and D (inclusive). There are three ways to choose X, and for each way to choose X, there are 4 ways to choose Y. This is maybe easier to see from the following list:

$$\begin{array}{cccc} 1A & 1B & 1C & 1D \\ 2A & 2B & 2C & 2D \\ 3A & 3B & 3C & 3D \end{array}$$

Note we can interpret this list in two ways: First, for each choice of X (for example, 2), there are 4 choices for Y, which we read across the middle row. Second, we can also interpret this by reading down the columns; for each choice for Y (for example, C), we have three choices for X. We can see these by reading down the third column.

Thus the total number of possible sequences XY is $3 \times 4 = 12$.

Question 2.1 *How many* 4*-digit numbers are there if every digit must be a* $5, 6, 7,$ *or* 8*? What if the last digit must be a* 5 *or* 6*?*

This is a pretty straightforward application of the multiplication principle. In the first case there are four choices for each spot, so we get $4 \times 4 \times 4 \times 4 = 4^4 = 256$ such numbers. In the second case the number of possible digits for the last spot is 2, so we get $4 \times 4 \times 4 \times 2 = 128$ numbers.

Exercise 1 *How many* 7*-digit numbers are there? Note that in order to be a* 7*-digit number, the first digit cannot be zero.*

Question 2.2 *How many five-letter words are there using only the letters* A *through* G*? What if the word must contain at least two distinct letters?*

In the first case, there are 7 choices for each spot, so we get $7^5 = 16,807$ words. In the second case, we need to subtract all five-letter words that are made up of just one letter from A through G. There are only seven of those, namely $AAAAA$, $BBBBB$, etc., so the count becomes $16,807 - 7 = 16,800$.

Exercise 2 *How many possible* 3*-letter words are there?*

Before doing more examples of the multiplication principle, we should discuss when to use addition, rather than multiplication, in a counting problem. Loosely speaking, the multiplication principle applies when you choose one thing AND another. In our first example, we counted all two-place sequences consisting of a number between 1 and 3 and a letter between A and F. So, we multiply, as choosing such a sequence involves a choice of both.

So when do we add? Well, addition corresponds to when you choose one thing OR another. For example, consider the following.

Question 2.3 *How many two-digit sequences are there consisting of either two digits between* 1 *and* 5 *(inclusive) or two identical digits greater than* 6.

We want to count all sequences that satisfy the first condition OR the second condition. Another way to think about this is that we are really counting two different sets of sequences. For the first type of sequence, we have five choices for each digit, meaning there are $5 \times 5 = 25$ such sequences. In the second case, we have three choices for the repeated number, and so there are 3 such sequences. Therefore, our total count is $25 + 3 = 28$ sequences.

This example is pretty simple, but it's good to remember this heuristic when going into more complicated counting problems, such as the counting of poker hands we consider later in the course.

Exercise 3 *How many* 3*-letter words are there that begin with an 'A' or a 'B'?*

Exercise 4 *How many* 4*-letter words are there that either begin with an A or end with a Q? (Be careful!)*

While the multiplication principle seems simple enough, it can actually be a bit subtle, as the next question will show.

Question 2.4 *How many ways are there to order four distinct objects?*

Here, something a bit different is going on. Let's choose A, B, C, and D to stand for our objects. We think of ordering these four objects as lining up these four letters in a row. We have four choices for our first spot; we can choose either A, B, C, or D. How many choices do we have for the second spot? Here's where it gets a little subtle: No matter our choice for the first spot, we will always have three choices for the second spot, *even though these choices may be different*! For example, if we choose A for the first spot, we have B, C, or D to choose for the second. If instead we choose B for the first spot, we have A, C, or D to choose for the second, etc.

Again, we write these choices out as a list:

$ABCD$	$ABDC$	$ACBD$	$ACDB$	$ADBC$	$ADCB$
$BACD$	$BADC$	$BCAD$	$BCDA$	$BDAC$	$BDCA$
$CABD$	$CADB$	$CBAD$	$CBDA$	$CDAB$	$CDBA$
$DABC$	$DACB$	$DBAC$	$DBCA$	$DCAB$	$DCBA$

If we look at the top row for example, we've already chosen A for the first spot. So we have three choices for the next spot. If we choose D for the next spot, that puts us in the last two entries of the row. Now we have two choices for the next spot (either B or C). Once we choose that entry, the final entry is fixed (it's whichever letter is left over).

So, how do we apply the multiplication principle to this? Well, we simply count up the number of choices for each spot. We have 4 choices for the first spot, 3 for the second, 2 for the third, and 1 for the fourth. Thus the total number of ways to arrange these letters is $4 \times 3 \times 2 \times 1 = 24$.

We can generalize this example in the obvious way, using n objects. The number of ways to order n (distinct) objects is $n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1$, since there are n choices for the first spot, $n-1$ choices for the second, etc. This number is known as n *factorial*, and is denoted by $n!$.

Fact 2.5 *The number of ways to order n distinct objects is*

$$n! = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1$$

Though it may not make immediate sense, we define $0!$ to be 1.

Question 2.6 *How many ways are there to place seven distinct books in a row?*

Placing the distinct books in a row is the same as ordering them, and so the answer is $7! = 5040$.

It's worth a remark here that cancelling factorials is pretty straightforward. When in doubt, simply write out the product the factorial represents. For example,

$$\frac{7!}{4!} = \frac{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{4 \times 3 \times 2 \times 1} = 7 \times 6 \times 5.$$

Exercise 5 *Simplify $\frac{n!}{(n-3)!}$, where $n \geq 3$.*

Exercise 6 *How many ways are there to order the letters A, B, C, D, and E?*

Question 2.7 *How many ways are there to order the letters A, B, C, D, and E if D and E must be the last two letters?*

We can think of such an order as first ordering the letters A, B, and C, and then ordering the letters D and E and placing that order after the first. Thus, there are $3! \times 2! = 12$ such orderings.

Exercise 7 *How many six-letter words are there in which each of the first three letters is A, B, or C?*

Exercise 8 *How many ways are there to order the letters A, B, C, D, E, and F if D, E, and F must be the last three letters in the order?*

Exercise 9 *How many ways are there to order the letters A, B, C, D, and E if either D or E must be the last letter?*

Lecture Two

Let's do another example using both the multiplication principle and factorials.

Question 2.8 *How many 7-letter words are there such that none of the first three letters is A, B, or C, and the last four letters are some ordering of the letters Q, R, S, and T?*

There are 3 choices for each of the first three letters, so there are 23^3 possible initial segments, and there are 4! orderings of Q, R, S, and T. Thus, there are $23^3 \times 4!$ such words.

The following exercises are a little more subtle, but use roughly the same principles.

Exercise 10 *Suppose a class consists of ten girls and ten boys. How many ways are there to create ten two-person groups, each of one boy and one girl?*

Exercise 11 *Suppose a code is made of some ordering of the letters A, B, and C, followed by a three digit number. For example, $CBA116$ is a permissible code. How many such codes are there?*

In fact, the factorial is a special case of what's known as a permutation, which we describe as follows.

Question 2.9 *How many ways are there to select k objects from n distinct objects, where the order of the selection matters?*

Though this question seems a bit more complicated than the factorial, it's roughly the same: We have n choices for the first object, $n-1$ for the second, and so on, all the way to $(n-k+1)$ choices for the kth object. This is known as the permutation $P(n,k)$:

Fact 2.10 *The number of ways to select k things from n distinct things, where the order of the selection matters, is*

$$P(n,k) = n \times (n-1) \times (n-2) \times \cdots \times (n-k+1) = \frac{n!}{(n-k)!}$$

Note that $P(n,n)$ is the number of ways to select n things from n, where order matters. But this is just the number of ways of ordering n distinct things, and indeed we have $P(n,n) = \frac{n!}{(n-n)!} = \frac{n!}{0!} = n!$.

Question 2.11 *How many three-letter words are there that have no repeated letters?*

We can visualize this as choosing three letters from the 26 possible letters, where the order of the selection matters (e.g., selecting the letters R, A, M in order is obviously different from selecting the letters A, R, M in order.

