Constr. Approx. (2006) 23: 103–120
DOI: 10.1007/s00365-005-0595-8

CONSTRUCTIVE APPROXIMATION

Weighted Energy Problem on the Unit Circle

Igor E. Pritsker

Abstract. We solve the weighted energy problem on the unit circle by finding the extremal measure and describing its support. Applications to polynomial and exponential weights are also included.

1. Introduction and Main Results

Let $w \not\equiv 0$ be a continuous nonnegative function on the unit circle $\mathbb{T} := \{z : |z| = 1\}$, and set

$$Q(z) := -\log w(z). \tag{1.1}$$

Let $\mathcal{M}(\mathbb{T})$ be the space of positive unit Borel measures supported on $\mathbb{T}$. For any measure $\mu \in \mathcal{M}(\mathbb{T})$, we define the energy functional

$$\begin{aligned} I_w(\mu) &:= \iint \log \frac{1}{|z-t|w(z)w(t)}\, d\mu(z)\, d\mu(t) \\ &= \iint \log \frac{1}{|z-t|}\, d\mu(z)\, d\mu(t) + 2\int Q(t)\, d\mu(t), \end{aligned} \tag{1.2}$$

and consider the minimum energy problem

$$V_w := \inf_{\mu \in \mathcal{M}(\mathbb{T})} I_w(\mu). \tag{1.3}$$

For a general reference on potential theory with external fields, or weighted potential theory, one should consult the book of Saff and Totik [14]. It follows from Theorem I.1.3 of [14] that V_w is finite, and that there exists a unique equilibrium measure $\mu_w \in \mathcal{M}(\mathbb{T})$ such that $I_w(\mu_w) = V_w$. Thus μ_w minimizes the energy functional (1.2) in the presence of the external field Q generated by the weight w. Furthermore, we have for the potential of μ_w that

$$U^{\mu_w}(z) + Q(z) \ge F_w, \qquad z \in \mathbb{T}, \tag{1.4}$$

and

$$U^{\mu_w}(z) + Q(z) = F_w, \qquad z \in S_w, \tag{1.5}$$

Date received: April 2, 2004. Date revised: November 20, 2004. Date accepted: January 28, 2005. Communicated by Vilmos Totik. Online publication: June 3, 2005.
AMS classification: Primary 31A15; Secondary 30C85, 45E05.
Key words and phrases: Weighted capacity, Potentials, External fields, Equilibrium measure.

where $U^{\mu_w}(z) := -\int \log|z-t|\,d\mu_w(t)$, $F_w := V_w - \int Q(t)\,d\mu_w(t)$, and $S_w :=$ supp μ_w (see Theorems I.1.3 and I.5.1 in [14]). The weighted capacity of $\mathbb{T}$ is defined by

$$\text{cap}(\mathbb{T}, w) := e^{-V_w}. \tag{1.6}$$

If $w \equiv 1$ on $\mathbb{T}$, then we obtain the classical logarithmic capacity $\text{cap}(\mathbb{T}) = 1$, and the equilibrium measure $dt/(2\pi)$, $e^{it} \in \mathbb{T}$.

The energy problems with external fields on subsets of the real line were treated in many papers, see [14] for a survey and references. The purpose of this paper is twofold: we provide a general solution to the weighted energy problem on the unit circle, and we also simplify the previously known arguments used in the real line case. Our method applies on the real line too, which leads to shorter proofs and generalizations of the results in [3] and [14].

We give below an explicit form of the equilibrium measure and describe its support for the weighted energy problem on $\mathbb{T}$. Throughout the paper, we use the notation $Q(t) := Q(e^{it})$.

Theorem 1.1. *Suppose that $Q \in C^{1+\varepsilon}(U)$, where U is an open neighborhood of S_w in $\mathbb{T}$. Then $d\mu_w(e^{i\theta}) = f(\theta)\,d\theta$, where $f \in L_\infty([0, 2\pi))$. Furthermore, the density $f(\theta)$ satisfies the equation*

$$f^2(\theta) = \left(\frac{Q'(\theta)}{\pi}\right)^2 - \frac{1}{\pi^2}\int_0^{2\pi} Q'(t) f(t) \cot\frac{\theta - t}{2}\,dt + \frac{1}{4\pi^2} \tag{1.7}$$

for a.e. $e^{i\theta} \in S_w$, where the integral in (1.7) *is understood in the principal value sense.*

Corollary 1.2. *Theorem* 1.1 *also holds with* (1.7) *replaced by*

$$f^2(\theta) = \frac{1}{\pi^2}\int_0^{2\pi} (Q'(\theta) - Q'(t)) f(t) \cot\frac{\theta - t}{2}\,dt - \left(\frac{Q'(\theta)}{\pi}\right)^2 + \frac{1}{4\pi^2} \tag{1.8}$$

for a.e. $e^{i\theta} \in S_w$.

Let $p(\theta)$ be the right-hand side of (1.8). *If $Q \in C^2(U)$, then $p \in C(U)$ and $f := \sqrt{p} \in C(S_w)$. Furthermore, S_w is the closure in $\mathbb{T}$ of the open set $\{e^{i\theta} \in \mathbb{T} : p(\theta) > 0\}$. Hence $f(\theta)$ vanishes at the endpoints of S_w.*

Moreover, if Q is real analytic on U, then S_w is a finite union of closed arcs of $\mathbb{T}$.

Corollary 1.3. *If $S_w = \mathbb{T}$ under the assumptions of Theorem* 1.1, *then*

$$f(\theta) = \frac{1}{2\pi} - \frac{1}{2\pi^2}\int_0^{2\pi} Q'(t) \cot\frac{\theta - t}{2}\,dt, \qquad e^{i\theta} \in \mathbb{T}.$$

In particular, f is Hölder continuous on $\mathbb{T}$.

We mention another instance when the structure of S_w is clear, which parallels the real line case (see Theorem IV.1.10(b) of [14]).

Proposition 1.4. *If $Q(t)$ is convex on $(\alpha, \beta) \subset \mathbb{R}$, $\beta - \alpha \leq 2\pi$, then the intersection of S_w with the arc $(e^{i\alpha}, e^{i\beta}) \subset \mathbb{T}$ is either an arc or empty set.*

The support S_w plays a crucial role in determining the equilibrium measure μ_w itself, as well as other components of this weighted energy problem. Indeed, if S_w is known, then μ_w can be found as a solution of the singular integral equation

$$\int \log \frac{1}{|z-\zeta|}\, d\mu(\zeta) - \log w(z) = F, \qquad z \in S_w,$$

where F is a constant (see (1.5) and [14, Chap. IV]). Applying the results of [8] (see also [7]), we solve this integral equation and obtain the following theorem.

Theorem 1.5. *Let $Q \in C^2(U)$, where U is an open neighborhood of S_w in $\mathbb{T}$. Assume that S_w consists of K arcs $\Gamma_k \subset \mathbb{T}$, $K \geq 1$, with the endpoints $a_k = e^{i\alpha_k}$ and $b_k = e^{i\beta_k}$ such that $\alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \cdots < \alpha_K < \beta_K$, $\beta_K - \alpha_1 < 2\pi$. Set $R(z) := \prod_{k=1}^{K}(z-a_k)(z-b_k)$, and consider the branch of $\sqrt{R(z)}$ defined in the domain $\overline{\mathbb{C}}\backslash S_w$ by $\lim_{z\to\infty}\sqrt{R(z)}/z^K = 1$. By the values of $\sqrt{R(z)}$ on S_w, we understand the limiting values from inside the unit disk. Let*

$$F(z) := \frac{\sqrt{R(z)}}{\pi i}\int_{S_w} \frac{g(\zeta)\, d\zeta}{\sqrt{R(\zeta)}(\zeta - z)}, \tag{1.9}$$

where

$$g(e^{it}) := \frac{i}{\pi} Q'(t) + \frac{1}{2\pi}, \qquad e^{it} \in S_w,$$

and the integral in (1.9) *is the Cauchy principal value. Then the density f of μ_w is given by*

$$f(t) = F(e^{it}), \qquad e^{it} \in S_w. \tag{1.10}$$

Furthermore, the following equations are satisfied

$$\int_{S_w} \frac{z^k g(z)\, dz}{\sqrt{R(z)}} = 0, \qquad k = 0, \ldots, K-1, \tag{1.11}$$

$$\int_{S_w} f(t)\, dt = 1, \tag{1.12}$$

and

$$\int_{\beta_k}^{\alpha_{k+1}} F(e^{it})\, dt = \frac{\alpha_{k+1} - \beta_k}{2\pi} + \frac{Q(\alpha_{k+1}) - Q(\beta_k)}{\pi}\, i, \qquad k = 1, \ldots, K, \tag{1.13}$$

where we assume that $\alpha_{K+1} = \alpha_1 + 2\pi$.

