Reverse Triangle Inequalities for Potentials

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Dedicated to George G. Lorentz, whose works have been a great inspiration

Abstract. We study the reverse triangle inequalities for suprema of logarithmic potentials on compact sets of the plane. This research is motivated by the inequalities for products of supremum norms of polynomials. We find sharp additive constants in the inequalities for potentials, and give applications of our results to the generalized polynomials.

We also obtain sharp inequalities for products of norms of the weighted polynomials $w^n P_n$, $\deg(P_n) \le n$, and for sums of suprema of potentials with external fields. An important part of our work in the weighted case is a Riesz decomposition for the weighted farthest-point distance function.

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1 Products of polynomials and sums of potentials

Let E be a compact set in the complex plane $\mathbb{C}$. Given the bounded above functions $f_j, j = 1, \ldots, m$, on E, we have by a standard inequality that

$$\sup_E \sum_{j=1}^m f_j \le \sum_{j=1}^m \sup_E f_j.$$

It is not possible to reverse this inequality for arbitrary functions, even if one introduces additive or multiplicative "correction" constants. However, we are able to prove the reverse inequalities for *logarithmic potentials*, with sharp additive constants. For a positive Borel measure μ with compact support in the plane, define its (subharmonic) potential [18, p. 53] by

$$p(z) := \int \log|z-t|\, d\mu(t).$$

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Let $\nu_j, j = 1, \ldots, m$, be positive compactly supported Borel measures with potentials p_j. We normalize the problem by assuming that the total mass of $\nu := \sum_{j=1}^m \nu_j$ is equal to 1, and consider the inequality

$$\sum_{j=1}^m \sup_E p_j \le C_E(m) + \sup_E \sum_{j=1}^m p_j. \tag{1.1}$$

Clearly, if (1.1) holds true, then $C_E(m) \ge 0$. One may also ask whether (1.1) holds with a constant C_E independent of m. The motivation for such inequalities comes directly from inequalities for the norms of products of polynomials. Indeed, if $P(z) = \prod_{j=1}^n (z - a_j)$ is a monic polynomial, then $\log |P(z)| = n \int \log |z - t| \, d\tau(t)$. Here, $\tau = \frac{1}{n} \sum_{j=1}^n \delta_{a_j}$ is the normalized counting measure in the zeros of P, with δ_{a_j} being the unit point mass at a_j. Let $\|P\|_E := \sup_E |P|$ be the uniform (sup) norm on E. Thus (1.1) takes the following form for polynomials $P_j, j = 1, \ldots, m$,

$$\prod_{j=1}^m \|P_j\|_E \le e^{nC_E(m)} \left\| \prod_{j=1}^m P_j \right\|_E,$$

where n is the degree of the product polynomial $\prod_{j=1}^m P_j$. We outline a brief history of such inequalities below.

Kneser [8] proved the first sharp inequality for norms of products of polynomials on $[-1, 1]$ (see also Aumann [1] for a weaker result)

$$\|P_1\|_{[-1,1]} \|P_2\|_{[-1,1]} \le K_{\ell,n} \|P_1 P_2\|_{[-1,1]}, \quad \deg P_1 = \ell, \ \deg P_2 = n - \ell, \tag{1.2}$$

where

$$K_{\ell,n} := 2^{n-1} \prod_{k=1}^{\ell} \left(1 + \cos \frac{2k-1}{2n}\pi\right) \prod_{k=1}^{n-\ell} \left(1 + \cos \frac{2k-1}{2n}\pi\right). \tag{1.3}$$

Observe that equality holds in (1.2) for the Chebyshev polynomial $t(x) = \cos n \arccos x = P_1(x)P_2(x)$, with a proper choice of the factors $P_1(x)$ and $P_2(x)$. Borwein [3] generalized this to the multifactor inequality

$$\prod_{j=1}^m \|P_j\|_{[-1,1]} \le 2^{n-1} \prod_{k=1}^{[\frac{n}{2}]} \left(1 + \cos \frac{2k-1}{2n}\pi\right)^2 \left\| \prod_{j=1}^m P_j \right\|_{[-1,1]}, \tag{1.4}$$

where n is the degree of $\prod_{j=1}^m P_j$. We remark that

$$2^{n-1} \prod_{k=1}^{[\frac{n}{2}]} \left(1 + \cos \frac{2k-1}{2n}\pi\right)^2 \sim (3.20991\ldots)^n \text{ as } n \to \infty. \tag{1.5}$$

Another inequality of this type for $E = D$, where $D := \{w : |w| \le 1\}$ is the closed unit disk, was proved by Gelfond [7, p. 135] in connection with the theory of transcendental numbers:

$$\prod_{j=1}^{m} \|P_j\|_D \le e^n \left\| \prod_{j=1}^{m} P_j \right\|_D . \tag{1.6}$$

Mahler [12] later replaced e by 2:

$$\prod_{j=1}^{m} \|P_j\|_D \le 2^n \left\| \prod_{j=1}^{m} P_j \right\|_D . \tag{1.7}$$

It is easy to see that the base 2 cannot be decreased, if $m = n$ and $n \to \infty$. However, Kroó and Pritsker [9] showed that, for any $m \le n$,

$$\prod_{j=1}^{m} \|P_j\|_D \le 2^{n-1} \left\| \prod_{j=1}^{m} P_j \right\|_D , \tag{1.8}$$

where equality holds in (1.8) for *each* $n \in \mathbb{N}$, with $m = n$ and $\prod_{j=1}^{m} P_j = z^n - 1$. On the other hand, Boyd [4, 5] proved that, given the number of factors m in (1.7), one has

$$\prod_{j=1}^{m} \|P_j\|_D \le (C_m)^n \left\| \prod_{j=1}^{m} P_j \right\|_D , \tag{1.9}$$

where

$$C_m := \exp\left(\frac{m}{\pi} \int_0^{\pi/m} \log\left(2\cos\frac{t}{2}\right) dt \right) \tag{1.10}$$

is asymptotically best possible for *each fixed* m, as $n \to \infty$.