So, to answer this question, we use the formula above:

$$P(26,3) = \frac{26!}{(26-3)!} = \frac{26!}{23!} = 26 \times 25 \times 24$$

This problem also allows us to do our first example of *overcounting*, which is when we solve a problem by counting *too many* things, then subtracting the things we don't want to count.

Question 2.12 *How many three-letter words are there with at least one repeated letter?*

We answer these questions using two different methods. For the first, we break the question into cases:

- Case 1: All three letters of the word are the same. There are clearly 26 such words like this (once we choose the one letter, we know the word).

- Case 2: One letter is repeated twice, and the third is different. There are three possible spots for the non-repeated letter, and 26 choices for it. Then there are 25 choices for the repeated letter, whose two occurrences must occupy the other two spots. So the number of words in this case is $3 \times 26 \times 25$.

Thus, there are $26 + 3 \times 26 \times 25 = 1976$ such words (ask yourself why we *added* the two cases, rather than multiplying them).

However, this problem can be solved much more easily by using the method of overcounting. Specifically, in this case, we will count all possible words, then subtract those we don't want to count (i.e., those that *don't* have repeated letters).

Clearly, there are 26^3 total three-letter words. Above, we found that there are $P(26,3) = \frac{26!}{23!} = 26 \times 25 \times 24$ three-letter words without repeated letters. Thus, there are

$$26^3 - (26 \times 25 \times 24) = 1976$$

three-letter words with at least one repeated letter.

Apart from being quicker, the second method is much better when we consider longer words, as shown by the following exercise.

Exercise 12 *How many five-letter words are there with at least one repeated letter?*

Exercise 13 *Expanding on Exercise 11, suppose we define a code to be any three letters followed by any three numbers.*

- *How many codes are there?*
- *How many codes are there in which the letters do not repeat?*
- *How many codes are there in which neither the letters nor the numbers repeat?*

Exercise 14 *How many codes are there in which there's at least one repetition of a number or letter (or both)?*

Lecture Three

Fact 2.13 *For the following question, we use the following basic probability principle. If there are n total objects and k of these objects are* special, *then if an object is chosen at random, the probability that the object is* special *is n/k.*

Question 2.14 *Suppose a music library consists of ten artists. When set to shuffle, the program selects an artist at random, and plays a song by them. What is the probability that, in playing three songs, at least one artist is repeated? What if ten songs are played?*

The number of total three artist sequences is $10^3 = 1,000$. The number of ways of choosing three distinct artists is $P(10,3) = 10 \times 9 \times 8 = 720$. As in Question 2.12, we subtract to get the number of sequences with at least one repeat, giving: $1,000 - 720 = 280$. So the probability of a repeat is $280/1000 = .28$. The case where 10 songs are played is done in the same way: There are 10^{10} total possible sequences, and $P(10,10) = 10!$ sequences with no repeat. Thus the probability of a repeat is

$$\frac{10^{10} - 10!}{10^{10}} \simeq .99964.$$

Following are a few more exercises using $P(n,k)$ and basic factorial properties.

Exercise 15 *Suppose you have a box of ten (distinguishable) chocolates. How many ways are there to give Alice, Bob, and Charlie one chocolate each?*

Exercise 16 *Suppose one of the chocolates is a cherry cordial. How many ways are there to do the above problem if neither Alice nor Charlie wants the cherry cordial?*

Let's do a few more exercises with these ideas.

Question 2.15 *How many ways are there to seat four girls and four boys in a row? What if no two people of the same gender can sit next to each other?*

The first part of the question is immediate; there are 8 people, hence $8! = 40,320$ ways for them to sit in a row. The next part of the question is a bit more subtle, and we can answer it in two ways. The first way is a fairly quick application of the multiplication principle: We have eight choices for the first person, but then only four choices for the second, as it has to be someone of the opposite gender. Similarly, we have three choices for the third person, three for the fourth, and so on. Thus, there are

$$8 \times 4 \times 3 \times 3 \times 2 \times 2 \times 1 \times 1 = 1152$$

such seating arrangements. Let's tackle this problem another way, though. Note there are only two ways to order the genders: GBGBGBGB or BGBGBGBG. Once one of these seating patterns is established, it's only a matter of ordering the boys and ordering the girls, and there are 4! ways to order each. Thus, we obtain $2 \times 4! \times 4! = 1152$ seating arrangements.

Exercise 17 *How many ways are there to seat five boys and six girls in a row if no two people of the same gender can sit next to each other?*

A general heuristic in counting is that problems tend to be more straightforward when all involved items are distinguishable. For example, we know that there are 5! anagrams of the word FRUIT, as each letter is different and so the number of anagrams is just the number of ways to order 5 distinct objects. However, there can't be 5! different anagrams of the word APPLE, since two of the letters are the same. If this is a little unclear at first, just ask how many different anagrams there are of AAAAA. There are 5 letters here, but clearly only one anagram; any way you have of shuffling these letters results in the same word, namely AAAAA.

To formalize this idea, we'll say that two items (or letters) are of the same *type* if they are indistinguishable.

Question 2.16 *How many permutations are there of n letters, where there are a_1-many items of type* 1*, a_2-letters of type* 2*, etc.?*

Asking the above question for the word APPLE, we can say that there is 1 item of type A, 2 of type P, 1 of type L, and 1 of type E.

As with many problems in combinatorics, we can get an idea for the solution by looking at small examples. Let's ask how many different anagrams there are of the word BOB. Just writing out some examples, we can see that there are three anagrams:

$$OBB \qquad BOB \qquad BBO$$

However, we want to come up with a formula for this case, rather than just counting all the anagrams by brute force. In order to this, let's use our method of overcounting as follows. Suppose the B's in BOB *were* distinguishable. We'll accomplish this here by writing B_1 and B_2. Then every letter in the word would be distinguishable, and we know that there are $3! = 6$ anagrams of a three-letter word, all of whose letters are distinguishable:

$$OB_1B_2 \qquad OB_2B_1 \qquad B_1OB_2 \qquad B_2OB_1 \qquad B_1B_2O \qquad B_2B_1O$$

If we were to remove the subscripts from the B's, we'd see that each anagram is counted twice, but why? Well, there are $2! = 2$ ways to shuffle B_1 and B_2, so if we look at one anagram, say OBB, there are two ways to arrange the two B's: OB_1B_2 and OB_2B_1. So we get the total number of anagrams of BOB by looking at the number of anagrams of B_1OB_2, and dividing by the number of ways to arrange the B's:

$$\frac{3!}{2!} = \frac{6}{2} = 3.$$

We can get a general answer to the above question using the same reasoning: There are $a_1!$ ways to arrange the objects of type 1, $a_2!$ ways to arrange the objects of type 2, and so on. So, looking at a fixed permutation of the objects, there would be $a_1!a_2!\cdots a_k!$ ways to obtain this fixed permutation if all the objects were made distinguishable.

We illustrate this with another example. Suppose we want all possible anagrams of the word BANANA. Here there is one item of type B, three items of type A, and two of type N. Let's focus on a particular anagram, say NBAANA, and do the same overcounting trick as above. We label all our letters, making each distinguishable: $BA_1N_1A_2N_2A_3$, and ask how many ways there are two obtain this anagram with the distinguishable letters. Well, there are $3! = 6$ ways to permute the A's, and $2! = 2$ ways to permute the N's, so we claim there are $6 \times 2 = 12$ different anagrams of the altered word that will correspond to the chosen anagram, NBAANA:

$$\begin{array}{lll}
N_1BA_1A_2N_2A_3 & N_1BA_1A_3N_2A_2 & N_1BA_2A_1N_2A_3 \\
N_1BA_2A_3N_2A_1 & N_1BA_3A_1N_2A_2 & N_1BA_3A_2N_2A_1 \\
N_2BA_1A_2N_1A_3 & N_2BA_1A_3N_1A_2 & N_2BA_2A_1N_1A_3 \\
N_2BA_2A_3N_1A_1 & N_2BA_3A_1N_1A_2 & N_2BA_3A_2N_1A_1
\end{array}$$

The same reasoning is easily seen to hold for any permutation of BANANA, meaning that we can count the number of anagrams of the original word as follows: The altered word has 6! different anagrams, but once we remove the subscripts, each anagram shows up 12 times, meaning the total number of anagrams of BANANA is

$$\frac{6!}{3!2!} = \frac{720}{12} = 60.$$

Generalizing this to our above question, we have the following fact.