Note that equations (1.11)–(1.13) may be used to find the endpoints of S_w.

2. Applications

Consider the weight function

$$w(z) := \prod_{j=1}^{J} |z - z_j|^{\lambda_j}, \tag{2.1}$$

where $\lambda_j > 0$, $z_j \in \mathbb{C}$, and $z_j \neq 0$, $j = 1, \ldots, J$. In some special cases, such weights on disks were previously treated in [13], with applications to the weighted polynomial approximation. In general, one can express the equilibrium measure μ_w for w of (2.1) as a linear combination of harmonic measures, which was done in [9] and [10]. The complete solution of the weighted energy problem on the unit circle given below was first found in [11], in connection with a number theoretic problem on heights of some subspaces of polynomials (see [2] for background).

Theorem 2.1. *For the weight w of* (2.1), *the support S_w consists of $K \leq J$ arcs. If $S_w = \mathbb{T}$, then we have that*

$$d\mu_w(e^{it}) = \frac{1}{2\pi}\left(1 + \sum_{j=1}^{J} \lambda_j - \sum_{j=1}^{J} \lambda_j \frac{||z_j|^2 - 1|}{|e^{it} - z_j|^2}\right) dt. \tag{2.2}$$

If $S_w \neq \mathbb{T}$, then we set

$$F_1(z) := \frac{\sqrt{R(z)}}{2\pi} \sum_{j=1}^{J} \lambda_j \left(\frac{z_j}{(z_j - z)\sqrt{R(z_j)}} + \frac{\bar{z}_j^{-1}}{(\bar{z}_j^{-1} - z)\sqrt{R(\bar{z}_j^{-1})}} \right),$$

where we use the notation of Theorem 1.5. *The equilibrium measure μ_w is given in this case by*

$$d\mu_w(e^{it}) = F_1(e^{it})\, dt, \qquad e^{it} \in S_w, \tag{2.3}$$

where the values of $\sqrt{R(z)}$ on S_w are the limiting values from inside the unit disk.

Furthermore, *the endpoints of S_w satisfy the equations*

$$\begin{aligned} \sum_{j=1}^{J} \lambda_j \left(\frac{z_j^k}{\sqrt{R(z_j)}} + \frac{\bar{z}_j^{-k}}{\sqrt{R(\bar{z}_j^{-1})}} \right) &= 0, \qquad k = 1, \ldots, K-1, \\ \sum_{j=1}^{J} \lambda_j \left(\frac{z_j^K}{\sqrt{R(z_j)}} + \frac{\bar{z}_j^{-K}}{\sqrt{R(\bar{z}_j^{-1})}} \right) &= 1 + \sum_{j=1}^{J} \lambda_j, \end{aligned} \tag{2.4}$$

and the equations

$$\int_{\beta_k}^{\alpha_{k+1}} F_1(e^{it})\, dt = 0, \qquad k = 1, \ldots, K. \tag{2.5}$$

Our second application is related to the exponential weights of the form

$$w(e^{i\theta}) = e^{-t_M(\theta)}, \qquad \text{where} \quad t_M(\theta) := \sum_{m=-M}^{M} c_m e^{im\theta} \tag{2.6}$$

is a real-valued trigonometric polynomial of degree M. A typical example of such a weight is given by $w(z) = |e^{-cz}|$, $c \in \mathbb{R}$, which was studied in [12] and [13] on disks and on the Szegő domain. The same weight appeared in a problem on the longest increasing subsequence of random permutations, see [1]. The general solution of the weighted energy problem on the unit circle is given below.

Theorem 2.2. *For the weight w of* (2.6), *the support S_w consists of $K \leq M$ arcs. If $S_w = \mathbb{T}$, then we have that*

$$d\mu_w(e^{it}) = \left(\frac{1}{2\pi} - \frac{1}{\pi}\sum_{m=-M}^{M} m\,\mathrm{sgn}(m) c_m e^{im\theta}\right)dt. \tag{2.7}$$

If $S_w \neq \mathbb{T}$, then we set

$$F_2(z) := \frac{\sqrt{R(z)}}{\pi}\left(\sum_{m=K}^{M} mc_m s_{m+1}(z) - \sum_{m=-M}^{-1} mc_m r_{-m-1}(z)\right),$$

in the notation of Theorem 1.5, *where*

$$\frac{1}{\sqrt{R(\zeta)}(\zeta - z)} = \sum_{k=K+1}^{\infty} \frac{s_k(z)}{\zeta^k} \quad \text{and} \quad \frac{1}{\sqrt{R(\zeta)}(\zeta - z)} = \sum_{k=0}^{\infty} r_k(z)\zeta^k,$$

respectively, near ∞ and near 0. *The measure μ_w is given by*

$$d\mu_w(e^{it}) = F_2(e^{it})\,dt, \qquad e^{it} \in S_w, \tag{2.8}$$

where the values of $\sqrt{R(z)}$ on S_w are the limiting values from inside the unit disk.

Furthermore, the endpoints of S_w satisfy (1.11) *with*

$$g(z) = \frac{1}{2\pi} - \frac{1}{\pi}\sum_{m=-M}^{M} mc_m z^m,$$

and (1.13) *with $F = F_2 + g$.*

Note that

$$r_k(z) = \frac{1}{k!}\frac{d^k}{d\zeta^k}\left(\frac{1}{\sqrt{R(\zeta)}(\zeta - z)}\right)\Bigg|_{\zeta=0}, \qquad k \geq 0,$$

and

$$s_k(z) = \frac{1}{k!}\frac{d^k}{d\zeta^k}\left(\frac{\zeta^{K+1}}{\sqrt{\zeta^{2K}R(1/\zeta)}(1 - z\zeta)}\right)\Bigg|_{\zeta=0}, \qquad k \geq 0.$$

Hence $s_k(z)$ is a polynomial in z of degree at most k, and $r_k(z)$ is a polynomial in $1/z$ of degree at most $k+1$. Also, it is clearly possible to evaluate the integrals in (1.11) (for this function g) by using the residues at 0 and ∞.

3. Proofs

The proof of Theorem 1.1 requires the following lemma:

Lemma 3.1. *The weighted equilibrium measure μ_w is absolutely continuous with respect to the arc length on $\mathbb{T}$, and*

$$d\mu_w(e^{it}) = f(t)\,dt, \qquad e^{it} \in \mathbb{T},$$

where $f \in L_\infty([0, 2\pi))$.

Proof. We shall show that U^{μ_w} is Lipschitz continuous in $\mathbb{C}$, which implies that the directional derivatives of U^{μ_w} exist a.e. on $\mathbb{T}$, and

$$d\mu_w(e^{it}) = -\frac{1}{2\pi}\left(\frac{\partial U^{\mu_w}}{\partial \mathbf{n}_-}(e^{it}) + \frac{\partial U^{\mu_w}}{\partial \mathbf{n}_+}(e^{it})\right) dt =: f(t)\,dt$$

for a.e. $e^{it} \in \mathbb{T}$, by Theorem II.1.5 of [14], where $\mathbf{n}_-$ and $\mathbf{n}_+$ are the outer and inner normals to $\mathbb{T}$. Clearly, the normal derivatives of U^{μ_w} are bounded by the Lipschitz constant, so that we obtain $f \in L_\infty([0, 2\pi))$.