For a compact set $E \subset \mathbb{C}$, a natural general problem is to find the *smallest* constant $M_E > 0$ such that

$$\prod_{j=1}^{m} \|P_j\|_E \le (M_E)^n \left\| \prod_{j=1}^{m} P_j \right\|_E \tag{1.11}$$

holds for arbitrary polynomials $\{P_j(z)\}_{j=1}^m$ with complex coefficients, where $n = \deg(\prod_{j=1}^m P_j)$. The solution of this problem is based on the logarithmic potential theory (cf. [18] and [20]). Let $\operatorname{cap}(E)$ be the *logarithmic capacity* of a compact set $E \subset \mathbb{C}$. For E with $\operatorname{cap}(E) > 0$, denote the *equilibrium measure* of E by μ_E. We remark that μ_E is a positive unit Borel measure supported on the outer boundary of E (see [20, p. 55]). Define

$$d_E(z) := \max_{t \in E} |z - t|, \qquad z \in \mathbb{C}, \tag{1.12}$$

which is clearly a positive and continuous function in $\mathbb{C}$. It is easy to see that the logarithm of this distance function is subharmonic in $\mathbb{C}$. Moreover, it has the following integral representation

$$\log d_E(z) = \int \log|z - t| d\sigma_E(t), \quad z \in \mathbb{C},$$

where σ_E is a positive unit Borel measure in $\mathbb{C}$ with unbounded support, see Lemma 5.1 of [14] and [10]. Further study of the representing measure σ_E is contained in the work of Gardiner and Netuka [6]. This integral representation is the key fact used by the first author to prove the following result.

Theorem 1.1 [14] *Let $E \subset \mathbb{C}$ be a compact set,* $\mathrm{cap}(E) > 0$. *Then the best constant M_E in* (1.11) *is given by*

$$M_E = \frac{\exp\left(\displaystyle\int \log d_E(z) d\mu_E(z)\right)}{\mathrm{cap}(E)}. \tag{1.13}$$

It is not difficult to see that M_E is invariant under the similarity transformations of the set E in the plane.

For the closed unit disk D, we have that $\mathrm{cap}(D) = 1$ and that $d\mu_D = d\theta/(2\pi)$, where $d\theta$ is the arclength on ∂D [20, p. 84]. Thus Theorem 1.1 yields

$$M_D = \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \log d_D(e^{i\theta})\ d\theta\right) = \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \log 2\ d\theta\right) = 2,$$

so that we immediately obtain Mahler's inequality (1.7).

If $E = [-1,1]$ then $\mathrm{cap}([-1,1]) = 1/2$ and $d\mu_{[-1,1]} = dx/(\pi\sqrt{1-x^2}),\ x \in [-1,1]$, which is the Chebyshev distribution (see [20, p. 84]). Using Theorem 1.1, we obtain

$$\begin{aligned} M_{[-1,1]} &= 2\exp\left(\frac{1}{\pi}\int_{-1}^{1} \frac{\log d_{[-1,1]}(x)}{\sqrt{1-x^2}}dx\right) = 2\exp\left(\frac{2}{\pi}\int_0^1 \frac{\log(1+x)}{\sqrt{1-x^2}}dx\right) \\ &= 2\exp\left(\frac{2}{\pi}\int_0^{\pi/2} \log(1+\sin t)dt\right) \approx 3.2099123, \end{aligned}$$

which gives the asymptotic version of Borwein's inequality (1.4)-(1.5).

Considering the above analysis of Theorem 1.1, it is natural to conjecture that the sharp universal bounds for M_E are given by

$$2 = M_D \le M_E \le M_{[-1,1]} \approx 3.2099123, \tag{1.14}$$

for any bounded non-degenerate continuum E, see [15]. This problem was treated in the recent papers of the first author and Ruscheweyh [16] and [17], where the lower bound $M_E \ge M_D = 2$ is proved for all compact sets E, and the upper bound is proved for certain special classes of continua. A general approach to this type of extremal problem was proposed by Baernstein, Laugesen and Pritsker [2]. We show in the next section that all results about M_E are directly applicable to the constants C_E and $C_E(m)$ in the inequality for potentials (1.1).

The assumption that E is of positive capacity is vital for our results. For example, when E consists of a finite number of points $\{z_j\}_{j=1}^N,\ N \ge 2$, then no inequality of the type (1.11) is possible with *any* constant. Indeed, if $m = n \geq N$ then we consider $P_j(z) = z - z_j,\ j = 1, \ldots, N$, and $P_j(z) \equiv 1, j > N$, which gives $\|P_j\|_E > 0,\ j = 1, \ldots, m$, but $\| \prod_{j=1}^m P_j\|_E = 0$.

For infinite countable sets E we have $\operatorname{cap}(E) = 0$, and the constants in the inequalities for norms of products of polynomials may grow arbitrarily fast.

Theorem 1.2 *Let $\{A_n\}_{n=1}^\infty$ be any increasing sequence satisfying $A_n \geq 1$. There exists an infinite countable set E such that*

$$\sup_{P_j} \frac{\prod_{j=1}^m \|P_j\|_E}{\| \prod_{j=1}^m P_j\|_E} \geq A_n, \quad n = \deg\left(\prod_{j=1}^m P_j\right) \in \mathbb{N}. \tag{1.15}$$

Thus one should expect faster-than-exponential growth of constants, if the assumption $\operatorname{cap}(E) > 0$ is lifted.

2 Main results

Our first inequality stated in Theorem 2.1 includes the constant C_E that is independent of the number of potentials m. In fact, Theorem 2.1 may be deduced from our Theorem 2.4, which takes m into account, and gives a sharp version of (1.1).

Theorem 2.1 *Let $E \subset \mathbb{C}$ be a compact set,* $\operatorname{cap}(E) > 0$. *Suppose that $\nu_j, j = 1, \ldots, m$, are positive compactly supported Borel measures with potentials p_j, such that the total mass of $\sum_{j=1}^m \nu_j$ is equal to 1. We have*

$$\sum_{j=1}^m \sup_E p_j \leq C_E + \sup_E \sum_{j=1}^m p_j, \tag{2.1}$$

where

$$C_E := \int \log d_E(z) d\mu_E(z) - \log \operatorname{cap}(E) \tag{2.2}$$

cannot be replaced by a smaller constant.

Since C_E is independent of m, it is possible to extend (2.1) to infinite sums of potentials. One should ensure the absolute convergence of the series $\sum_{j=1}^\infty p_j$ on E for this purpose.

We note that C_E is invariant under the similarity transforms of the plane. It is obvious from (1.13) that $C_E = \log M_E$. Hence the results of [2, 16, 17] apply here, and we obtain the following.

Corollary 2.2 *Let $E \subset \mathbb{C}$ be an arbitrary compact set,* $\operatorname{cap}(E) > 0$. *Then $C_E \geq \log 2$, where equality holds if and only if $\partial U \subset E \subset U$, where U is a closed disk.*

Corollary 2.3 *Let $E \subset \mathbb{C}$ be a connected compact set, but not a single point. Suppose that $z, w \in E$ satisfy* $\operatorname{diam} E = |z - w|$ *and the line segment $[z, w]$ joining z to w lies in E. If E is contained in the disk with diameter $[z, w]$, then*

$$C_E \leq \frac{2}{\pi} \int_0^2 \frac{\log(2+x)}{\sqrt{4-x^2}}\, dx = C_{[-2,2]} \approx \log 3.2099123.$$

Furthermore, this inequality holds for any centrally symmetric continuum E that contains its center of symmetry.