Fact 2.17 *The number of permutations of n objects, with a_1 objects of type 1, a_2 objects of type 2, ..., a_k objects of type k is*

$$\frac{n!}{a_1!a_2!\cdots a_k!}$$

For example, suppose we wanted to count the number of anagrams of the word COMBINATORICS. Note that there are two C's, two O's, two I's, and every other letter appears just once. Since there are 13 total letters, the above formula gives

$$\frac{13!}{2!2!2!}$$

anagrams of COMBINATORICS. Note that we usually omit the occurrences of 1! from the denominator, since these don't change our count.

Exercise 18 *How many distinct anagrams are there of the word ANAGRAM?*

Lecture Four

The next problem sums up a lot of the techniques we've been using. It also shows how complicated a seemingly simple counting problem can get even with just one or two added assumptions.

Question 2.18 *Suppose Frank has three French books, four German books, and two English books.*

1. *How many ways are there for him to arrange these books on a shelf if they are all distinguishable?*
2. *What if the French books are indistinguishable from one another, but all the other books are distinguishable?*
3. *What if books of the same language are indistinguishable?*
4. *What if all books are distinguishable, but the German books must be together?*
5. *What if all books are distinguishable, but all books of the same language must be together?*
6. *What if all books are indistinguishable, but all books of the same language must be together?*

Part (1) is pretty quick. There are nine total books, and all are distinguishable, so there are $9!$ ways to order them.

Parts (2) and (3) follow quickly from Fact 2.17 as well. If only the French books are indistinguishable, then there are $9!/3!$ many ways to arrange the books. If all books of the same language are indistinguishable, then the count is $9!/(3!4!2!)$.

Part (4) is where it gets a little tricky. How do we count all the cases where the German books are together? Well, think of it this way: Picture the German books as one "block." How many ways are there to order the five other books, plus this one block? Since everything is distinguishable, there are $6!$ ways to order the five books and the block of German books. However, we are not quite done, since the German books can still be ordered *within* the block. Since there are four German books, there are $4!$ ways to do this. Thus, the final answer is $6!4!$.

Exercise 19 *Solve Part (5) of the above.*

Part (6) is somewhat quicker, given the above view of consecutive books as a "block." Since there are three blocks, and the order of the books within the blocks doesn't matter (as books of the same language are indistinguishable), the answer to this part is simply $3!$.

Exercise 20 *How many anagrams of the word BANANA are there if all three A's cannot be consecutive? (For example, BANAAN is allowed, but BAAANN is not).*

The general principle going on here is the following: Suppose we want to count a set of objects. Often we can obtain an initial count of the objects. Call this number α. If, in this initial count, each distinct object is counted β times, then the correct count is α/β. This is essentially what's going on with our above permutations of indistinct objects.

Given this heuristic, we can give the following variation on Exercise 10.

Exercise 21 **($\star$)** *Suppose a class consists of twenty people. How many ways are there to create ten two-person groups?*

3 $\binom{n}{k}$ and the Binomial Theorem

Now we get to one of the most important statistics in basic combinatorics. The following question can be seen as the "order doesn't matter" version of Question 2.9

Question 3.1 *How many ways are there to select k objects from n distinct objects, where the order of the selection does not matter?*

Another way to view the above question is in terms of sets. Recall that a *set* is simply an unordered collection of objects (usually called the set's *elements*). We write our sets between curly brackets $\{$ and $\}$. So, for example, the set $\{2, 5, a, q\}$ is the same as the set $\{2, a, q, 5\}$, as the order in which we write the elements of a set does not matter.

A set A is a *subset* of a set B if every element of A is an element of B. For example, $\{2, q\}$ is a subset of $\{2, 5, a, q\}$, whereas $\{2, 7\}$ is not, since 7 is not an element of the larger set. The number of elements in a set is known as its *size* or *cardinality.*

Perhaps the first question one should have here is the following:

Question 3.2 *How many subsets are there of a set of size n?*

There is a unique set with no elements, called the *empty set*, and denoted $\emptyset$. A set with one element thus has two subsets: the emptyset, and the set itself. If the set has two elements (say the set is $\{a, b\}$), it has four subsets: $\emptyset, \{a\}, \{b\}, \{a, b\}$. A set with three elements ends up having eight subsets. So hopefully we notice a pattern: Sets with n elements seem to have 2^n subsets.

This is a case where a numerical observation can actually guide our combinatorial reasoning: The count 2^n is the same as the number of ways to flip n

on/off switches. Indeed, every subset of an n-element set can be constructed by asking, for each of the n elements, "Is this element in the subset?" There are thus 2^n ways to answer this question for every element of the set, hence we have the following:

Fact 3.3 *There are 2^n subsets of a set of size n.*

However, we can ask more about these subsets:

Question 3.4 *How many size-k subsets can we choose from a set of n distinct objects?*

This is a statistic that occurs quite often in "real-world" problems. For example, the number of possible poker hands is the number of ways to select 5 cards from 52 (distinct) cards, since the order in which you select the cards of a given hand is immaterial.

To illustrate, we can ask how many size-2 subsets there are of $\{2, 5, a, q\}$. It turns out there are six such subsets: $\{2, 5\}, \{2, a\}, \{2, q\}, \{5, a\}, \{5, q\}$, and $\{a, q\}$. Remind yourself that the elements of a set are unordered, so we haven't listed, for example, $\{q, 2\}$ in the above count, as it is the same as the set $\{2, q\}$.

Surprisingly, the above question is answered fairly easily, using our already established formula for $P(n, k)$:

$$P(n, k) = \frac{n!}{(n-k)!}.$$

Specifically, $P(n, k)$ counts the number of ways to select k objects from n distinct objects, where order matters. If we wish to count *unordered* selections, simply note that each unordered subset of size k is counted $k!$ times by $P(n, k)$. Thus, we simply divide by $k!$ to answer the above questions.

Fact 3.5 *The number of ways to select k objects from n distinct objects is:*

$$\frac{P(n, k)}{k!} = \frac{n!}{(n-k)!k!}.$$

This number is denoted $\binom{n}{k}$, and pronounced "n choose k."

Let's look at a few basic cases. Note that $\binom{n}{0} = \frac{n!}{n!0!} = 1$ (recall that $0! = 1$). This makes sense, as there's only one way to select zero things from n things, namely by selecting nothing.

Similarly, $\binom{n}{n} = \frac{n!}{0!n!} = 1$, as there is only one way to select n things from n things, namely by selecting everything.

This is a good habit to pick up; when confronted with a new definition, it often helps to check it in some easy cases. We can also calculate $\binom{n}{1}$. This is the number of ways to select 1 thing from n things, so we know that it should equal n. Indeed,

$$\binom{n}{1} = \frac{n!}{(n-1)!1!} = \frac{n \times (n-1) \times (n-1) \times \cdots \times 2 \times 1}{(n-1) \times (n-2) \times \cdots \times 2 \times 1} = n.$$

In our example above, we found that there were 6 size-2 subsets of a set of size 4. Indeed,

$$\binom{4}{2} = \frac{4!}{2!2!} = \frac{24}{4} = 6.$$

Let's start with a simple question.

Question 3.6 *Suppose Fabrizio has ten watches and three briefcases. How many ways can he select four watches and two briefcases to bring on holiday?*

There are $\binom{10}{4} = 210$ ways to select four watches, and $\binom{3}{2} = 3$ ways to select the briefcases. Since he's selecting *both*, there are $210 \times 3 = 630$ possible selections.

Let's do a similar exercise.

Exercise 22 *Suppose Alice has seven bangles, six scarves, and five pairs of earrings to wear. How many ways are there for her to select exactly three bangles, one scarf, and one pair of earrings?*

Lecture Five

Question 3.7 *How many possible poker hands are there? How many poker hands are a flush (but not a straight flush)?*

The first part of the question is pretty quick: There are 5 cards in a hand, and 52 cards from which to choose, thus the number of possible hands is:

$$\binom{52}{5} = \frac{52!}{47!5!} = 2598960.$$

To answer the second part, we need to consider what choices are made in selecting a flush. First, we need to select the suit of the flush, and there are four such choices. After selecting the suit, we need to select the five denominations of the cards. There are 13 possible denominations, so the total number of flush hands is

$$4 \times \binom{13}{5} = 5148.$$

However, this also includes straight flushes, and we need to subtract this number. Indeed, there are 4 choices for the suit of a straight flush, and 10 choices for the card that starts the flush (Ace, $2, 3, \ldots, 10$). So there are

$$4 \times 10 = 40$$

straight flushes, and thus there are

$$5148 - 40 = 5108$$

flushes that are not straight flushes.