Recall that $Q(t) = -\log w(e^{it})$ is a $C^{1+\varepsilon}$ function in an open neighborhood U of S_w in $\mathbb{T}$. We can modify $w(e^{it})$ so that for the resulting function $v(e^{it})$ we still have

$$U^{\mu_w}(z) - \log v(z) = F_w, \qquad z \in S_w,$$

and

$$U^{\mu_w}(z) - \log v(z) \geq F_w, \qquad z \in \mathbb{T},$$

and we also have $\log v(e^{it}) \in C^{1+\varepsilon}(\mathbb{T})$. Theorem I.3.3 of [14] then implies that $\mu_v = \mu_w$ and $F_v = F_w$. Thus we can work with v instead. Indeed, such modification is possible by (1.4) and (1.5), because S_w is a compact set contained in U. Hence we can find an open cover $O \subset U$ for S_w, consisting of finitely many open arcs. Then we set $v|_O = w|_O$, and modify w to v on $\mathbb{T}\setminus O$ (consisting of finitely many closed arcs) in such a way that (see (1.4))

$$U^{\mu_w}(z) - \log v(z) \geq F_w, \qquad z \in \mathbb{T}\setminus O,$$

and $\log v(e^{it}) \in C^{1+\varepsilon}(\mathbb{T})$.

Let u be a solution of the Dirichlet problem in the unit disk D for the boundary data $\log v(e^{it}) + F_w$. Then $u \in C^{1+\varepsilon}(\overline{D})$ by Privalov's theorem (see §5 of Chap. IX in [5]). Since $u|_{\mathbb{T}} \leq U^{\mu_w}|_{\mathbb{T}}$ by our construction, we obtain that

$$u(z) \leq U^{\mu_w}(z), \qquad z \in \overline{D},$$

as U^{μ_w} is superharmonic. In the proof of Lipschitz continuity of U^{μ_w}, we first consider $z \in S_w$ and $\zeta \in \mathbb{C}$, and follow an idea of Götz [6]. Since $u|_{S_w} = U^{\mu_w}|_{S_w}$, it is immediate that

$$U^{\mu_w}(z) - U^{\mu_w}(\zeta) \leq u(z) - u(\zeta) \leq C|z - \zeta|, \qquad z \in S_w, \quad \zeta \in \overline{D},$$

where C is the Lipschitz constant for u on $\overline{D}$. Note that $U^{\mu_w}(1/\bar{\zeta}) = U^{\mu_w}(\zeta) + \log|\zeta|,\ \zeta \neq 0$, because $S_w \subset \mathbb{T}$. Hence we have from the above estimate that

$$U^{\mu_w}(z) - U^{\mu_w}(\zeta) \leq (C+1)|z-\zeta|, \qquad z \in S_w, \quad \zeta \in \mathbb{C}. \tag{3.1}$$

In order to prove a matching estimate from below, we consider a nearest point $\zeta^* \in S_w$ for ζ, i.e., $\operatorname{dist}(\zeta, S_w) = |\zeta - \zeta^*| =: r$. Then

$$\begin{aligned} U^{\mu_w}(z) - U^{\mu_w}(\zeta) &= u(z) - u(\zeta^*) + U^{\mu_w}(\zeta^*) - U^{\mu_w}(\zeta) \\ &\geq -C|z-\zeta^*| + U^{\mu_w}(\zeta^*) - U^{\mu_w}(\zeta), \qquad z \in S_w, \quad \zeta \in \mathbb{C}. \end{aligned} \tag{3.2}$$

Using the area mean-value inequality, we obtain that

$$\begin{aligned} U^{\mu_w}(\zeta^*) &\geq \frac{1}{\pi(2r)^2} \int_{D_{2r}(\zeta^*)} U^{\mu_w}(x+iy)\,dx\,dy \\ &= \frac{1}{4\pi r^2} \int_{D_{2r}(\zeta^*)\setminus D_r(\zeta)} U^{\mu_w}(x+iy)dx\,dy + \tfrac{1}{4} U^{\mu_w}(\zeta), \end{aligned} \tag{3.3}$$

where the second term comes from the mean-value property for the harmonic function U^{μ_w} in $D_r(\zeta)$. Note that $U^{\mu_w}(\xi) \geq U^{\mu_w}(\zeta^*) - (C+1)|\zeta^* - \xi|,\ \xi \in \mathbb{C}$, by (3.1). Hence (3.3) implies that

$$U^{\mu_w}(\zeta^*) \geq \frac{4\pi r^2 - \pi r^2}{4\pi r^2}\big(U^{\mu_w}(\zeta^*) - (C+1)2r\big) + \tfrac{1}{4}\, U^{\mu_w}(\zeta),$$

and that

$$U^{\mu_w}(\zeta^*) - U^{\mu_w}(\zeta) \geq -6(C+1)r.$$

Applying this in (3.2), we have

$$\begin{aligned} U^{\mu_w}(z) - U^{\mu_w}(\zeta) &\geq -C|z-\zeta^*| - 6(C+1)|\zeta - \zeta^*| \\ &\geq -C|z-\zeta| - 7(C+1)|\zeta-\zeta^*| \geq -8(C+1)|z-\zeta|. \end{aligned}$$

Consequently,

$$|U^{\mu_w}(z) - U^{\mu_w}(\zeta)| \leq 8(C+1)|z-\zeta|, \qquad z \in S_w, \quad \zeta \in \mathbb{C}. \tag{3.4}$$

We now show that (3.4) is true for any $z, \zeta \in \mathbb{C}$. Observe that

$$\sup\{|U^{\mu_w}(z) - U^{\mu_w}(\zeta)| : |z-\zeta| \leq \delta,\ z, \zeta \in \mathbb{C}\} = |U^{\mu_w}(z_0) - U^{\mu_w}(\zeta_0)|$$

for some $z_0, \zeta_0 \in \mathbb{C},\ |z_0 - \zeta_0| \leq \delta$, because

$$\lim_{\substack{z,\zeta\to\infty \\ |z-\zeta|\leq\delta}} (U^{\mu_w}(z) - U^{\mu_w}(\zeta)) = 0.$$

Consider $h(\xi) := U^{\mu_w}(\xi) - U^{\mu_w}(\xi - z_0 + \zeta_0)$, which is continuous on $\overline{\mathbb{C}}$ (see Theorem I.5.1 of [14]) and harmonic in $\overline{\mathbb{C}}\setminus S_w$. By the maximum–minimum principle, we have

$$|U^{\mu_w}(z_0) - U^{\mu_w}(\zeta_0)| = |h(z_0)| \leq \max_{\xi\in S_w} |h(\xi)| = |h(\xi_0)| \leq 8(C+1)|z_0 - \zeta_0|,$$

where $\xi_0 \in S_w$, and where the last inequality follows from (3.4). Thus

$$\sup\{|U^{\mu_w}(z) - U^{\mu_w}(\zeta)| : |z-\zeta| \le \delta,\ z, \zeta \in \mathbb{C}\} \le 8(C+1)\delta,$$

i.e., U^{μ_w} is Lipschitz continuous in $\mathbb{C}$. ∎

Proof of Theorem 1.1. The equilibrium equation (1.5) and Lemma 3.1 give that

$$\int_0^{2\pi} f(t) \log \left| 2 \sin \frac{\theta - t}{2} \right| dt = Q(\theta) - F_w, \qquad e^{i\theta} \in S_w.$$