We conjecture in line with (1.14) (see [15, 16]) that $C_E \leq C_{[-2,2]}$ for all non-degenerate continua E.

We now explore the dependence of $C_E(m)$ in (1.1) on m. The key results for a polynomial analog are due to Boyd [4, 5] for the unit disk, see (1.9)-(1.10). The polynomial case for general sets was touched upon in [14], and developed further in [17].

A closed set $S \subset E$ is called *dominant* if

$$d_E(z) = \max_{t \in S} |z - t| \quad \text{for all } z \in \operatorname{supp}(\mu_E). \tag{2.3}$$

When E has at least one finite dominant set, we define a *minimal dominant* set $\mathfrak{D}_E$ as a dominant set with the smallest number of points $\operatorname{card}(\mathfrak{D}_E)$. Of course, E might not have finite dominant sets at all, in which case we can take any dominant set as the minimal dominant set with $\operatorname{card}(\mathfrak{D}_E) = \infty$, e.g., $\mathfrak{D}_E = \partial E$.

Theorem 2.4 *Let $E \subset \mathbb{C}$ be compact,* $\operatorname{cap}(E) > 0$. *Suppose that $\nu_j, j = 1, \ldots, m$, are positive compactly supported Borel measures with potentials p_j, such that the total mass of $\sum_{j=1}^m \nu_j$ is equal to 1. Then*

$$\sum_{j=1}^m \sup_E p_j \leq C_E(m) + \sup_E \sum_{j=1}^m p_j, \tag{2.4}$$

where

$$C_E(m) := \max_{c_k \in \partial E} \int \log \max_{1 \leq k \leq m} |z - c_k|\, d\mu_E(z) - \log \operatorname{cap}(E) \tag{2.5}$$

cannot be replaced by a smaller constant for each fixed $m \geq 2$. Furthermore, if $m < \operatorname{card}(\mathfrak{D}_E)$ then $C_E(m) < C_E$, while $C_E(m) = C_E$ for $m \geq \operatorname{card}(\mathfrak{D}_E)$. When $\mathfrak{D}_E$ is infinite, $C_E(m) < C_E$ holds for all $m \in \mathbb{N},\ m \geq 2$.

Since $|z - c_k| \leq d_E(z),\ c_k \in \partial E$, it is clear from (2.2) and (2.5) that $C_E(m) \leq C_E$ for all E and all $m \in \mathbb{N}$. Thus Theorem 2.1 is an immediate consequence of Theorem 2.4. If the sets $\{c_k\}_{k=1}^m$ are dense in ∂E as $m \to \infty$, then $\lim_{m\to\infty} \max_{1 \leq k \leq m} |z - c_k| = d_E(z),\ z \in \mathbb{C}$. Hence $\lim_{m\to\infty} C_E(m) = C_E$. However, the following result shows that we always have strict inequality for smooth sets.

Corollary 2.5 *If $E \subset \mathbb{C}$ is a compact set bounded by finitely many C^1-smooth curves, then $C_E(m) < C_E$ for all $m \in \mathbb{N},\ m \geq 2$.*

On the other hand, we have $C_E(m) = C_E$ for $m \geq s$ for every polygon with s vertices. Furthermore, not all vertices may belong to the minimal dominating set. For example, if E is an obtuse triangle, then $\mathfrak{D}_E$ consists of only two vertices that are the endpoints of the longest side. Hence $C_E(m) = C_E$ for $m \geq 2$ as in the segment case. Any circular arc of angular measure at most π has its endpoints as the minimal dominating set, which gives $C_E(m) = C_E$ for $m \geq 2$ here too. However, if the angular measure of this arc is greater than π, then one immediately obtains that $\mathfrak{D}_E$ is infinite, and $C_E(m) < C_E$ for all $m \geq 2$.

Finding the exact values of $C_E(m)$ for general sets is very complicated. It is analogous to finding solutions of discrete energy problems. Following Boyd [4, 5], we give the values of $C_D(m)$, where D is a disk, see (1.9)-(1.10).

Corollary 2.6 *If E is a closed disk D, then*

$$C_D(m) = \frac{m}{\pi} \int_0^{\pi/m} \log\left(2\cos\frac{t}{2}\right) dt, \quad m \geq 2.$$

It is easy to see that $C_D(m) < C_D = \log 2,\ m \geq 2$.

We conclude this section with an application of our results for potentials to generalized polynomials of the form $P_j(z) = \prod_{k=1}^{k_j} |z - z_{k,j}|^{r_k}$, where $k_j \in \mathbb{N}$ and $z_{k,j} \in \mathbb{C},\ r_k > 0,\ k = 1, \ldots, k_j$. Let $n_j := \sum_{k=1}^{k_j} r_k$ be the degree of the generalized polynomial P_j.

Corollary 2.7 *Let $E \subset \mathbb{C}$ be a compact set,* $\mathrm{cap}(E) > 0$. *If $P_j,\ j = 1, \ldots, m$, are the generalized polynomials of the corresponding degrees n_j, then*

$$\prod_{j=1}^m \|P_j\|_E \leq e^{nC_E(m)} \left\| \prod_{j=1}^m P_j \right\|_E \leq e^{nC_E} \left\| \prod_{j=1}^m P_j \right\|_E,$$

where $n = \sum_{j=1}^m n_j$, and where C_E and $C_E(m)$ are defined by (2.2) *and* (2.5) *respectively.*

We remind the reader that $C_E = \log M_E$, so that the above corollary extends Theorem 1.1.