Since the selection of any of the possible poker hands is equally likely, this means that the probability of being dealt a flush is:

$$\frac{5108}{2598960} \simeq 0.00197.$$

In other words, the probability of being dealt a flush is approximately $1/500$.

The numbers $\binom{n}{k}$ are often referred to as "binomial coefficients," for a reason that we'll see in a bit.

Let's solve a previous problem using binomial coefficients. We've already calculated that there are 60 anagrams of the word BANANA, but let's find this answer using binomial coefficients. Think of our anagram as filling six blanks. We first choose the blank that the B will occupy. There are $\binom{6}{1} = 6$ ways to do this. Then we select the three blanks (from the remaining 5) which the A's will occupy. There will be two blanks left over, and thus the N's will occupy these. So, the number of anagrams of BANANA is given by:

$$\binom{6}{1}\binom{5}{3} = \frac{6!}{5!1!} \times \frac{5!}{2!3!} = 60.$$

Note that we could have done this problem in several different orders. For example, we could have chosen the blanks for the A's, then chosen the blanks for the N's. This would have given

$$\binom{6}{3}\binom{3}{2} = \frac{6!}{3!3!} \times \frac{3!}{1!2!} = 60.$$

Exercise 23 *How many ways are there to line up three vanilla yogurts, six strawberry yogurts, and one peach yogurt, where yogurts of the same flavor are indistinguishable? Prove your answer in two ways.*

We also note that there is a symmetry in the binomial coefficients. Namely,

$$\binom{n}{k} = \binom{n}{n-k}$$

for all n and k. To see this, simply write out the expressions for each:

$$\binom{n}{n-k} = \frac{n!}{(n-(n-k))!(n-k)!} = \frac{n!}{k!(n-k)!} = \binom{n}{k}.$$

However, this fact has a much nicer combinatorial proof: $\binom{n}{k}$ counts the number of k-element subsets of a set of size n. However, when we select a subset of size k, we could either consider that set, or the set of elements left over, of which there are $n-k$.

In the above, we discuss what we call a *combinatorial proof*. We usually prefer combinatorial proofs, but what do we mean by this? Well, if we look at the first proof of the above fact, it's simply a matter of plugging in the values and checking that the expressions are equal. However, this tells us nothing about *why* the equation is true. Our second proof, which is combinatorial, actually examines the sets corresponding to $\binom{n}{k}$ and $\binom{n}{n-k}$, and sheds more light on the *reason* why the equation holds.

What's the benefit of a combinatorial proof? Apart from aesthetics, one is more likely to remember a result if they know a combinatorial proof for it. For example, the picture from the above fact, in which we select k items from n and have $n-k$ items left over, can be easily remembered as the reason why that equation holds.

Exercise 24 ($\star$) *Give a combinatorial proof for the following:*

$$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$$

Question 3.8 *How many ways are there to split a group of ten people into two groups of five people each?*

In order to be able to write down examples, let's first consider a smaller case, where you're splitting a group of four people into two two-person groups. There are $\binom{4}{2} = 6$ ways to select one group of two people from the four. The remaining two can be considered in the other group. However, there's one small problem: Suppose the people are represented by A, B, C, and D. If we select A and B for the first group, then C and D are in the other group. However, if we select C and D for the first group, then A and B are in the other group. But as we don't care about the order of the groups, these two cases are actually the same. Thus, each way of splitting into two groups is counted twice in the above count. So, the total number of ways to divide four people into two two-person groups is $\binom{4}{2}/2 = 3$. Indeed, the three ways are AB and CD, AC and BD, and AD and BC.

The same exact reasoning holds to answer our above question, giving $\binom{10}{5}/2 = 126$.

Exercise 25 *How many ways are there to form two groups of four people out of nine people (so that one person is necessarily excluded)?*

Exercise 26 *Suppose Clark has a box of ten (distinguishable) chocolates. How many ways are there for him to select four chocolates from the box? What if the box contains two specific chocolates, and Clark doesn't want to select* both *of these chocolates (but selecting one of them is okay)?*

Depending on how much you remember of high school algebra, the binomial coefficients may look familiar to you. We'll start by expanding the polynomial $(1+t)^2$:

$$(1+t)^2 = 1 + 2t + t^2.$$

As combinatorists, we should ask ourselves *why* the coefficients shown above appear; i.e., why are the constant term and the coefficient of t^2 both 1, while the coefficient of t is 2?

Well, if we write $(1+t)^2 = (1+t)(1+t)$, we see that there's only one way to get a constant term, namely by selecting 1 from each of the factors. Similarly, there's only one way to obtain a t^2 in the product, namely by selecting t from each of the factors.

$(1+t)^4$. We first write

$$(1+t)^4 = (1+t)(1+t)(1+t)(1+t) = \_\_ + \_\_\, t + \_\_\, t^2 + \_\_\, t^3 + \_\_\, t^4$$

We now just need to find the above coefficients of the various powers of t. Let's start with the constant term. In order to obtain a constant, we cannot select a

t from any of the factors. There is clearly just one way to do this, meaning the constant term is 1.

Now in order to get a t, we just need to select a t from exactly one of the four factors (and 1's from the three remaining factors). This is the number of ways to select one thing from four, i.e. $\binom{4}{1}$.

Similarly, how do we get a t^2? We need to select two of the factors to take t's from, and there are $\binom{4}{2}$ ways to do this. Continuing, there are $\binom{4}{3}$ ways to select t's from three of the four factors, and $\binom{4}{4}$ ways to select a t from every factor. Thus, we have

$$(1+t)^4 = \binom{4}{0} + \binom{4}{1}t + \binom{4}{2}t^2 + \binom{4}{3}t^3 + \binom{4}{4}t^4$$

This is a special case of what is known as the Binomial Theorem, which we'll state in full generality below. In high school algebra, these numbers are introduced as *Pascal's Triangle*, a triangular array of binomial coefficients.

Now we can state the Binomial Theorem. The proof of this theorem follows exactly the same reasoning as the above: The coefficient of t^k in the expansion of the polynomial $(1+t)^n$ is the number of ways of choosing k factors from n, which is $\binom{n}{k}$.

Theorem 3.9 (The Binomial Theorem) *For any $n \geq 0$,*

$$\begin{aligned}(1+t)^n &= \binom{n}{0} + \binom{n}{1}t + \binom{n}{2}t^2 + \cdots \binom{n}{n-1}t^{n-1} + \binom{n}{n}t^n \\ &= \sum_{k=0}^{n} \binom{n}{k} t^k.\end{aligned}$$

If you do not recall summation notation (i.e., the expression using the $\sum$), don't worry; it's only shorthand for the preceding expression.

The Binomial Theorem shows that $(1+t)^n$ is an example of a *generating function*, which is a polynomial whose coefficients have some combinatorial significance. Generating functions can sometimes make arguments surprisingly easy, which we will show with the following.

Recall that there's a unique set with no elements, called the *empty set*, and written $\emptyset$.

Lecture Six

Fact 3.10 *For any non-empty set, the number of subsets of even size equals the number of subsets of odd size.*

Before proving this fact, let's look at a few examples. If the set has only one element, then it has two subsets: itself, and the empty set. The empty set has size 0, which is even, and the whole set has size 1, which is odd.

Okay, that wasn't too interesting. Let's look at a set of size 4, say $\{a, b, c, d\}$. (It's easier if we name the elements, but obviously that won't affect our counting). There are eight subsets of even size: $\emptyset, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}$, and $\{a, b, c, d\}$, and eight subsets of odd size: $\{a\}, \{b\}, \{c\}, \{d\}, \{a, b, c\}, \{a, b, d\}$, $\{a, c, d\}$, and $\{b, c, d\}$. And, sure enough,

$$\binom{4}{0} + \binom{4}{2} + \binom{4}{4} = 1 + 6 + 1 = 8 = 4 + 4 = \binom{4}{1} + \binom{4}{3}.$$

On the outset, this seems like a difficult thing to prove in general. We need to show that, for any n,

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots$$

Note that the sides of the equation are not never-ending, it's just that we don't know where to write $\binom{n}{n}$, since which side this term is on depends on if n is even or odd.