Differentiating this equation with respect to θ, we obtain that

$$\frac{1}{2} \int_0^{2\pi} f(t) \cot \frac{\theta - t}{2}\, dt = Q'(\theta)$$

for a.e. $\theta,\ e^{i\theta} \in S_w$. The justification of differentiation under the integral is done as in Lemma 2.45 of [3]. Indeed, consider the harmonic conjugate of f (see [4, Chap. 3])

$$\tilde{f}(\theta) = \frac{1}{2\pi} \int_0^{2\pi} f(t) \cot \frac{\theta - t}{2}\, dt,$$

where the integral is understood in the principal value sense. Since $f \in L_\infty([0, 2\pi))$, we have that $\tilde{f} \in L_p([0, 2\pi))$ for any $p < \infty$, by M. Riesz's theorem. From the Fundamental Theorem of Calculus and Fubini's theorem, we obtain that

$$\int_0^{2\pi} f(t) \log \left| 2 \sin \frac{\theta - t}{2} \right| dt = \pi \int_0^{\theta} \tilde{f}(s)\, ds + c, \qquad e^{i\theta} \in \mathbb{T},$$

where c is a constant. It follows that

$$\pi \int_0^{\theta} \tilde{f}(s)\, ds = Q(\theta) - F_w - c, \qquad e^{i\theta} \in S_w.$$

Using the Fundamental Theorem of Calculus again, we can differentiate the above equation, so that

$$\tilde{f}(\theta) = \frac{Q'(\theta)}{\pi} \qquad \text{a.e. on } S_w. \tag{3.5}$$

Consider the analytic in D function

$$H(z) := \frac{1}{2\pi} \int_0^{2\pi} f(t) \frac{e^{it} + z}{e^{it} - z}\, dt, \qquad z \in D,$$

and recall that $H(z) = u(z) + i\tilde{u}(z),\ \tilde{u}(0) = 0$, where u and $\tilde{u}$ have the boundary values (see [4])

$$u|_{\mathbb{T}} = f \qquad \text{and} \qquad \tilde{u}|_{\mathbb{T}} = \tilde{f}.$$

Clearly, the function

$$-iH^2(z) = 2u(z)\tilde{u}(z) + i(\tilde{u}^2(z) - u^2(z)), \qquad z \in D,$$

is analytic in D. Hence the harmonic conjugate of $2u\tilde{u}$ is

$$(2u\tilde{u})\tilde{} = \tilde{u}^2 - u^2 + c,$$

where c is selected by the standard convention $(2u\tilde{u})\tilde{}(0) = 0 = \tilde{u}^2(0) - u^2(0) + c$. Since $\tilde{u}(0) = 0$, we have

$$c = u^2(0) = \left(\frac{1}{2\pi}\int_0^{2\pi} f(t)\,dt\right)^2 = \left(\frac{\mu_w(\mathbb{T})}{2\pi}\right)^2 = \frac{1}{4\pi^2}.$$

Passing to the boundary values, we obtain that

$$2(f\tilde{f})\tilde{} = \tilde{f}^2 - f^2 + \frac{1}{4\pi^2} \qquad \text{a.e. on } \mathbb{T},$$

and that

$$\text{(3.6)} \quad \frac{2}{\pi}Q'\tilde{f} - \left(\frac{Q'}{\pi}\right)^2 - 2(f\tilde{f})\tilde{} + \frac{1}{4\pi^2} = f^2 - \left(\tilde{f} - \frac{Q'}{\pi}\right)^2 \qquad \text{a.e. on } U.$$

Observe that the right-hand side of (3.6) gives a decomposition for the left-hand side on U into the positive part f^2 and the negative part $-(\tilde{f} - Q'/\pi)^2$, because of (3.5) and $f(t) = 0,\ e^{it} \notin S_w$. Hence

$$\text{(3.7)} \qquad \begin{aligned} f^2(\theta) &= \left(\frac{Q'(\theta)}{\pi}\right)^2 - \frac{2}{\pi}(fQ')\tilde{}(\theta) + \frac{1}{4\pi^2} \\ &= \left(\frac{Q'(\theta)}{\pi}\right)^2 - \frac{1}{\pi^2}\int_0^{2\pi} Q'(t)f(t)\cot\frac{\theta - t}{2}\,dt + \frac{1}{4\pi^2} \end{aligned}$$

for a.e. θ such that $e^{i\theta} \in S_w$. ■

Proof of Corollary 1.2. Using (3.5), we obtain that

$$\begin{aligned} &\frac{1}{\pi^2}\int_0^{2\pi} (Q'(\theta) - Q'(t))f(t)\cot\frac{\theta - t}{2}\,dt - \left(\frac{Q'(\theta)}{\pi}\right)^2 + \frac{1}{4\pi^2} \\ &\quad = \frac{Q'(\theta)}{\pi^2}\int_0^{2\pi} f(t)\cot\frac{\theta - t}{2}\,dt - \frac{1}{\pi^2}\int_0^{2\pi} Q'(t)f(t)\cot\frac{\theta - t}{2}\,dt \\ &\qquad - \left(\frac{Q'(\theta)}{\pi}\right)^2 + \frac{1}{4\pi^2} \\ &\quad = \left(\frac{Q'(\theta)}{\pi}\right)^2 - \frac{1}{\pi^2}\int_0^{2\pi} Q'(t)f(t)\cot\frac{\theta - t}{2}\,dt + \frac{1}{4\pi^2} \qquad \text{for a.e. } e^{i\theta} \in S_w, \end{aligned}$$

which is the right-hand side of (1.7).

If $Q \in C^2(U)$, then $p \in C(U)$, because the function

$$\Phi(\theta, t) := (Q'(\theta) - Q'(t))\cot\frac{\theta - t}{2}$$

can be extended continuously to $U \times U$ by setting $\Phi(\theta, \theta) := 2Q''(\theta)$. Hence f has a continuous extension on S_w by (1.8), which satisfies $f(\theta) = \sqrt{p(\theta)},\ e^{i\theta} \in S_w$. Using this extension in (3.6), we obtain that

$$p = \frac{2}{\pi} Q' \tilde{f} - \left(\frac{Q'}{\pi}\right)^2 - 2(f\tilde{f})\tilde{} + \frac{1}{4\pi^2} = f^2 - \left(\tilde{f} - \frac{Q'}{\pi}\right)^2$$

everywhere on U. Therefore, $S_w = \overline{\{e^{i\theta} : p(\theta) > 0\}}$.

It is immediate to see now that the real analyticity of $Q(t)$ on U implies that of $p(t)$. If we assume that S_w has infinitely many arcs, then $p(t)$ must have infinitely many zeros on $S_w \subset U$, and hence it must vanish identically. ∎

Proof of Corollary 1.3. Recall that by the Hilbert inversion formula we have

$$(\tilde{f})\tilde{} = -f + \frac{1}{2\pi} \int_0^{2\pi} f(t)\, dt \qquad \text{a.e. on } \mathbb{T},$$

see [4, Chap. III] and [8, §28]. But the latter integral is equal to the mass $\mu_w(\mathbb{T}) = 1$. Hence we obtain, from (3.5) for $S_w = \mathbb{T}$, that

$$f(\theta) = \frac{1}{2\pi} - \frac{1}{\pi}(Q')\tilde{}(\theta) = \frac{1}{2\pi} - \frac{1}{2\pi^2} \int_0^{2\pi} Q'(t) \cot \frac{\theta - t}{2}\, dt,$$

for a.e. $e^{it} \in \mathbb{T}$. Since Q' is Hölder continuous on $\mathbb{T}$, we conclude that the same is true for its conjugate $(Q')\tilde{}$ by [4, Chap. III]. Therefore, f has a Hölder continuous extension to $\mathbb{T}$, and the above equation holds for all θ. ∎