3 Weighted polynomials and potentials

In this section, we assume that $E \subset \mathbb{C}$ is any closed set, which is not necessarily bounded. Let $w : E \to [0, \infty)$ be an *admissible weight function* [19, p. 26] in the sense of potential theory with external fields. This means that

- w is upper semicontinuous on E
- $\mathrm{cap}\left(\{z \in E : w(z) > 0\}\right) > 0$

- If E is unbounded then $\lim_{|z|\to\infty, z\in E} |z|w(z) = 0$

It is implicit that $\mathrm{cap}(E) > 0$ in this case. We study certain analogs of our main results for weighted polynomials of the form $w^k(z)P(z)$, $\deg(P) \le k$, as well as for potentials with external fields. In order to state such analogs, we need the notions of the weighted equilibrium measure μ_w and the modified Robin's constant F_w. Recall that μ_w is a positive unit Borel measure supported on a compact set $S_w \subset E$, that is characterized by the inequalities

$$\int \log|z-t|\,d\mu_w(t) + \log w(z) + F_w \ge 0, \quad z \in S_w = \mathrm{supp}(\mu_w),$$

and

$$\int \log|z-t|\,d\mu_w(t) + \log w(z) + F_w \le 0, \quad \text{for q.e. } z \in E,$$

where q.e. (quasi everywhere) means that the above inequality holds with a possible exceptional set of zero capacity (cf. Theorem 1.3 of [19, p. 27]). We refer to [19] for a detailed survey of potential theory with external fields. The *weighted farthest-point distance function*

$$d_E^w(z) := \sup_{t\in E} w(t)|z-t|, \quad z \in \mathbb{C}, \tag{3.1}$$

plays an important role in our results, resembling the role of its predecessor $d_E(z)$ defined in (1.12).

Theorem 3.1 *Let $E \subset \mathbb{C}$ be a closed set, and let w be an admissible weight on E. If $P_j,\ j = 1, \ldots, m$, are polynomials of the corresponding degrees n_j, then*

$$\prod_{j=1}^m \|w^{n_j} P_j\|_E \le e^{nC_E^w(m)} \left\| w^n \prod_{j=1}^m P_j \right\|_E \le e^{nC_E^w} \left\| w^n \prod_{j=1}^m P_j \right\|_E, \tag{3.2}$$

where $n = \sum_{j=1}^m n_j$. The constant

$$C_E^w(m) := \sup_{c_k\in E} \int \log \max_{1\le k\le m} w(c_k)|z-c_k|\,d\mu_w(z) + F_w \tag{3.3}$$

cannot be replaced by a smaller value for each fixed $m \ge 2$. Also,

$$C_E^w := \int \log d_E^w(z)\,d\mu_w(z) + F_w \tag{3.4}$$

cannot be replaced by a smaller value independent of m.

If w is continuous and E has positive capacity at each of its points, then any weighted polynomial of the form $w^k P$, $\deg(P) \le k$, attains its norm on S_w, which is often a *proper* subset of E (cf. [13] and Section III.2 of [19]). More generally, the norm is always attained on $S_w \cup \overline{R}_w \subset E$, where $R_w := \{z \in E : \int \log|z-t|\,d\mu_w(t) + \log w(z) + F_w > 0\}$, see Theorem

2.7 of [19, p. 158]. Thus all sup norms in Theorem 3.1 may be replaced by the norms on $S_w \cup \overline{R}_w$. As a consequence for the weighted distance function d_E^w, we observe that for any $z \in \mathbb{C}$ there exists $\zeta_z \in S_w \cup \overline{R}_w$ such that

$$d_E^w(z) = \|w(\cdot)(z-\cdot)\|_E = \|w(\cdot)(z-\cdot)\|_{S_w\cup\overline{R}_w} = w(\zeta_z)|z-\zeta_z|.$$

We give a couple of examples of the constant C_E^w for specific sets and weights below. It is clear that any example of this kind heavily depends on the knowledge of the weighted equilibrium measure μ_w and the modified Robin's constant F_w. In addition, one should be able to compute the weighted distance function d_E^w.

Examples

1. Incomplete polynomials of G. G. Lorentz: Let $E = [0,1]$ and $w(x) = x$. It is known that

$$d\mu_w(x) = \frac{2}{\pi x}\sqrt{\frac{x-1/4}{1-x}}, \quad x \in S_w = [1/4, 1],$$

see [19, p. 243]. We also have that $F_w = 8\log 2 - 3\log 3$ by [19, p. 206]. Furthermore, it follows from a direct calculation that

$$d_E^w(x) = \begin{cases} 1-x, & 1/4 \le x \le 2(\sqrt{2}-1), \\ x^2/4, & 2(\sqrt{2}-1) \le x \le 1. \end{cases}$$

The approximate numerical value obtained from (3.4) is $C_E^w \approx 1.037550517$, so that (3.2) gives

$$\prod_{j=1}^m \|x^{n_j}P_j(x)\|_{[0,1]} \le (2.8222954)^n \left\| x^n \prod_{j=1}^m P_j(x)\right\|_{[0,1]},$$

where $\deg P_j \le n_j$ and $n = n_1 + \ldots + n_m$. (The polynomials $x^{n_j}P_j(x)$ are special examples of incomplete polynomials, a subject that was introduced by G. G. Lorentz in [11].) Note that the above inequality is a significant improvement of the Borwein-Kneser inequality (1.4)-(1.5) applied to the polynomials $x^{n_j}P_j(x)$ on $[0,1]$. Indeed, since the degree of $x^{n_j}P_j(x)$ equals $2n_j$, we obtain from (1.4)-(1.5) (or from (1.13)) that

$$\prod_{j=1}^m \|x^{n_j}P_j(x)\|_{[0,1]} \le (10.303537)^n \left\| x^n \prod_{j=1}^m P_j(x)\right\|_{[0,1]},$$

where the constant comes from $(M_{[0,1]})^2 \approx (3.2099123)^2 < 10.303537$.

2. Let $E = \mathbb{C}$ and $w(z) = e^{-|z|}$. In this case, we have [19, p. 245] that

$$d\mu_w(re^{i\theta}) = \frac{1}{2\pi}\, dr\, d\theta, \quad r \in [0,1],\ \theta \in [0, 2\pi),$$

$d_E^w(z) = e^{|z|-1}$ for $z \in S_w = \{z : |z| \le 1\}$, and $F_w = 1$. Here we explicitly find that $C_E^w = 1/2$ and consequently, from (3.2),

$$\prod_{j=1}^m \|e^{-n_j|z|}P_j(z)\|_{\mathbb{C}} \le e^{n/2} \left\| e^{-n|z|} \prod_{j=1}^m P_j(z)\right\|_{\mathbb{C}}.$$

With the notation of Theorems 2.1-2.4, we let $\alpha_j := \nu_j(\mathbb{C})$ be the total mass of the measure ν_j. For the potentials with external fields $p_j(z) + \alpha_j \log w(z)$, we have the following estimates.