Now let's bring everything in the above equation to the left hand side. Thus, we want to show that

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots + (-1)^n \binom{n}{n} = 0.$$

However, if we simply plug in $t = -1$ to the Binomial Theorem, we obtain:

$$(1 + (-1))^n = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots + (-1)^n \binom{n}{n}.$$

Since $(1 - 1)^n = 0^n = 0$, this proves the fact.

The following exercise is a bit more challenging, though of the same flavor.

Exercise 27 ($\star$) *Suppose you have a group of n people from which to choose a committee. How many ways are there to choose a committee, and then choose some subset of the people you've chosen to be "elite" members?*

For example, we may choose Alice, Bob, and Dave to be our committee, and then choose Alice and Dave to be the elite members. Note that either set is allowed to be empty: we could choose a committee of zero people (in which case *there would necessarily be zero elite members), and we could always choose zero elite members from any committee.*

Prove your answer in two ways: Using the Binomial Theorem, and using a straightforward counting argument.

Let's do another problem with the Binomial Theorem.

Question 3.11 *Among n people, how many ways are there to choose a committee and a chair of that committee?*

Let's take a straightforward approach to this problem, by just summing up the ways based on the size of the committee. If the committee has zero people, there are zero ways to do this (as a chair cannot be chosen). If the committee has 1 person, then that person must be the chair. If the committee has 2 people, there are 2 choices for the chair. In general, for a k-person committee, there are k choices for who can be chair. Thus, we want to count

$$0\binom{n}{0} + 1\binom{n}{1} + 2\binom{n}{2} + \cdots n\binom{n}{n}.$$

Looking at the statement of the Binomial Theorem, this looks like we want to "bring down" the exponent of x, and then plug in $x = 1$. Well, the operation that does that is the derivative. Indeed, if we differentiate the statement of the Binomial Theorem, we get

$$\begin{aligned} n(1+t)^{n-1} &= \binom{n}{1} + 2t\binom{n}{2} + 3t^2\binom{n}{3} + \cdots nt^{n-1}\binom{n}{n} \\ &= \sum_{k=1}^{n} kt^{k-1}\binom{n}{k}. \end{aligned}$$

Plugging in the value $t = 1$, we get

$$n2^{n-1} = 1\binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \cdots + n\binom{n}{n},$$

meaning the answer to our question is $n2^{n-1}$.

However, we should look at the simplicity of that answer and wonder if there's another way to obtain it. Indeed, if we choose the chair of the committee *first*, there are n ways to do that. Then there are 2^{n-1} possible subsets of the remaining $n-1$ people to form the non-chair members of the committee.

Lecture Seven

Many proofs with generating functions have a similar feel to the above; by plugging in the correct value, the result simply falls out of the expression! For example, we can come up with another proof for the following important fact:

Exercise 28 *Use the Binomial Theorem to prove that an n element set has 2^n subsets.*

In general, it's good practice to see problems that can be solved in several different ways, such as the following.

Question 3.12 *Suppose you have a box of 9 distinct chocolates. How many ways are there to distribute two of these chocolates among your friends Alice and Bob?*

To answer the above, we can consider three cases: Either Alice gets both chocolates, Bob gets both chocolates, or they get one chocolate each.

If one person gets both chocolates, the order of the chocolates selected clearly doesn't matter, and there are $\binom{9}{2} = 36$ ways to select 2 chocolates from 9. If Alice and Bob get one chocolate each, then the order of the selection matters (say, for instance, that you give the first chocolate to Alice and the second to Bob). The number of ways to select two things from 9 where order matters is $P(9,2) = 9 \times 8 = 72$. Summing up these cases, the answer to our question is:

$$\binom{9}{2} + \binom{9}{2} + P(9,2) = 144.$$

Let's look at another way of doing this problem: Imagine the chocolates as labeled 1 through 9, and now imagine that we've made two "copies" of each chocolate, the first labeled A and the second labeled B. Thus, our chocolates are labeled $1A, 1B, 2A, 2B$, and so on. If we select a chocolate labeled A we give it to Alice, and if we select one labeled B we give it to Bob. Then any way of selecting two chocolates from these 18 corresponds to a distribution of chocolates to Alice and Bob, except if we choose the same number. For example, selecting both $5A$ and $5B$ does not correspond to a distribution, since there's only one "real" chocolate labeled 5. There are $\binom{18}{2}$ ways to select two chocolates from 18, and 9 ways to select two that have the same numerical label. This gives

$$\binom{18}{2} - 9 = 153 - 9 = 144.$$

Let's practice our binomial coefficient calculations with a few more poker hands.

Question 3.13 *How many poker hands consist of exactly one pair?*

This question is a touch trickier than it may seem. We first need to choose the denomination of the pair, and there are 13 ways to do this. Then we need to choose the suits of the two cards of like denominations, and there are $\binom{4}{2} = 6$ ways to do this. Now we need to choose the other three cards. We don't want to choose the same denomination as the paired cards, and we also don't want another pair. Therefore, we need to choose three distinct denominations from the remaining 12, and there are $\binom{12}{3} = 220$ ways to do this. Now we need to choose the suits of these three cards. Imagine lining the cards up; there are 4 choices for the suit of each, and thus there are 4^3 ways to assign suits to all three cards. Thus, the number of poker hands with exactly one pair is

$$13 \times \binom{4}{2} \times \binom{12}{3} \times 4^3 = 1098240.$$

For the next exercise, recall that a *full house* is a hand where three of the cards are of one denomination, and the two remaining cards also form a pair.

Exercise 29 *How many poker hands are full houses, where the denomination of the pair is lower than the denomination of the three-of-a-kind? (Recall that we consider the Ace as the highest denomination.)*

The next example shows that a counting problem can sometimes be more easily solved when it's put in the right context.

First we need a definition: A *composition* of a positive integer n is a sum of the form $a_1 + a_2 + \ldots + a_k = n$, where each a_i is a positive integer. For example, 4 has eight compositions:

$$\begin{aligned} 1+1+1+1 &= 4 \\ 1+1+2 &= 4 \\ 1+2+1 &= 4 \\ 2+1+1 &= 4 \\ 1+3 &= 4 \\ 2+2 &= 4 \\ 3+1 &= 4 \\ 4 &= 4 \end{aligned}$$

Note that the order of the parts in a composition matters; e.g., $1+2+1$ is a different composition of 4 than $2+1+1$.

Question 3.14 *How many compositions of n into k parts are there? How many total compositions are there of n?*

This is a question that initially seems difficult, however it actually becomes pretty straightforward once we look at it correctly. Let's first address the case of compositions of 4. We use something colloquially called a *stars and bars* argument. We draw four stars:

$$\star\star\star\star$$

A composition of four then corresponds to a way of inserting dividers or *bars* into these. For example, the composition $1+2+1$ corresponds to

$$\star|\star\star|\star$$

whereas the composition $1+3$ corresponds to

$$\star|\star\star\star$$

For example, we can use this approach to count the number of compositions of 4 into three parts. We already listed these above, but we wish to find a formula we can generalize. To get a composition of 4 into three parts, we need to place two bars. Moreover, there are three possible places for these bars. Since the bars are indistinguishable, the order in which we select the bars' positions clearly does not matter. Thus, there are $\binom{3}{2} = 3$ compositions of 4 into three parts. Generalizing, we have

Fact 3.15 *The number of compositions of n into k parts is*

$$\binom{n-1}{k-1}$$

What about the total number of compositions of n? Well, that would be the total number of ways of placing any number of bars in the $n-1$ available spots. Thus, this is just the total number of subsets of the set of $n-1$ spots, so we have:

Fact 3.16 *The total number of compositions of n is 2^{n-1}.*

Note this works with our count of compositions of 4, since $2^{4-1} = 8$.