Proof of Proposition 1.4. We first note that

$$u(\theta) := U^{\mu_w}(e^{i\theta}) = -\int \log |e^{i\theta} - e^{it}|\, d\mu_w(t) = -\int \log \left| 2 \sin \frac{\theta - t}{2} \right| d\mu_w(t)$$

is a strictly convex function of θ on each arc of $\mathbb{T} \backslash S_w$. Indeed, we have

$$u''(\theta) = \frac{1}{4} \int \csc^2 \frac{\theta - t}{2}\, d\mu_w(t) > 0, \qquad e^{i\theta} \in \mathbb{T} \backslash S_w,$$

so that the claim follows. Suppose that $S_w \cap (e^{i\alpha}, e^{i\beta})$ is not an arc. Then there are two points $\theta_1, \theta_2 \in (\alpha, \beta)$ such that $e^{i\theta_1}, e^{i\theta_2} \in S_w$, and $e^{i\theta} \notin S_w$ for $\theta_1 < \theta < \theta_2$. Since $u(\theta) + Q(\theta)$ is a strictly convex function on (θ_1, θ_2), which takes values $u(\theta_1) + Q(\theta_1) = u(\theta_2) + Q(\theta_2) = F_w$ by (1.5), we have that $u(\theta) + Q(\theta) < F_w$ for $\theta \in (\theta_1, \theta_2)$. But this contradicts (1.4), which is true for any $e^{i\theta} \in \mathbb{T}$. ∎

Proof of Theorem 1.5. We start with the observation that

$$\frac{e^{it} + e^{i\theta}}{e^{it} - e^{i\theta}} = i \cot \frac{\theta - t}{2}.$$

Hence we obtain for the singular Schwarz integral (understood as the principal value) that

$$\frac{1}{2\pi}\int_0^{2\pi} f(t)\frac{e^{it}+e^{i\theta}}{e^{it}-e^{i\theta}}dt = i\tilde{f}(\theta) \qquad \text{for a.e. } \theta \in [0, 2\pi).$$

It follows from (3.5) that

$$\frac{1}{2\pi}\int_0^{2\pi} f(t)\frac{e^{it}+e^{i\theta}}{e^{it}-e^{i\theta}}dt = \frac{i}{\pi}Q'(\theta) \qquad \text{a.e. on } S_w,$$

and

$$\frac{1}{\pi i}\int_0^{2\pi} \frac{f(t)\,d(e^{it})}{e^{it}-e^{i\theta}} - \frac{1}{2\pi} = \frac{i}{\pi}Q'(\theta) \qquad \text{a.e. on } S_w.$$

Consequently, the density function f satisfies the following singular integral equation with Cauchy kernel:

$$\frac{1}{\pi i}\int_{S_w} \frac{f(z)\,dz}{z-\zeta} = \frac{i}{\pi}Q'(\theta) + \frac{1}{2\pi}, \qquad \zeta = e^{i\theta} \in S_w, \tag{3.8}$$

where we set $f(z) = f(e^{it}) := f(t)$. Since S_w consists of finitely many arcs Γ_k, $k = 1, \ldots, K$, and f is a continuous function vanishing at the endpoints of S_w by Corollary 1.2, we obtain from the results of [8, Chap. 11, §88] (see also [7]) that f must be the unique solution of (3.8) given by the singular integral

$$f(z) = \frac{\sqrt{R(z)}}{\pi i}\int_{S_w} \frac{g(\zeta)\,d\zeta}{\sqrt{R(\zeta)}(\zeta - z)}, \qquad z \in S_w, \tag{3.9}$$

where $g(\zeta)$ denotes the right-hand side of (3.8), and $\sqrt{R(z)}$ is defined in the statement of Theorem 1.5. Furthermore, vanishing of f at the endpoints of S_w implies that g must satisfy the following moment conditions

$$\int_{S_w} \frac{z^k g(z)\,dz}{\sqrt{R(z)}} = 0, \qquad k = 0, \ldots, K-1,$$

see [8, pp. 251, 256] and [7, p. 14]. Hence (1.9)–(1.11) are proved. Equation (1.12) simply expresses the fact that $d\mu_w(e^{it}) = f(t)\,dt$ is a probability measure.

Observe that the equilibrium equation (1.5) gives

$$U^{\mu_w}(a_{k+1}) - U^{\mu_w}(b_k) = Q(\beta_k) - Q(\alpha_{k+1}), \qquad k = 1, \ldots, K. \tag{3.10}$$

On the other hand, we have that

$$\begin{aligned} U^{\mu_w}(a_{k+1}) - U^{\mu_w}(b_k) &= \int_{\beta_k}^{\alpha_{k+1}} \frac{d}{dt}\,U^{\mu_w}(e^{it})\,dt \\ &= -\frac{1}{2}\int_{\beta_k}^{\alpha_{k+1}}\int_0^{2\pi} f(u)\cot\frac{t-u}{2}\,du\,dt \\ &= -\frac{1}{2i}\int_{\beta_k}^{\alpha_{k+1}}\int_0^{2\pi} f(u)\frac{e^{iu}+e^{it}}{e^{iu}-e^{it}}\,du\,dt \\ &= -\frac{\pi}{i}\int_{\beta_k}^{\alpha_{k+1}}\left(\frac{1}{\pi i}\int_{S_w}\frac{F(e^{iu})\,d(e^{iu})}{e^{iu}-e^{it}} - \frac{1}{2\pi}\right)dt. \end{aligned}$$

Note that $\lim_{r\to 1-}\sqrt{R(re^{iu})} = -\lim_{r\to 1+}\sqrt{R(re^{iu})}$ for $e^{iu}\in S_w$. Thus the limiting boundary values of $\sqrt{R(z)}$ on the arcs of S_w, from inside and outside the unit circle, are opposite in sign. Hence the same is true for the function $F(z)$, which is analytic in $\overline{\mathbb{C}}\setminus S_w$. Passing to the contour integral over both sides of the cut S_w, we obtain by the Cauchy integral theorem that

$$\frac{1}{\pi i}\int_{S_w}\frac{F(e^{iu})d(e^{iu})}{e^{iu}-e^{it}} = \frac{1}{2\pi i}\oint_{S_w}\frac{F(e^{iu})d(e^{iu})}{e^{iu}-e^{it}} = F(e^{it}) - \lim_{z\to\infty} F(z).$$

The latter limit is equal to 0, in fact, which follows from the moment conditions (1.11). Indeed, we have in a neighborhood of ∞ that

$$\begin{aligned} F(z) &= \frac{\sqrt{R(z)}}{\pi i}\int_{S_w}\frac{g(\zeta)\,d\zeta}{\sqrt{R(\zeta)}(\zeta-z)} = -\frac{\sqrt{R(z)}}{\pi i}\sum_{k=0}^{\infty} z^{-k-1}\int_{S_w}\frac{\zeta^k g(\zeta)\,d\zeta}{\sqrt{R(\zeta)}} \\ &= -\frac{\sqrt{R(z)}}{\pi i}\sum_{k=K}^{\infty} z^{-k-1}\int_{S_w}\frac{\zeta^k g(\zeta)\,d\zeta}{\sqrt{R(\zeta)}}, \end{aligned}$$

which gives $F(\infty)=0$. Consequently,

$$\begin{aligned} U^{\mu_w}(a_{k+1}) - U^{\mu_w}(b_k) &= -\frac{\pi}{i}\int_{\beta_k}^{\alpha_{k+1}}\left(F(e^{it}) - \frac{1}{2\pi}\right)dt \\ &= -\frac{\pi}{i}\int_{\beta_k}^{\alpha_{k+1}} F(e^{it})\,dt + \frac{\alpha_{k+1}-\beta_k}{2i}. \end{aligned}$$

Combining this equation with (3.10), we prove (1.13). ■

Proof of Theorem 2.1. Note that S_w cannot contain the zeros of w by (1.5), i.e., we have that

$$Q(\theta) = -\sum_{j=1}^{J}\lambda_j \log|e^{i\theta} - z_j|$$

is infinitely differentiable in a neighborhood of S_w in $\mathbb{T}$. Thus the results of Section 1 apply here. It is clear that

$$Q'(\theta) = -\sum_{j=1}^{J}\frac{\lambda_j r_j \sin(\theta-\varphi_j)}{1+r_j^2-2r_j\cos(\theta-\varphi_j)} = -\sum_{j=1}^{J}\frac{\lambda_j r_j\sin(\theta-\varphi_j)}{|e^{i\theta}-z_j|^2},$$

where $z_j = r_j e^{i\varphi_j}$, $j=1,\dots,J$. Applying elementary trigonometric identities, we obtain that

$$Q'(\theta) - Q'(t) = \sum_{j=1}^{J}\frac{2\lambda_j r_j^2\sin(\theta-t) - 2\lambda_j r_j(1+r_j^2)\cos\left(\frac{\theta+t}{2}-\varphi_j\right)\sin\frac{\theta-t}{2}}{|e^{i\theta}-z_j|^2|e^{it}-z_j|^2}$$

and that

$$(Q'(\theta) - Q'(t)) \cot \frac{\theta - t}{2} = \sum_{j=1}^{J} \frac{2\lambda_j r_j^2 (\cos(\theta - t) + 1) - \lambda_j r_j (1 + r_j^2)(\cos(\theta - \varphi_j) + \cos(t - \varphi_j))}{|e^{i\theta} - z_j|^2 |e^{it} - z_j|^2}.$$