Theorem 3.2 *Let $E \subset \mathbb{C}$ be a closed set, and let w be an admissible weight on E. Suppose that $\nu_j, j = 1, \ldots, m$, are positive compactly supported Borel measures with potentials p_j, such that $\nu_j(\mathbb{C}) = \alpha_j$ and $\sum_{j=1}^m \alpha_j = 1$. Then*

$$\sum_{j=1}^{m} \sup_E (\alpha_j \log w + p_j) \le C_E^w(m) + \sup_E \left(\log w + \sum_{j=1}^{m} p_j \right)$$
$$\le C_E^w + \sup_E \left(\log w + \sum_{j=1}^{m} p_j \right). \tag{3.5}$$

The constants C_E^w and $C_E^w(m)$ are defined by (3.3) *and* (3.4) *respectively. They are sharp here in the same sense as in Theorem 3.1.*

Using a well known connection between polynomials and potentials of discrete measures, we observe that Theorem 3.1 is a direct consequence of Theorem 3.2. For each polynomial $P_j(z) = \prod_{k=1}^{n_j} (z - z_{k,j}),\ j = 1, \ldots, m$, we associate the zero counting measure $\nu_j := \frac{1}{n} \sum_{k=1}^{n_j} \delta_{z_{k,j}}$. Since

$$\frac{1}{n} \log |P_j(z)| = \int \log |z - t|\, d\nu_j(t) = p_j(z) \quad \text{and} \quad \frac{1}{n} \log \|w^{n_j} P_j\|_E = \sup_E \left(\frac{n_j}{n} \log w + p_j \right),$$

it is clear that (3.5) gives the log of (3.2) in this notation. Another immediate observation is that Theorem 3.2 implies (2.1) and (2.4), if we set $w \equiv 1$ on E.

The key ingredient in our proofs of Theorems 3.1 and 3.2 is the following Riesz representation for $\log d_E^w(z)$, which may be of independent interest.

Theorem 3.3 *Let $E \subset \mathbb{C}$ be a closed set. Suppose that $w : E \to [0, \infty)$ is upper semi-continuous on E, and that $w \not\equiv 0$ on E. If E is unbounded then we also assume that $\lim_{|z| \to \infty, z \in E} |z| w(z) = 0$. The function*

$$\log d_E^w(z) := \sup_{t \in E} (\log w(t) + \log |z - t|) = \log \|w(\cdot)(z - \cdot)\|_E, \quad z \in \mathbb{C}, \tag{3.6}$$

is subharmonic in $\mathbb{C}$, and

$$\log d_E^w(z) = \int \log |z - t|\, d\sigma_E^w(t) + \sup_E \log w, \quad z \in \mathbb{C}, \tag{3.7}$$

where σ_E^w is a positive unit Borel measure.

Note that we relaxed conditions on the weight w in Theorem 3.3 by not requiring the set $\{z \in E : w(z) > 0\}$ be of positive capacity. Such weights are called quasi-admissible in [19]. Since the proofs of Theorems 3.1 and 3.3 only require (3.7) for a finite set $E = \{c_k\}_{k=1}^m$, we give a short and transparent proof of this special case. The complete general proof of Theorem 3.3 will appear in our forthcoming work, together with a comprehensive study of the weighted distance function.

We remark that d_E^w is Lipschitz continuous in $\mathbb{C}$, which readily follows from triangle inequality. Indeed, we have that $|z_1 - t| \le |z_1 - z_2| + |z_2 - t|$ for all $z_1, z_2 \in \mathbb{C}$ and all $t \in E$. Hence

$$d_E^w(z_1) = \sup_{t\in E} w(t)|z_1 - t| \le |z_1 - z_2| \sup_{t\in E} w(t) + \sup_{t\in E} w(t)|z_2 - t| = |z_1 - z_2| \sup_E w + d_E^w(z_2)$$

and

$$|d_E^w(z_2) - d_E^w(z_1)| \le |z_2 - z_1| \sup_E w,$$

after interchanging z_1 and z_2. If the set $\{z \in E : w(z) > 0\}$ is not a single point, then d_E^w is strictly positive in $\mathbb{C}$, and $\log d_E^w$ is also Lipschitz continuous in $\mathbb{C}$. In particular, this always holds for admissible weights.

4 Proofs

Proof of Theorem 1.2. Without loss of generality we assume that $A_n \to \infty$ as $n \to \infty$. Consider the sequence $x_1 = 1$ and $x_n = 1/(2A_n)$, $n \ge 2$, and let $E := \{x_n\}_{n=1}^\infty \cup \{0\}$. Thus E is a compact subset of $[0,1]$. Define $P_j(x) := x - x_j$, $j \in \mathbb{N}$. Note that $\|P_1\|_E = 1$, $\|P_j\|_E = 1 - x_j \ge \frac{1}{2}$, $j \ge 2$, and

$$\left\| \prod_{j=1}^n P_j \right\|_E = \prod_{j=1}^n x_j \le \frac{1}{2^{n-1}A_n}, \quad n \in \mathbb{N}.$$

Hence

$$\frac{\prod_{j=1}^n \|P_j\|_E}{\left\|\prod_{j=1}^n P_j\right\|_E} = \frac{\prod_{j=2}^n (1 - x_j)}{\prod_{j=1}^n x_j} \ge \frac{2^{-n+1}}{2^{-n+1}A_n^{-1}} = A_n.$$

■

Proofs of Theorems 2.1 and 2.4. Since any subharmonic potential p_k is upper semicontinuous, it attains a supremum on the compact set E. Furthermore, we can assume that the supremum is attained on ∂E, by the maximum principle for subharmonic functions [18, p. 29]. Thus for any $k = 1, \ldots, m$, there exists $c_k \in \partial E$ such that

$$\sup_E p_k = p_k(c_k).$$

Applying Lemma 2 of [5] to the set $\{c_k\}_{k=1}^m$, we obtain for the function

$$d_m(z) := \max_{1\le k\le m} |z - c_k|, \qquad z \in \mathbb{C},$$

that

$$\log d_m(z) = \int \log|z-t|d\sigma_m(t), \qquad z \in \mathbb{C},$$

where σ_m is a probability measure on $\mathbb{C}$. Let $\nu := \sum_{k=1}^m \nu_k$, so that ν is a unit measure with the potential $p(z) = \int |z-t|\, d\nu(t) = \sum_{k=1}^m p_k(z)$. We use the integral representation of $\log d_m$ and Fubini's theorem in the following estimate:

$$\begin{aligned}\sum_{k=1}^m \sup_E p_k &= \sum_{k=1}^m p_k(c_k) = \sum_{k=1}^m \int \log|c_k - z|\, d\nu_k(z) \le \int \log d_m(z)\, d\nu(z) \\ &= \int\int \log|z-t|\, d\sigma_m(t)\, d\nu(z) = \int p(t)\, d\sigma_m(t). \end{aligned} \tag{4.1}$$