4 Choices with Repeats

Let's start with a problem that can be solved pretty easily using what we've already done. Suppose a donut shop stocks ten kinds of donuts. How many ways are there to select three distinct donuts? This is immediate, since the order of donuts doesn't matter, there are $\binom{10}{3} = 120$ such ways.

What if we just want to select any three donuts, where donuts of the same type are indistinguishable? Then we need to be a touch more careful. There are a few cases to examine. The first case is that all three donuts are distinct, and we've already handled this case. Next, we handle the case where two donuts are alike, and the third is different. There are ten ways to select the "doubled" donut, and nine ways to select the third donut. Finally, the last case is that all three donuts are the same type, and there are obviously ten ways to make such a selection. So, the total number of ways to select any three donuts is:

$$\binom{10}{3} + 10 \times 9 + 10 = 220.$$

It is easy to see how complicated this case-by-case analysis can get if we increase the number of donuts taken. So we need a more methodical way to tackle problems like this. Before doing this, let's look at a related problem.

A *monic monomial* (or simply *monomial*, for our purposes) is a product of variables, each to some positive exponent. For example, x^2yz^4 and $w^2y^3z^6$ are monic monomials. A natural question is the following:

Question 4.1 *How many monic monomials are there of degree k, using at most n variables?*

For example, there are twenty degree 3 monomials in 4 variables: $w^3, w^2x, w^2y, w^2z, wx^2, wxy, wxz, wy^2, wyz, wz^2, x^3, x^2y, x^2z, xy^2, xyz, xz^2, y^3, y^2z, yz^2$, and z^3.

If we think about this for a second, this is really just the same question as with the donuts: Imagine each variable is a different type of donut. Then a monomial of degree k corresponds to a way of selecting k variables, or donuts. So, our general question can be phrased as the following:

Question 4.2 *How many ways are there to select k objects from an unlimited supply of objects of n types, where objects of the same type are indistinguishable?*

Believe it or not, a stars and bars argument works here as well. However, we need to modify the argument a bit. Let's look at a more manageable example with the donuts. Say there are four possible donut types: Anise, Bourbon, Crawdad, and Durian, and we wish to select five donuts. One such selection may look like, for example, $AABDD$ (after all, who in their right mind would select a crawdad donut?!). How might this correspond to a stars and bars argument? Well, let's replace the A's with stars: $\star\star BDD$. Since these come first, we know they must correspond to A's. We next want to replace the B with a star, but we don't want that star to be confused for an A, so we put a bar: $\star\star\,|\,\star DD$. We now want to replace the D's with stars, but there's a slight problem: After putting a bar, how will we know the stars correspond to D's and not C's? The answer is that we also consider the number of C's, namely zero, by putting one bar down and then another bar immediately after it. After replacing the D's with stars as well, we get

$$\star\star\,|\,\star\,||\,\star\star$$

Let's try this process on another selection. If we wish to convert $BBDDD$ to a word of stars and bars, we get:

$$|\star\star||\star\star\star$$

To review, since there are zero A's, we begin the word with a bar. We then put two consecutive stars for the two B's, a bar to pass us to the C's, zero stars (as there are zero C's), another bar to pass us to the D's, and finally three stars for the three D's.

So, the answer to question 4.2 is the same as the number of "words" using k stars and $n-1$ bars. How many such words are there? Well, we can think of such a word as filling $n+k-1$ blanks. Once we select which blanks contain the stars, the word is determined. So, how many ways can we select spots for the k stars? It's simply the number of ways to select k places from $n+k-1$ blanks, which is

$$\binom{n+k-1}{k}$$

Before summing this up in a fact, we need another definition. If A is a set of distinct elements, a *multisubset* of A is a set of elements of A in which elements may be selected multiple times. For example, $\{a, a, b, b, b, c, e\}$ is a submultiset of $\{a, b, c, d, e\}$. In particular, note that submultisets are allowed to be larger than the original set. All our above cases are really just computing the number of submultisets of a certain size from a given set. For example, the number of possible selections of five donuts from four types is the number of submultisets of the set $\{A, B, C, D\}$.

Put together with our knowledge of binomial coefficients, this makes the following fact.

Fact 4.3 *Let X be a set of n distinct elements. Then the number of subsets of X of size k is*

$$\binom{n}{k}$$

and the number of submultisets of X of size k is

$$\binom{n+k-1}{k}.$$

Lecture Eight

In the example that led off this section, we wanted to know how many ways there are to choose three donuts from 10 available types. If the types are not allowed to be repeated (as we first considered), then there are $\binom{10}{3}$ ways. But if repetition is allowed, the number of possible selections is

$$\binom{10+3-1}{3} = \binom{12}{3} = 220,$$

an answer we found earlier using ad hoc methods.

We also asked how many monomials there are of degree 3, using at most four variables. Well this corresponds to a way of selecting three variables, with repetition, from the available four. So the number of such monomials is:

$$\binom{4+3-1}{3} = 20.$$

Now let's recall our illustrating example that allowed us to formulate our count of multisets, namely the case of selecting an order of donuts from among four types, A, B, C, and D. We now know how to count the number of orders of any number of donuts. For example, we know there are

$$\binom{4+5-1}{5} = 56$$

orders of five donuts. Let's make the problem a bit more complicated, though.

Question 4.4 *How many ways are there to select seven donuts from four types (A, B, C, and D)? What if you want at least one donut of type A? What if you want at least two donuts of type A and one of type D?*

The answer to the first part is immediate from our main result of this section:

$$\binom{4+7-1}{7} = \binom{10}{7} = 120.$$

We can get an answer for the next part by subtracting those which do not include any donuts of type A. Such a selection corresponds to selecting seven donuts from the remaining three types, and there are

$$\binom{3+7-1}{7} = \binom{9}{7} = 36$$

ways to do this, meaning the answer to the second part is $120 - 36 = 84$.

However, it is clear that we need to modify our approach for the third part, since it would be really messy to count up all the selections that fail to have two donuts of type A *or* fail to have one donut of type D. So let's think about this problem differently: Suppose we've already chosen the two donuts of type A and the one donut of type D. Then we have four donuts left to choose, and four types, so there are

$$\binom{4+4-1}{4} = \binom{7}{4} = 35$$

such selections!

In fact, the second part could've been solved this way as well: If we select our one required donut of type A, then there are six more donuts to select from the four types, and there are

$$\binom{4+6-1}{6} = \binom{9}{6} = 84$$

ways to do this.

Exercise 30 *How many degree-9 monomials are there in the variables v, w, x, y, z that are divisible by x^2yz^3? For example, $vx^3y^2z^3$ would be such a monomial.*

Exercise 31 *Suppose there are seven available types of bagels. How many ways are there to order a dozen bagels, where there are no more than 4 sesame bagels?*

Lecture Nine

5 Generating Functions

Let's look at a simple problem, put in terms of the Binomial Theorem. Suppose there are three distinguishable types of donuts. How many ways are there to choose two different donuts? Without using the Binomial Theorem, the answer is pretty straightforward: $\binom{3}{2}$. However, using the Binomial Theorem, we can view this problem as finding the coefficient of t^2 in the product

$$(1+t)(1+t)(1+t) = \binom{3}{0} + \binom{3}{1}t + \binom{3}{2}t^2 + \binom{3}{3}t^3 = 1 + 3t + 3t^2 + t^3$$

Each parenthesized term corresponds to a type of donut; the 1 corresponds to not choosing that type, and the t corresponds to choosing it.

In this way, we could write

$$\begin{aligned}(1+t)(1+t)(1+t) = {} & (\text{number of ways to choose 0 donuts}) \\ & + (\text{number of ways to choose 1 donuts})t \\ & + (\text{number of ways to choose 2 donuts})t^2 \\ & + (\text{number of ways to choose 3 donuts})t^3\end{aligned}$$

So, what's the benefit of this approach? Well, loosely speaking, the polynomial multiplication does the work for us. We don't need to do any reasoning; we just look at the coefficient of the appropriate power of t. For example, let's alter the above problem as follows: Suppose that we're allowed to choose zero, one, or two donuts of the first type. So now our polynomials become

$$(1+t+t^2)(1+t)(1+t),$$

where selecting the t^2 corresponds to choosing two donuts of the first type. Multiplying this out, we obtain:

$$1 + 3t + 4t^2 + 3t^3 + t^4.$$

Combinatorially, we could figure out that there are three ways to choose three donuts: We could choose one donut of each type, two donuts of the first type and one of the second, or two donuts of the first type and one of the third. However, as before, the polynomials do the reasoning for us; we simply look at the coefficient of t^3.