If we insert the above representations in (1.8), it becomes clear that

$$f^2(\theta) = \frac{L_{2J}(e^{i\theta})}{\pi^2 \prod_{j=1}^{J} |e^{i\theta} - z_j|^4}, \qquad e^{i\theta} \in S_w,$$

where $L_{2J}(e^{i\theta})$ is a trigonometric polynomial of degree at most $2J$. Since $L_{2J}(e^{i\theta})$ has at most $4J$ zeros on $[0, 2\pi)$, and $S_w = \overline{\{e^{i\theta} : L_{2J}(e^{i\theta}) > 0\}}$, we conclude that S_w consists of at most $2J$ arcs of $\mathbb{T}$. Naturally, L_{2J} and f vanish at the endpoints of those arcs.

Alternatively, we can write

$$\begin{aligned} Q'(\theta) &= -\sum_{j=1}^{J} \frac{\lambda_j r_j \sin(\theta - \varphi_j)}{1 + r_j^2 - 2r_j \cos(\theta - \varphi_j)} = \frac{1}{2i} \sum_{j=1}^{J} \frac{\lambda_j (z_j e^{-i\theta} - \bar{z}_j e^{i\theta})}{|e^{i\theta} - z_j|^2} \\ &= \frac{1}{2i} \sum_{j=1}^{J} \lambda_j \frac{z_j - \bar{z}_j \zeta^2}{(\zeta - z_j)(1 - \bar{z}_j \zeta)}, \qquad \zeta = e^{i\theta}. \end{aligned}$$

Thus we obtain from Theorem 1.5 that

$$\begin{aligned} g(\zeta) &= \frac{1}{2\pi} \sum_{j=1}^{J} \lambda_j \frac{z_j - \bar{z}_j \zeta^2}{(\zeta - z_j)(1 - \bar{z}_j \zeta)} + \frac{1}{2\pi} \\ &= \frac{1 + \sum_{j=1}^{J} \lambda_j}{2\pi} + \frac{1}{2\pi} \sum_{j=1}^{J} \frac{\lambda_j z_j}{\zeta - z_j} + \frac{1}{2\pi} \sum_{j=1}^{J} \frac{\lambda_j \bar{z}_j^{-1}}{\zeta - \bar{z}_j^{-1}}. \end{aligned}$$

This form of g is convenient for evaluation of the integrals as in (1.9).

We first consider the case $S_w = \mathbb{T}$. Then we have from (3.8) that

$$\frac{1}{\pi i} \int_{\mathbb{T}} \frac{f(z)\, dz}{z - \zeta} = g(\zeta), \qquad \zeta \in \mathbb{T}.$$

Applying the inversion formula to this Cauchy singular integral, we obtain

$$f(z) = \frac{1}{\pi i} \int_{\mathbb{T}} \frac{g(\zeta)\, d\zeta}{\zeta - z}, \qquad z \in \mathbb{T},$$

by §27 of [8]. It follows from Plemelj's formulas (see [8, §17]) that

$$f(z) = \lim_{\substack{|\xi|<1 \\ \xi \to z}} \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{g(\zeta)\, d\zeta}{\zeta - \xi} + \lim_{\substack{|\xi|>1 \\ \xi \to z}} \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{g(\zeta)\, d\zeta}{\zeta - \xi},$$

where both integrals are taken in the counterclockwise direction. Evaluating the above integrals via residues and passing to the limits, we immediately have that

$$\begin{aligned} f(z) = \frac{1+\sum_{j=1}^{J}\lambda_j}{2\pi} &+ \frac{1}{2\pi}\sum_{|z_j|>1}\frac{\lambda_j z_j}{z-z_j} + \frac{1}{2\pi}\sum_{|z_j|<1}\frac{\lambda_j \bar{z}_j^{-1}}{z-\bar{z}_j^{-1}} \\ &- \frac{1}{2\pi}\sum_{|z_j|<1}\frac{\lambda_j z_j}{z-z_j} - \frac{1}{2\pi}\sum_{|z_j|>1}\frac{\lambda_j \bar{z}_j^{-1}}{z-\bar{z}_j^{-1}} \\ = \frac{1+\sum_{j=1}^{J}\lambda_j}{2\pi} &+ \frac{1}{2\pi}\sum_{|z_j|>1}\lambda_j\left(\frac{z_j}{z-z_j} - \frac{\bar{z}_j^{-1}}{z-\bar{z}_j^{-1}}\right) \\ &+ \frac{1}{2\pi}\sum_{|z_j|<1}\lambda_j\left(\frac{\bar{z}_j^{-1}}{z-\bar{z}_j^{-1}} - \frac{z_j}{z-z_j}\right). \end{aligned}$$

Thus (2.2) follows by a simple algebraic manipulation. Another proof of (2.2) can be produced by using Corollary 1.3 and finding the harmonic conjugate of Q' from its trigonometric form.

We now assume that S_w consists of $K \geq 1$ proper arcs of $\mathbb{T}$, and apply (1.9) of Theorem 1.5. As in the proof of Theorem 1.5, we observe that the limiting boundary values of $\sqrt{R(\zeta)}$ on those arcs, from inside and outside the unit circle, are opposite in sign. Hence the same is true for the function $g(\zeta)/(\sqrt{R(\zeta)}(\zeta - z))$, which is analytic in $\overline{\mathbb{C}}\setminus S_w$ except for the simple poles at z_j and $\bar{z}_j^{-1}$. Here, we used the natural extension of $g(\zeta)$ from S_w to $\mathbb{C}$ as a rational function. Passing to the contour integral over both sides of the cut S_w, and computing the residues at z_j and $\bar{z}_j^{-1}$, we obtain that

$$\begin{aligned} f(z) &= \frac{\sqrt{R(z)}}{2\pi i}\oint_{S_w}\frac{g(\zeta)\,d\zeta}{\sqrt{R(\zeta)}(\zeta - z)} \\ &= \frac{\sqrt{R(z)}}{2\pi}\sum_{j=1}^{J}\frac{\lambda_j z_j}{\sqrt{R(z_j)}(z_j - z)} + \frac{\sqrt{R(z)}}{2\pi}\sum_{j=1}^{J}\frac{\lambda_j \bar{z}_j^{-1}}{\sqrt{R(\bar{z}_j^{-1})}(\bar{z}_j^{-1} - z)}, \quad z \in S_w. \end{aligned}$$