It is known [14] that the support of σ_m is unbounded, so that we need to estimate the growth of p in the plane by its supremum on the set E. This is analogous to the Bernstein-Walsh lemma for polynomials [18, p. 156]. Let $g(t) := \displaystyle\int \log|t-z|\, d\mu_E(z) - \log \mathrm{cap}(E),\ t \in \mathbb{C}$, and note that $g(t) \ge 0,\ t \in E$, by Frostman's theorem [18, p. 59]. On the other hand, we trivially have that $p(t) - \sup_E p \le 0,\ t \in E$, which gives

$$g(t) \ge p(t) - \sup_E p,\ t \in E.$$

By the Principle of Domination (see [19, p. 104]), we deduce that the last inequality holds everywhere:

$$p(t) \le \sup_E p + g(t),\ t \in \mathbb{C}.$$

This inequality applied in (4.1) yields

$$\begin{aligned}\sum_{k=1}^m \sup_E p_k &\le \int \left(\sup_E p + g(t)\right) d\sigma_m(t) = \sup_E p + \int g(t)\, d\sigma_m(t) \\ &= \sup_E p + \int \left(\int \log|z-t|\, d\mu_E(z) - \log \mathrm{cap}(E)\right) d\sigma_m(t) \\ &= \sup_E p + \int\int \log|z-t|\, d\mu_E(z)\, d\sigma_m(t) - \log \mathrm{cap}(E) \\ &= \sup_E p + \int\int \log|z-t|\, d\sigma_m(t)\, d\mu_E(z) - \log \mathrm{cap}(E) \\ &= \sup_E p + \int \log d_m(z) d\mu_E(z) - \log \mathrm{cap}(E),\end{aligned}$$

where we consecutively used $\sigma_m(\mathbb{C}) = 1$, the representation of g via the potential of μ_E, Fubini's theorem, and the integral representation for $\log d_m$. Hence (2.4) follows from the above estimate by taking maximum over all possible m-tuples of $c_k \in \partial E,\ k = 1, \ldots, m$. (Note that $\log d_m$ is a continuous function in the variables c_k, so that $\displaystyle\int \log d_m(z) d\mu_E(z)$ is continuous too.) Furthermore, (2.1) is immediate after observing that $d_m(z) \le d_E(z),\ z \in \mathbb{C}$.

Suppose that $\int \log d_m(z) d\mu_E(z)$ attains its maximum on $(\partial E)^m$ for some set $c_k^* \in \partial E,\ k = 1, \ldots, m$. We now show that $C_E(m)$ cannot be replaced by a smaller constant for a fixed $m \geq 2$. Let

$$d_m^*(z) := \max_{1\le k\le m} |z - c_k^*|, \quad z \in \mathbb{C},$$

and define the sets

$$S_1 := \{z \in \operatorname{supp} \mu_E : |z - c_1^*| = d_m^*(z)\}$$

and

$$S_k := \{z \in \operatorname{supp} \mu_E \setminus \cup_{j=1}^{k-1} S_j : |z - c_k^*| = d_m^*(z)\}, \quad k = 2, \ldots, m.$$

It is clear that

$$\operatorname{supp} \mu_E = \bigcup_{k=1}^m S_k \quad \text{and} \quad S_k \bigcap S_j = \emptyset,\ k \neq j.$$

Hence the measures $\nu_k^* := \mu_E|_{S_k}$ give the decomposition

$$\mu_E = \sum_{k=1}^m \nu_k^*.$$

If E is regular, then $\int \log |z - t|\, d\mu_E(z) = \log \operatorname{cap}(E),\ t \in E$, by Frostman's theorem [18, p. 59]. Thus we obtain that

$$\begin{aligned}\sum_{k=1}^m \sup_E p_k^* &\geq \sum_{k=1}^m p_k^*(c_k^*) = \sum_{k=1}^m \int \log |c_k^* - z|\, d\nu_k^*(z) = \sum_{k=1}^m \int \log d_m^*(z)\, d\nu_k^*(z)\\ &= \int \log d_m^*(z)\, d\mu_E(z) - \log \operatorname{cap}(E) + \sup_{t\in E} \int \log |z - t|\, d\mu_E(z).\\ &= C_E(m) + \sup_E \sum_{k=1}^m p_k^*.\end{aligned}$$

Hence equality holds in (2.4) in this case.

An alternative proof that $C_E(m)$ cannot be replaced by a smaller constant for any set E (that does not require E to be regular) may be given by using the n-th Fekete points $\mathcal{F}_n = \{a_{l,n}\}_{l=1}^n$ of E [18, p. 152]. Let $\{c_k^*\}_{k=1}^m$ be the maximizers of $\int \log d_m(z) d\mu_E(z)$ on $(\partial E)^m$, as before. We define a subset $\mathcal{F}_{k,n} \subset \{a_{l,n}\}_{l=1}^n$, associated with each point $c_k^*,\ k = 1, \ldots, m$, so that $a_{l,n} \in \mathcal{F}_{k,n}$ if

$$d_m^*(a_{l,n}) = |a_{l,n} - c_k^*|, \quad 1 \leq l \leq n. \tag{4.2}$$

In the case that (4.2) holds for more than one c_k^*, we assign $a_{l,n}$ to only one set $\mathcal{F}_{k,n}$, to avoid an overlap of these sets. It is clear that, for any $n \in \mathbb{N}$,

$$\bigcup_{k=1}^m \mathcal{F}_{k,n} = \mathcal{F}_n \quad \text{and} \quad \mathcal{F}_{k_1,n} \bigcap \mathcal{F}_{k_2,n} = \emptyset,\ k_1 \neq k_2.$$

Define the measures

$$\nu_{k,n}^* := \frac{1}{n} \sum_{a_{l,n} \in \mathcal{F}_{k,n}} \delta_{a_{l,n}},$$

so that for their potentials

$$p_{k,n}^*(z) = \frac{1}{n} \sum_{a_{l,n} \in \mathcal{F}_{k,n}} \log|z - a_{l,n}|, \quad k = 1, \ldots, m,$$

we have

$$\sup_E p_{k,n}^* \geq \frac{1}{n} \sum_{a_{l,n} \in \mathcal{F}_{k,n}} \log|c_k^* - a_{l,n}| = \frac{1}{n} \sum_{a_{l,n} \in \mathcal{F}_{k,n}} \log d_m^*(a_{l,n}), \quad k = 1, \ldots, m.$$