This is a special case of a *generating function*, defined as follows.

Definition 5.1 *Suppose $A = \langle a_0, a_1, a_2, \ldots \rangle$ is a sequence. The* generating function *of A is the power series*

$$G_A(t) = a_0 + a_1 t + a_2 t^2 + \cdots$$

Note that if all but finitely many a_i's are zero, then $G_A(t)$ is a polynomial. However this is not required in the definition.

For example, the generating function for the sequence $A = \langle 1, 3, 0, 5, 8, 0, 0, \ldots \rangle$ is given by

$$G_A(t) = 1 + 3t + 5t^3 + 8t^4.$$

Note that the sequence A is not required to be one of any combinatorial significance. However, the usefulness of generating functions is given by the following result.

Observation 5.2 *Let $A = \langle a_0, a_1, \ldots \rangle$ and $B = \langle b_0, b_1, \ldots \rangle$ be two sequences, and suppose $C = \langle c_0, c_1, \ldots \rangle$ is a sequence such that, for all n,*

$$c_n = a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \cdots a_{n-1} b_1 + a_n b_0.$$

Then $G_C(t) = G_A(t) G_B(t)$.

Really what this Observation is saying is that if $G_A(t)$ is the generating function for performing one task and $G_B(t)$ is the generating function for performing another task, then $G_A(t)G_B(t)$ is the generating function for performing *both* tasks.

For example, $(1 + t + t^2)$ is the generating function for selecting zero, one, or two donuts of type one, and $(1 + t)$ is the generating function for selecting zero or one donut of type two, so

$$(1 + t + t^2)(1 + t) = 1 + 2t + 2t^2 + t^3$$

is the generating function of choosing donuts of both types. This says that there's one way to choose no donuts, two ways to choose one, two ways to choose two, and one way to choose three. Again, we could've obtained these results without generating functions, but the language of generating functions allows us to compute it directly, without counting up cases.

Let's do another example. Suppose we're allowed to select zero, one, two, or three donuts of each of two types, then an even number of the third type, not exceeding six. The generating function is then

$$(1 + t + t^2 + t^3)^2 (1 + t^2 + t^4 + t^6)$$

$$= 1 + 2t + 4t^2 + 6t^3 + 7t^4 + 8t^5 + 8t^6 + 8t^7 + 7t^8 + 6t^9 + 4t^{10} + 2t^{11} + t^{12}$$

It's occasionally a good idea to check a couple of the coefficients when doing computations such as this one. For example, the above suggests that there are 4 ways to select ten donuts. Indeed, we could select four donuts of the third type, and then we'd need to select three donuts each of the first two types. Otherwise we could select six donuts of the third type, and then there are three ways to select four donuts from types one and two: one of type one and three of type two, three of type one and one of type two, or two each of types one and two.

Multiplying out the polynomials here is still a chore, but note that this is something a computer algebra system can do. That is, it's a way of *encoding* the combinatorics in simple multiplication that can be easily relegated to computation.

Exercise 32 *Write the generating function for selecting one, three, or five donuts of type one, and zero, two, or three donuts of type two.*

As we've seen, generating functions can be somewhat annoying to multiply out. Fortunately, there are many techniques to help us deal with polynomials.

First note that $(1-t)(1+t+t^2+\cdots+t^n) = 1-t^{n+1}$. Dividing both sides by $(1-t)$ gives us

$$1+t+t^2+\cdots+t^n = \frac{1-t^{n+1}}{1-t}$$

If you've had Calculus II you've probably seen the following, known as the *geometric series identity*:

$$1+t+t^2+\cdots = \frac{1}{1-t}$$

The actual mathematical definition of power series (i.e., infinite polynomials) is a little subtle. Luckily, we don't need to worry about this for our purposes; we can just take identities like the above as a given.

Now let's look at how the geometric series identity can be expanded upon. Consider the following expression:

$$\frac{1}{(1-t)^n} = \underbrace{(1+t+t^2+\cdots)(1+t+t^2+\cdots)\cdots(1+t+t^2+\cdots)}_{n \text{ factors}}$$

Let's ask what the coefficient is of t^k? Well, we need to pick k t's from among the n different factors, and t's picked from the same factor are indistinguishable. So, the coefficient of t^k is just the number of ways to select k things from n different types, namely

$$\binom{n+k-1}{k}.$$

Thus, we have the following identity:

$$\begin{aligned}\frac{1}{(1-t)^n} &= \binom{n+0-1}{0} + \binom{n+1-1}{1}t + \binom{n+2-1}{2}t^2 + \cdots \\ &= \binom{n-1}{0} + \binom{n}{1}t + \binom{n+1}{2}t^2 + \cdots\end{aligned}$$

We collect these identities along with the Binomial Theorem in the following list.

Observation 5.3

$$(1+t)^n = \binom{n}{0} + \binom{n}{1}t + \binom{n}{2}t^2 + \cdots + \binom{n}{n}t^n$$

$$\frac{1-t^{n+1}}{1-t} = 1 + t + t^2 + \cdots + t^n$$

$$\frac{1}{1-t} = 1 + t + t^2 + t^3 + \cdots$$

$$\frac{1}{(1-t)^n} = \binom{n-1}{0} + \binom{n}{1}t + \binom{n+1}{2}t^2 + \cdots$$

Let's use generating functions to solve another counting problem.

Question 5.4 *How many ways are there to select ten donuts from seven types, if we must select either one, five, or six donuts of the seventh type?*

To answer this question we need to find the coefficient of x^{10} in the following generating function:

$$(1 + t + t^2 + t^3 + \cdots)^6(t + t^5 + t^6) = \frac{1}{(1-t)^6}(t + t^5 + t^6).$$

How can we get a t^{10} from the above product? Let's name the two factors above; set $f(t) = \frac{1}{(1-t)^6}$ and $g(t) = (t + t^5 + t^6)$. We can get t^{10} in three ways: by multiplying the t^9 term from $f(t)$ by the t from $g(t)$, by multiplying the t^5 term from $f(t)$ by the t^5 from $g(t)$, or by multiplying the t^4 term from $f(t)$ by the t^6 from $g(t)$. By Observation 5.3, these give the coefficient of t^{10} in the product as

$$\binom{6+8}{9} + \binom{6+4}{5} + \binom{6+3}{4} = 2380.$$

Of course, the solution to this problem also admits a combinatorial interpretation as follows: The first binomial coefficient is the number of ways to select nine donuts from six types, so this corresponds to the case in which we select one donut of the seventh, the second binomial coefficient corresponds to the case in which we select five donuts from the first six types and five from the seventh, and the last binomial coefficient corresponds to the case in which we select four donuts from the first six types and six from the seventh.

Exercise 33 *Redo Question 5.4, but for the case where* 12 *donuts are selected. (Do this exercise in the same way that we did the problem; don't simply multiply out the generating function!)*

Lecture Ten

The following problem can also be done combinatorially, though the approach is a lot more nuanced than in the last problem.