Thus (2.3) is proved. Our next goal is to show that the number of intervals for S_w is at most J. Observe that the previous equation gives

$$\begin{aligned} f(t) &= \frac{\sqrt{R(e^{it})}}{\pi}\sum_{j=1}^{J}\frac{\lambda_j}{2}\left(\frac{-z_j}{\sqrt{R(z_j)}(e^{it} - z_j)} + \frac{1}{\sqrt{R(\bar{z}_j^{-1})}(1 - \bar{z}_j e^{it})}\right) \\ &= \frac{\sqrt{R(e^{it})}P(e^{it})}{\pi\prod_{j=1}^{J}(e^{it} - z_j)(1 - \bar{z}_j e^{it})} = \frac{\sqrt{R(e^{it})}P(e^{it})}{\pi e^{iJt}\prod_{j=1}^{J}|e^{it} - z_j|^2}, \qquad e^{it} \in S_w, \end{aligned}$$

where P is an algebraic polynomial. Comparing this with the previously obtained form for $f^2(t)$, we conclude that

$$R(e^{it})P^2(e^{it}) = e^{i2Jt}L_{2J}(e^{it}), \tag{3.11}$$

for $e^{it} \in S_w$, so that these polynomials in e^{it} coincide. It follows that $\deg(RP^2) \leq 4J$, i.e., $2K + 2\deg P \leq 4J$ and

$$K + \deg P \leq 2J. \tag{3.12}$$

Since $S_w = \overline{\{e^{i\theta} : L_{2J}(e^{it}) > 0\}}$, we have that $L_{2J}(e^{it})$ takes real values for all t, and that $L_{2J}(e^{it}) \le 0$ for $e^{it} \in \mathbb{T}\backslash S_w$. Equation (3.11) and the one before it suggest that

$$\sqrt{R(e^{it})}P(e^{it})e^{-iJt} \ge 0, \qquad e^{it} \in S_w, \tag{3.13}$$

and that $\sqrt{R(e^{it})}P(e^{it})e^{-iJt}$ is pure imaginary on $\mathbb{T}\backslash S_w$. Assume that $P(e^{it}) \ne 0$ for $t \in (\beta_k, \alpha_{k+1})$, that is, P does not vanish on $\mathbb{T}$ between a pair of neighboring arcs of S_w. Note that the argument of $\sqrt{R(e^{it})}$ decreases by $\pi/2$ when we pass over an endpoint of S_w (when moving in the positive direction on $\mathbb{T}$), while the argument of $P(e^{it})e^{-iJt}$ remains continuous everywhere on $\mathbb{T}$, except for the zeros of P. Hence we have that the values of $\sqrt{R(e^{it})}P(e^{it})e^{-iJt}$ should have opposite signs on these neighboring arcs of S_w, which immediately contradicts (3.13). It follows that $P(e^{it})$ has a zero on $\mathbb{T}$ between each pair of arcs of S_w, so that $\deg P \ge K$. Finally, $2K \le K + \deg P \le 2J$, by (3.12), and $K \le J$.

In the remaining part, we prove (2.4) and (2.5). Applying Theorem 1.5, we see that g must satisfy the moment conditions of (1.11). Again, these integrals are found by passing to the contour integrals over both sides of the cut S_w, and computing the residues at z_j, $\bar{z}_j^{-1}$, and ∞:

$$\begin{aligned}\frac{1}{\pi i}\int_{S_w}\frac{z^k g(z)\,dz}{\sqrt{R(z)}} &= \frac{1}{2\pi i}\oint_{S_w}\frac{z^k g(z)\,dz}{\sqrt{R(z)}} \\ &= \frac{1}{2\pi}\sum_{j=1}^{J}\lambda_j\left(\frac{z_j^{k+1}}{\sqrt{R(z_j)}} + \frac{\bar{z}_j^{-k-1}}{\sqrt{R(\bar{z}_j^{-1})}}\right) \\ &\quad - \frac{1+\sum_{j=1}^{J}\lambda_j}{2\pi}\lim_{z\to\infty}\frac{z^{k+1}}{\sqrt{R(z)}}.\end{aligned}$$

Note that

$$\lim_{z\to\infty}\frac{z^{k+1}}{\sqrt{R(z)}} = \begin{cases}0, & k = 0,\dots,K-2,\\ 1, & k = K-1,\end{cases}$$

so that (2.4) follows from (1.11).

We deduce (2.5) from (1.13). For this purpose, we evaluate $F(z)$ for $z \in \mathbb{C}\backslash S_w$, by using the residues at z_j, $\bar{z}_j^{-1}$, and z:

$$\begin{aligned}F(z) &= \frac{\sqrt{R(z)}}{2\pi i}\oint_{S_w}\frac{g(\zeta)\,d\zeta}{\sqrt{R(\zeta)}(\zeta - z)} \\ &= \frac{\sqrt{R(z)}}{2\pi}\sum_{j=1}^{J}\frac{\lambda_j z_j}{\sqrt{R(z_j)}(z_j - z)} + \frac{\sqrt{R(z)}}{2\pi}\sum_{j=1}^{J}\frac{\lambda_j \bar{z}_j^{-1}}{\sqrt{R(\bar{z}_j^{-1})}(\bar{z}_j^{-1} - z)} + g(z) \\ &= F_1(z) + g(z).\end{aligned}$$

Hence

$$\int_{\beta_k}^{\alpha_{k+1}} F_1(e^{it})\,dt + \int_{\beta_k}^{\alpha_{k+1}} g(e^{it})\,dt = \frac{\alpha_{k+1}-\beta_k}{2\pi} + \frac{Q(\alpha_{k+1}) - Q(\beta_k)}{\pi}\,i, \tag{3.14}$$

where $k = 1, \dots, K$, by (1.13). We next compute that

$$\begin{aligned}\int_{\beta_k}^{\alpha_{k+1}} g(e^{it})\,dt &= \int_{\beta_k}^{\alpha_{k+1}} \left(\frac{1}{2\pi}\sum_{j=1}^{J}\lambda_j\left(\frac{z_j}{e^{it}-z_j}+\frac{\bar{z}_j^{-1}}{e^{it}-\bar{z}_j^{-1}}\right)+\frac{1+\sum_{j=1}^{J}\lambda_j}{2\pi}\right)dt \\ &= \frac{1}{2\pi i}\sum_{j=1}^{J}\lambda_j\left(z_j\int_{b_k}^{a_{k+1}}\frac{dz}{z(z-z_j)}+\bar{z}_j^{-1}\int_{b_k}^{a_{k+1}}\frac{dz}{z(z-\bar{z}_j^{-1})}\right) \\ &\quad+\frac{\alpha_{k+1}-\beta_k}{2\pi}\left(1+\sum_{j=1}^{J}\lambda_j\right).\end{aligned}$$

It is immediate to see that

$$\begin{aligned}&\frac{1}{2\pi i}\sum_{j=1}^{J}\lambda_j\left(z_j\int_{b_k}^{a_{k+1}}\frac{dz}{z(z-z_j)}+\bar{z}_j^{-1}\int_{b_k}^{a_{k+1}}\frac{dz}{z(z-\bar{z}_j^{-1})}\right) \\ &\qquad=\sum_{j=1}^{J}\frac{\lambda_j}{2\pi i}\left(\log\frac{a_{k+1}-z_j}{b_k-z_j}+\log\frac{a_{k+1}-\bar{z}_j^{-1}}{b_k-\bar{z}_j^{-1}}-2\log\frac{a_{k+1}}{b_k}\right) \\ &\qquad=\sum_{j=1}^{J}\frac{\lambda_j}{2\pi i}\left(\log\frac{a_{k+1}|a_{k+1}-z_j|^2}{b_k|b_k-z_j|^2}-2\log\frac{a_{k+1}}{b_k}\right) \\ &\qquad=\frac{Q(\alpha_{k+1})-Q(\beta_k)}{\pi}\,i-\frac{\alpha_{k+1}-\beta_k}{2\pi}\sum_{j=1}^{J}\lambda_j.\end{aligned}$$

Substituting the results of the above computations in (3.14), we immediately obtain (2.5). ∎

Proof of Theorem 2.2. We follow essentially the same scheme as in the previous proof. Note that $Q(\theta) = t_M(\theta)$ is infinitely differentiable on $\mathbb{T}$. It is clear that

$$Q'(\theta) = \sum_{m=-M}^{M} imc_m e^{im\theta}$$

and

$$g(e^{i\theta}) = \frac{1}{2\pi} - \frac{1}{\pi}\sum_{m=-M}^{M} mc_m e^{im\theta},$$

where g is defined in Theorem 1.5. Inserting $Q'(\theta)$ into (1.8), we observe that the right-hand side $p(\theta)$ is a trigonometric polynomial of degree at most $2M$. Thus we have from Theorem 1.1 and Corollary 1.2 that $d\mu_w(e^{i\theta}) = f(\theta)\,d\theta$ with $f^2(\theta) = L_{2M}(e^{i\theta})$, where $L_{2M}(z)$ is a Laurent polynomial of degree at most $2M$. It follows that S_w consists of at most $2M$ arcs, because L_{2M} has at most $4M$ zeros on $\mathbb{T}$.