It follows from the weak* convergence of $\nu_n^* := \sum_{k=1}^m \nu_{k,n}^* = \frac{1}{n} \sum_{l=1}^n \delta_{a_{l,n}}$ to μ_E, as $n \to \infty$, that

$$\begin{aligned}
\liminf_{n\to\infty} \sum_{k=1}^m \sup_E p_{k,n}^* &\geq \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^n \log d_m^*(a_{k,n}) \\
&= \int \log d_m^*(z) d\mu_E(z).
\end{aligned}$$

Also, we have for the potential p_n^* of ν_n^* that [18, p. 155]

$$\lim_{n\to\infty} \sup_E p_n^* = \lim_{n\to\infty} \log \left\| \prod_{k=1}^n (z - a_{k,n}) \right\|_E^{1/n} = \log \operatorname{cap}(E),$$

which gives

$$\liminf_{n\to\infty} \sum_{k=1}^m \sup_E p_{k,n}^* \geq C_E(m) + \lim_{n\to\infty} \sup_E p_n^*.$$

Hence (2.4) turns into asymptotic equality as $n \to \infty$, with $m \geq 2$ being fixed.

A similar argument with Fekete points shows that (2.1) turns into asymptotic equality when $m = n \to \infty$.

Since $d_m(z) \leq d_E(z)$ for any $z \in \mathbb{C}$, we immediately obtain that $C_E(m) \leq C_E$. Suppose that $m < \operatorname{card}(\mathfrak{D}_E)$. Then there is $z_0 \in \operatorname{supp}(\mu_E)$ such that $d_m^*(z_0) < d_E(z_0)$. As both functions are continuous, the same strict inequality holds in a neighborhood of z_0, so that $\int \log d_m^*(z)\, d\mu_E(z) < \int \log d_E(z)\, d\mu_E(z)$ and $C_E(m) < C_E$. When $\mathfrak{D}_E$ is infinite, this argument gives that $C_E(m) < C_E,\ m \geq 2$. Assume now that $\mathfrak{D}_E$ is finite and that $m \geq \operatorname{card}(\mathfrak{D}_E)$. Then $d_m^*(z) = d_E(z)$ for all $z \in \operatorname{supp}(\mu_E)$, because one of the possible choices of the points $\{c_k\}_{k=1}^m \subset \partial E$ includes points of the set $\mathfrak{D}_E$. It is immediate that $\int \log d_m^*(z)\, d\mu_E(z) = \int \log d_E(z)\, d\mu_E(z)$ and $C_E(m) = C_E$ in this case.

■

Proof of Corollary 2.2. Use $C_E = \log M_E$ and apply Theorem 2.5 of [16].

■

Proof of Corollary 2.3. The first part when E is contained in the disk with diameter $[z, w]$ follows from $C_E = \log M_E$ and Corollary 2.2 of [16]. The second part for centrally symmetric E is a consequence of Corollary 6.3 from [2].

■

Proof of Corollary 2.5. We need to show that the minimal dominant set is infinite, hence the result follows from Theorem 2.4. Suppose to the contrary that $\mathfrak{D}_E = \{\zeta_l\}_{l=1}^s$ is finite. Let $J \subset \partial E$ be a smooth closed Jordan curve. Define

$$J_l := \{z \in J : d_E(z) = |z - \zeta_l|\}. \quad l = 1, \ldots, s.$$

It is clear that $J = \cup_{l=1}^s J_l$. Observe that the segment $[z, \zeta_l],\ z \in J_l$, is orthogonal to ∂E at ζ_l. Hence each J_l is contained in the normal line to ∂E at $\zeta_l,\ l = 1, \ldots, s$. We thus obtain that J is contained in a union of s straight lines, so that J cannot have a continuously turning tangent, which contradicts the smoothness assumption.

■

Proof of Corollary 2.6. Apply Theorem 1 of [5], and use that $C_D(m) = \log C_m$, where C_m is given in (1.10).

■

Proof of Corollary 2.7. For $P_j(z) = \prod_{k=1}^{k_j} |z - z_{k,j}|^{r_k}$, define the zero counting measures

$$\nu_j = \frac{1}{n} \sum_{k=1}^{k_j} r_k \delta_{z_{k,j}}, \quad j = 1, \ldots, m,$$

where δ_z is a unit point mass at z. We obtain that

$$p_j(z) = \int \log|z - t|\, d\nu_j(t) = \frac{1}{n} \log|P_j(z)|, \quad j = 1, \ldots, m.$$

Hence Corollary 2.7 follows by combining (2.1) and (2.4).

■

Proofs of Theorems 3.1 and 3.2. We follow some ideas used to prove Theorem 1.1 in [14] and Theorems 2.1-2.4 of this paper, augmented with certain necessary facts on weighted potentials and distance function. Note that admissibility of the weight w implies $\lim\limits_{z\to\infty, z\in E} \log w(z) - \log|z| = -\infty$. Combining this observation with upper semicontinuity of $\log w$ and of the potentials p_j, we conclude that there exist points $c_j \in E$ satisfying

$$\sup_E(\alpha_j \log w + p_j) = \alpha_j \log w(c_j) + p_j(c_j), \quad j = 1, \ldots, m.$$

Consider the weighted distance function

$$d_m^w(z) := \max_{1\le j\le m} w(c_j)|z - c_j|, \qquad z \in \mathbb{C},$$

and write by Theorem 3.3 (with $E = \{c_j\}_{j=1}^m$ there) that

$$\log d_m^w(z) = \int \log|z-t| d\sigma_m^w(t) + \max_{1\le j\le m} \log w(c_j), \qquad z \in \mathbb{C}, \tag{4.3}$$

where σ_m^w is a probability measure on $\mathbb{C}$. Consider the unit measure $\nu := \sum_{k=1}^m \nu_k$ and its potential $p(z) = \displaystyle\int |z-t|\, d\nu(t) = \sum_{k=1}^m p_k(z)$. Using (4.3) and Fubini's theorem, we have

$$\begin{aligned}
\sum_{j=1}^m \sup_E(\alpha_j \log w + p_j) &= \sum_{j=1}^m (\alpha_j \log w(c_j) + p_j(c_j)) \\
&= \sum_{j=1}^m \left(\alpha_j \log w(c_j) + \int \log|c_j - z|\, d\nu_j(z)\right) \\
&\le \int \log d_m^w(z)\, d\nu(z) \\
&= \int\int \log|z-t|\, d\sigma_m^w(t)\, d\nu(z) + \max_{1\le j\le m} \log w(c_j) \\
&= \int p(t)\, d\sigma_m^w(t) + \max_{1\le j\le m} \log w(c_j).
\end{aligned} \tag{4.4}$$

