Throughout, let G be a bipartite graph with vertex partition $X \sqcup Y$. Recall that this means every edge of G has one endpoint in X and one in Y. For a vertex v, let $N(v)$ denote the vertices adjacent to v. For a subset $A \subseteq X$, let $N(A) = \bigcup_{v \in A} N(v)$.

Recall that a *matching* is a set of disjoint edges. A matching M on G is *complete* if every vertex of X is contained in some edge of M.

If some $A \subseteq X$ satisfies $|A| > |N(A)|$, then it is obvious no complete matching on G can exist. Thus the condition $|A| \leq |N(A)|$ for all $A \subseteq X$ is a necessary condition for G to possess a complete matching. Hall's Marriage Theorem says that this condition is also sufficient.

Theorem 0.1 (Hall's Marriage Theorem)**.** *Let G be as above, and suppose $|A| \leq |N(A)|$ for all $A \subseteq X$. Then there exists a complete matching on G.*

Proof. Let M be a matching on G, and suppose M is not complete. We show that we can create a matching M' with one more edge than M.

Color the edges belonging to M red, and color all other edges blue. Since M is not complete, let $x_0 \in X$ be a vertex not contained in an edge of M. We want to find a path to x_0 from some $y \in Y$ such that the edges of this path alternate in color, and such that y is not contained in an edge of M.

By the hypotheses, there must be some $y_1 \in Y$ adjacent to x_0. If y_1 is not incident to a red edge, we're done. Otherwise, let (x_1, y_1) be a red edge. Again by the hypotheses, $|N(\{x_0, x_1\})| \geq 2$, so let $y_2 \neq y_1$ be a vertex in $N(\{x_0, x_1\})$. Note that y_2 must be connected to some x_i for $i = 0$ or 1.

We continue this process: If y_i is not incident to a red edge, we stop. Otherwise, we let x_i be the element of X such that (x_i, y_i) is a red edge, and we define y_{i+1} to be an element of $N(\{x_0, x_1, \dots, x_i\}) \setminus \{y_1, y_2, \dots, y_i\}$, meaning y_{i+1} is adjacent to some x_j for $j \leq i$.

As G is finite, this process must eventually terminate with some y_t that is not the endpoint of a red edge. For each y_i, let $\alpha(i) < i$ be an index such that $(x_{\alpha(i)}, y_i)$ is an edge. Note that this edge must be blue, since $x_{\alpha(i)}$ is already contained in the red edge $(x_{\alpha(i)}, y_{\alpha(i)})$.

Now our alternating path is given by the edges

$$(x_{\alpha(t)}, y_t), (x_{\alpha(t)}, y_{\alpha(t)}), (x_{\alpha^2(t)}, y_{\alpha(t)}), (x_{\alpha^3(t)}, y_{\alpha^2(t)}) \dots (x_0, y_m),$$

where $\alpha(m) = 0$.

This path starts at x_0 and ends at y_t, alternating blue and red edges. Moreover, neither x_0 nor y_t is incident to a red edge.

Construct a new matching M' as follows: change the red edges of this path to blue, and the blue edges of this path to red. Let M' be the set of all red edges after this procedure. It is easy to see that M' is still a matching. Since our path was of odd length, we have $|M'| = |M| + 1$. □

In the accompanying animated gif, we have $t = 5$, and the values of α are given by $\alpha(2) = 0, \alpha(3) = \alpha(4) = 2$, and $\alpha(5) = 3$.