When $S_w = \mathbb{T}$, we obtain (2.7) from Corollary 1.3, after substituting the conjugate

$$(Q')\tilde{}(\theta) = \frac{1}{2\pi}\int_0^{2\pi} Q'(t)\cot\frac{\theta-t}{2}\,dt = \sum_{m=-M}^{M} m\,\mathrm{sgn}(m)c_m e^{im\theta}.$$

Suppose that S_w consists of $K \geq 1$ proper arcs of $\mathbb{T}$, and apply (1.9) of Theorem 1.5. Using that the limiting boundary values of $\sqrt{R(\zeta)}$ on S_w are opposite in sign, we again pass to the contour integral over both sides of the cut S_w:

$$(3.15)\qquad \begin{aligned} F_2(z)=F(z) &= \frac{\sqrt{R(z)}}{2\pi i}\oint_{S_w}\frac{g(\zeta)\,d\zeta}{\sqrt{R(\zeta)}(\zeta-z)} \\ &= \frac{\sqrt{R(z)}}{2\pi i}\oint_{S_w}\frac{1}{\sqrt{R(\zeta)}(\zeta-z)}\left(\frac{1}{2\pi}-\frac{1}{\pi}\sum_{m=-M}^{M} mc_m\zeta^m\right)d\zeta \\ &= -\frac{\sqrt{R(z)}}{\pi}\sum_{m=-M}^{M}\frac{mc_m}{2\pi i}\oint_{S_w}\frac{\zeta^m\,d\zeta}{\sqrt{R(\zeta)}(\zeta-z)}, \qquad z\in S_w. \end{aligned}$$

The latter contour integrals are found by the Cauchy integral theorem and evaluation of residues at ∞ and at 0, with the help of the series expansions for $1/(\sqrt{R(\zeta)}(\zeta - z))$. This immediately gives the stated form of $F_2(z) = F(z),\ z \in S_w$, so that (2.8) follows from (1.10). The expansion coefficients $r_k(z)$ and $s_k(z)$ can be expressed in the standard way

$$r_k(z) = \frac{1}{k!}\frac{d^k}{d\zeta^k}\left(\frac{1}{\sqrt{R(\zeta)}(\zeta-z)}\right)\bigg|_{\zeta=0}, \qquad k\geq 0,$$

and

$$s_k(z) = \frac{1}{k!}\frac{d^k}{d\zeta^k}\left(\frac{\zeta^{K+1}}{\sqrt{\zeta^{2K}R(1/\zeta)}(1-z\zeta)}\right)\bigg|_{\zeta=0}, \qquad k\geq 0.$$

It transpires now that $s_k(z)$ is a polynomial in z of degree at most k, and that $r_k(z)$ is a polynomial in $1/z$ of degree at most $k+1$. Hence we have

$$F_2(z) = \frac{P(z)\sqrt{R(z)}}{z^M}, \qquad z\in S_w,$$

where $P(z)$ is a polynomial in z. Comparing this with the previously obtained form for $f^2(t)$, we conclude that

$$R(e^{it})P^2(e^{it}) = e^{i2Mt}L_{2M}(e^{it}), \qquad e^{it}\in S_w,$$

so that these polynomials in e^{it} coincide. It follows that $\deg(RP^2) \leq 4M$, i.e., $2K + 2\deg P \leq 4M$ and

$$(3.16)\qquad K + \deg P \leq 2M.$$

Since $S_w = \overline{\{e^{i\theta} : L_{2M}(e^{it}) > 0\}}$, we have that $L_{2M}(e^{it})$ takes real values for all t, and that $L_{2M}(e^{it}) \leq 0$ for $e^{it} \in \mathbb{T}\backslash S_w$. Therefore,

$$(3.17)\qquad \sqrt{R(e^{it})}P(e^{it})e^{-iMt} \geq 0, \qquad e^{it}\in S_w,$$

being the density of μ_w, and $\sqrt{R(e^{it})}P(e^{it})e^{-iMt}$ is pure imaginary on $\mathbb{T}\backslash S_w$. Assume that P does not vanish on $\mathbb{T}$ between a pair of neighboring arcs of S_w. Note again that the argument of $\sqrt{R(e^{it})}$ decreases by $\pi/2$ when we pass over an endpoint of S_w, while the argument of $P(e^{it})e^{-iMt}$ is continuous on $\mathbb{T}$, except for possible zeros of P. Hence the values of $\sqrt{R(e^{it})}P(e^{it})e^{-iMt}$ should have opposite signs on these neighboring arcs of S_w, which immediately contradicts (3.17). It follows that $P(e^{it})$ has a zero on $\mathbb{T}$ between each pair of arcs of S_w, so that $\deg P \geq K$. Finally, $2K \leq K + \deg P \leq 2M$, by (3.16), and $K \leq M$.

Repeating the same evaluation as in (3.15), but for $z \notin S_w$, we obtain that

$$F(z) = \frac{\sqrt{R(z)}}{2\pi i} \oint_{S_w} \frac{g(\zeta)\,d\zeta}{\sqrt{R(\zeta)}(\zeta - z)} = F_2(z) + g(z). \qquad \blacksquare$$

Acknowledgment. The author thanks the referee for numerous suggested improvements.

References

1. J. Baik, P. Deift, K. Johansson (1990): *On the distribution of the length of the longest increasing subsequence of random permutations*. J. Amer. Math. Soc., **12**:1119–1178.
2. E. Bombieri, J. D. Vaaler (1987): *Polynomials with low height and prescribed vanishing*. In: Analytic Number Theory and Diophantine Problems (Stillwater, OK, 1984), pp. 53–73. Boston, MA: Birkhäuser.
3. P. Deift, T. Kreicherbauer, K. T.-R. McLaughlin (1998): *New results on the equilibrium measure for logarithmic potentials in the presence of an external field*. J. Approx. Theory, **95**:388–475.
4. J. B. Garnett (1981): Bounded Analytic Functions. New York: Academic Press.
5. G. M. Goluzin (1969): Geometric Theory of Functions of a Complex Variable. Providence, RI: American Mathematical Society.
6. M. Götz (2000): *On the distribution of weighted extremal points on a surface in* $\mathbb{R}^d$, $d \geq 3$. Potential Anal., **13**:345–359.
7. N. I. Muskhelishvili (1941): *Application of integrals of Cauchy type to a class of singular integral equations*. Trans. Inst. Math. Tbilissi, **10**:1–43 (Russian).
8. N. I. Muskhelishvili (1992): Singular Integral Equations. New York: Dover.
9. I. E. Pritsker (to appear): *Small polynomials with integer coefficients*. J. Analyse Math. Available electronically at http://www.math.okstate.edu/~igor/intcheb.ps
10. I. E. Pritsker (to appear): *The Gelfond–Schnirelman method in prime number theory*. Canad. J. Math. Available electronically at http://www.math.okstate.edu/~igor/gsm.pdf
11. I. E. Pritsker, J. D. Vaaler (to appear): *The height of subspaces of polynomials*. Manuscript.
12. I. E. Pritsker, R. S. Varga (1997): *The Szegő curve, zero distribution and weighted approximation*. Trans. Amer. Math. Soc., **349**:4085–4105.
13. I. E. Pritsker, R. S. Varga (1998): *Weighted polynomial approximation in the complex plane*. Constr. Approx., **14**:475–492.
14. E. B. Saff, V. Totik (1997): Logarithmic Potentials with External Fields. Berlin: Springer-Verlag.

I. E. Pritsker
Department of Mathematics
Oklahoma State University
Stillwater, OK 74078
USA
igor@math.okstate.edu