Question 5.5 *How many ways are there to select 25 fritters from four types, with between three and eight of each type?*

Our generating function here is a bit more complicated:

$$(t^3 + t^4 + \cdots + t^8)^4 = t^{12}(1 + t + \cdots + t^5)^4 = t^{12}(1 - t^6)^4 \frac{1}{(1-t)^4}.$$

Let $f(t) = (1 - t^6)^4$ and $g(t) = \frac{1}{(1-t)^4}$. We want to find the coefficient of t^{25} in the above, which amounts to finding the coefficient of t^{13} in the product $f(t)g(t)$. The only way we can get a t^{13} as a product of terms in $f(t)$ and $g(t)$ is as $1 \cdot t^{13}, t^6 \cdot t^7$, or $t^{12} \cdot t$. If we plug $-t^6$ into the Binomial Theorem, we obtain:

$$(1 - t^6)^4 = \binom{4}{0} - \binom{4}{1}t^6 + \binom{4}{2}t^{12} - \binom{4}{3}t^{18} + \binom{4}{4}t^{24}.$$

Thus, the coefficient of t^{13} in $f(t)g(t)$ (which is the coefficient of t^{25} in the full generating function), is

$$\binom{4}{0}\binom{4+12}{13} - \binom{4}{1}\binom{4+6}{7} + \binom{4}{2}\binom{4}{1} = 104.$$

Exercise 34 *How many ways are there to select twenty bialys from six types, if we have to choose between one and five bialys of the first five types, and either one or three bialys of the sixth type?*

Exercise 35 *How many ways are there to split seven cherry cordials, eight caramels, and thirteen toffees between two people if each person gets fourteen candies, and at least two of each candy type?*

Generating functions can also be helpful in proving certain combinatorial identities. For example, we show the following:

Fact 5.6 *For any n, we have the following identity:*

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \cdots \binom{n}{n}^2 = \binom{2n}{n}$$

To prove this fact, first use the symmetry $\binom{n}{k} = \binom{n}{n-k}$ of binomial coefficients, to transform the left-hand side of the above equation into:

$$\binom{n}{0}\binom{n}{n} + \binom{n}{1}\binom{n}{n-1} + \binom{n}{2}\binom{n}{n-2} + \cdots + \binom{n}{n}\binom{n}{0}.$$

But this is clearly the coefficient of t^n in the product $(1+t)^n(1+t)^n = (1+t)^{2n}$. The Binomial Theorem then gives the coefficient of t^n in $(1+t)^{2n}$ as $\binom{2n}{n}$, which proves the fact.

Exercise 36 ($\star$) *Come up with a combinatorial proof of the above fact.*

Generating functions are very helpful when dealing with integer partitions, defined as follows:

Definition 5.7 *An* integer partition *of n is a sum $a_1 + a_2 + \cdots + a_k = n$ of positive integers with $a_1 \geq a_2 \geq \cdots a_k$. The numbers a_i are called the* parts *of the partition.*

For example, the integer partitions of 4 are $4, 3+1, 2+2, 2+1+1$, and $1+1+1+1$. Note that the integer partitions of n shouldn't be confused with the *compositions* of n, in which order matters. We can think of the partitions of n as being the compositions once we drop the assumption that order matters (and so we just order the entries to be weekly decreasing).

Unlike with the case of compositions, the partitions of n don't have a nice clean formula to enumerate them. So, it's often more convenient to work with the generating function for all partitions, or for partitions satisfying certain constraints.

For each n, let d_n be the number of partitions of n into distinct parts. For example, 4 has two partitions using distinct parts: 4 and $3+1$, so $d_4 = 2$.

Question 5.8 *How do we find the generating function*

$$D(t) = d_0 + d_1 t + d_2 t^2 + d_3 t^3 + \cdots$$

for the partitions into distinct parts?

The arithmetic of generating functions allows us to construct $D(t)$ in an interesting way; rather than focus on a particular d_n, we focus on the possible choices for the values of the parts. That is, we can either choose zero parts equal to 1, or one part equal to 1. We can't choose any more than one 1, since we're focusing on distinct partitions. Thus the part of $D(t)$ corresponding to choosing the 1's is $(1+t)$. Similarly, we can choose zero 2's or one 2, which corresponds to the binomial $(1+t^2)$. Continuing like this, we have

$$D(t) = (1+t)(1+t^2)(1+t^3)(1+t^4)\cdots$$

This is a bit weird, since we haven't really defined what an infinite product is. For example, what is the coefficient of t in the following infinite product?!

$$(1+t)(1+t)(1+t)(1+t)\cdots$$

Fact 5.9 *We say that an infinite product is well-defined if, for any $n > 0$, only finitely many parenthesized terms of the product contain terms of the form t^k for $0 < k \leq n$.*

For example, the formula for $D(t)$ is a well-defined infinite product, as only the first n binomials can contain terms of the form t^k for $0 < k \leq n$.

Lecture Eleven

Let's adapt our calculation for $D(t)$ to come up with the generating function for all partitions. We think about it in the same way: We can choose zero 1's, one 1, two 1's, and so on. This corresponds to the infinite series

$$1 + t + t^2 + t^3 + \cdots .$$

Similarly, we can choose zero 2's, one 2, two 2's, and so on, and this gives us the series

$$1 + t^2 + t^4 + t^6 + \cdots .$$

Continuing this for all possible part values, we have the following.

Theorem 5.10 *The generating function for all integer partitions is given by*

$$P(t) = (1+t+t^2+\cdots)(1+t^2+t^4+\cdots)(1+t^3+t^6+\cdots)(1+t^4+t^8+\cdots)\cdots$$

Using the identity $1 + t^k + t^{2k} + t^{3k} + \cdots = \frac{1}{1-t^k}$*, we get the following alternate expression for* $P(t)$*:*

$$P(t) = \frac{1}{(1-t)(1-t^2)(1-t^3)(1-t^4)\cdots}.$$

Exercise 37 *For each* n*, let* de_n *denote the number of partitions of* n *into a distinct parts, all of which are even. For example,* $de_6 = 2$*, since* 6 *has two partitions into distinct even parts:* 6 *and* $4 + 2$*. Find an expression for the generating function* $DE(t) = de_0 + de_1 t + de_2 t^2 + \cdots$*.*

We can also use generating functions to prove a basic fact about binary representations of integers.

Theorem 5.11 *For any* n*, there is a unique way to write* n *as a sum of distinct powers of* 2*.*

Another way to say what we want to prove is that every n has exactly one partition into distinct powers of 2. The generating function for partitions into distinct powers of 2 is

$$F(t) = (1 + t)(1 + t^2)(1 + t^4)(1 + t^8)\cdots .$$

To prove the theorem, we just want to show that

$$F(t) = 1 + t + t^2 + t^3 + \cdots = \frac{1}{1-t},$$

or equivalently that $(1-t)F(t) = 1$.

Indeed, we just multiply the expression $(1-t)F(t)$ term by term to obtain the following:

$$
\begin{aligned}
(1-t)F(t) &= (1-t)(1+t)(1+t^2)(1+t^4)(1+t^8)\cdots \\
&= (1-t^2)(1+t^2)(1+t^4)(1+t^8)(1+t^{16}) \\
&= (1-t^4)(1+t^4)(1+t^8)(1+t^{16})(1+t^{32}) \\
&\ \ \vdots \\
&= 1.
\end{aligned}
$$

Exercise 38 *Show using generating functions that every n has a unique representation*

$$n = c_0 3^0 + c_1 3^1 + c_2 3^2 + \cdots + c_k 3^k$$

where each c_i is $0, 1,$ or 2.

We can also prove a surprising theorem about partitions from analyzing the associated generating functions:

Theorem 5.12 *For any n, the number of partitions of n into only odd parts is equal to the number of partitions of n into distinct parts.*

For example, 6 has four partitions into odd parts, namely $5+1, 3+3, 3+1+1+1$, and $1+1+1+1+1+1$, and four partitions into distinct parts, $6, 5+1, 4+2$, and $3+2+1$.

So how do we prove the above? Well we've already found the generating function for partitions into distinct parts to be

$$D(t) = (1+t)(1+t^2)(1+t^3)(1+t^4)\cdots$$

so we'd like to find the generating function for partitions into odd parts. If we can then show that these two generating functions are equal, that will prove the theorem.

The generating function $O(t)$ for partitions into odd parts is fairly straightforward, given what we've done so far. All we need to do is take $P(t)$ and remove the factors corresponding to even parts:

$$O(t) = \frac{1}{(1-t)(1-t^3)(1-t^5)(1-t^7)\cdots}$$

Using the difference of squares, we have, for all k,

$$(1+t^k) = \frac{(1-t^{2k})}{(1-t^k)}.$$

We plug this in to get an alternate expression for $D(t)$:

$$D(t) = \frac{(1-t^2)(1-t^4)(1-t^6)(1-t^8)(1-t^{10})\cdots}{(1-t)(1-t^2)(1-t^3)(1-t^4)(1-t^5)\cdots}.$$

Since every parenthesized term from the top cancels with one on the bottom, we get

$$D(t) = \frac{1}{(1-t)(1-t^3)(1-t^5)(1-t^7)(1-t^9)\cdots} = O(t),$$

which proves the theorem!