We now need an estimate of p in $\mathbb{C}$ via the sup of $\log w + p$ on E. Obviously, $\log w(t) + p(t) \le \sup_E(\log w + p)$ for $t \in S_w$, as $S_w \subset E$. We also know from Theorem 1.3 of [19, p. 27] that

$$\int \log|t-z|\, d\mu_w(z) + F_w \ge -\log w(t), \quad t \in S_w.$$

This gives

$$p(t) \le \sup_E(\log w + p) - \log w(t) \le \sup_E(\log w + p) + \int \log|t-z|\, d\mu_w(z) + F_w, \quad t \in S_w.$$

Hence we have the desired estimate

$$p(t) \le \sup_E(\log w + p) + \int \log|t-z|\, d\mu_w(z) + F_w, \quad t \in \mathbb{C},$$

by the Principle of Domination [19, p. 104]. We proceed with inserting the above inequality into (4.4), and estimate as follows

$$
\begin{aligned}
\sum_{j=1}^{m} \sup_E(\alpha_j \log w + p_j) \le & \int \left(\sup_E(\log w + p) + \int \log|t-z|\, d\mu_w(z) + F_w \right) d\sigma_m^w(t) \\
& + \max_{1\le j\le m} \log w(c_j) \\
& = \sup_E(\log w + p) + F_w + \max_{1\le j\le m} \log w(c_j) \\
& + \int\int \log|z-t|\, d\sigma_m^w(t)\, d\mu_w(z) \\
& = \sup_E(\log w + p) + F_w + \int \log d_m^w(z)\, d\mu_w(z),
\end{aligned}
$$

where we again used $\sigma_m^w(\mathbb{C}) = \mu_w(\mathbb{C}) = 1$, the representation for $\log d_m^w$, and Fubini's theorem. Hence the first inequality in (3.5) follows by taking sup over m-tuples of $c_j \in E,\ j = 1, \ldots, m$. The second inequality is immediate from $d_m^w(z) \le d_E^w(z),\ z \in \mathbb{C}$.

It was explained after the statement of Theorem 3.2 that Theorem 3.1 is its special case. In particular, we have that (3.5) for the zero counting measures ν_j of polynomials P_j implies (3.2). Thus (3.2) is also proved. On the other hand, if we show that the constants $C_E^w(m)$ and C_E^w are sharp in Theorem 3.1, then they are obviously sharp in Theorem 3.2 too. Hence we select this path and prove sharpness for the weighted polynomial case, i.e., for discrete measures in weighted Fekete points.

Since $\log d_m^w(z)$ is an upper semicontinuous function of $c_j \in E,\ j = 1, \ldots, m$, we have that $\displaystyle\int \log d_m^w(z)\, d\mu_E(z)$ is also upper semicontinuous in those variables, and hence attains its maximum on E^m for some set $c_j^* \in E,\ j = 1, \ldots, m$. We now show that $C_E(m)$ cannot be replaced by a smaller constant for each fixed m, by adapting the proof of Theorem 4.1 from [14]. Let

$$
d_m^*(z) := \max_{1\le j\le m} w(c_j^*)|z - c_j^*|, \qquad z \in \mathbb{C}.
$$

Consider the weighted n-th Fekete points $\mathcal{F}_n = \{a_{l,n}\}_{l=1}^n$ for the weight w on E, and the corresponding polynomials (cf. Section III.1 of [19])

$$
F_n(z) = \prod_{l=1}^{n} (z - a_{l,n}), \quad n \in \mathbb{N}.
$$

We define a subset $\mathcal{F}_{j,n} \subset \{a_{l,n}\}_{l=1}^n$ associated with each point $c_j^*,\ j = 1, \ldots, m$, so that $a_{l,n} \in \mathcal{F}_{j,n}$ if

$$
d_m^*(a_{l,n}) = w(c_j^*)|a_{l,n} - c_j^*|, \quad 1 \le l \le n. \tag{4.5}
$$

If (4.5) holds for more than one c_j^*, then we include $a_{l,n}$ into only one set $\mathcal{F}_{j,n}$, to avoid an overlap of these sets. It is clear that, for any $n \in \mathbb{N}$,

$$
\bigcup_{j=1}^{m} \mathcal{F}_{j,n} = \{a_{l,n}\}_{l=1}^n \quad \text{and} \quad \mathcal{F}_{k_1,n} \bigcap \mathcal{F}_{k_2,n} = \emptyset,\ k_1 \ne k_2.
$$

We next introduce the factors of $F_n(z)$ by setting

$$F_{j,n}(z) := \prod_{a_{l,n}\in\mathcal{F}_{j,n}} (z - a_{l,n}), \quad j = 1, \ldots, m,$$

so that

$$\|w^{n_j} F_{j,n}\|_E \geq w^{n_j}(c_j^*) \prod_{a_{l,n}\in\mathcal{F}_{j,n}} |c_j^* - a_{l,n}| = \prod_{a_{l,n}\in\mathcal{F}_{j,n}} d_m^*(a_{l,n}), \quad j = 1, \ldots, m,$$

where $n_j := \deg(F_{j,n})$. Since the normalized counting measures $\nu_{\mathcal{F}_n}$ in the weighted Fekete points converge to the weighted equilibrium measure μ_w in the weak* topology, see Theorem 1.3 in [19, p. 145], it follows that

$$\begin{aligned}
\liminf_{n\to\infty} \left(\prod_{j=1}^{m} \|w^{n_j} F_{j,n}\|_E\right)^{1/n} &\geq \lim_{n\to\infty} \left(\prod_{l=1}^{n} d_m^*(a_{l,n})\right)^{1/n} \\
&= \lim_{n\to\infty} \exp\left(\frac{1}{n}\sum_{j=1}^{n} \log d_m^*(a_{k,n})\right) \\
&= \exp\left(\int \log d_m^*(z)\, d\mu_w(z)\right),
\end{aligned}$$

because $\log d_m^*(z)$ is continuous in $\mathbb{C}$. We also have that $\lim\limits_{n\to\infty} \|w^n F_n\|_E^{1/n} = e^{-F_w}$ by Theorem 1.9 of [19, p. 150], which gives

$$\liminf_{n\to\infty} \left(\frac{\prod_{j=1}^m \|w^{n_j} F_{j,n}\|_E}{\|w^n F_n\|_E}\right)^{1/n} \geq e^{C_E(m)}.$$

To show that C_E cannot be replaced by a smaller constant independent of m, one should essentially repeat the above argument with $m = n \to \infty$.
■

Proof of Theorem 3.3.
We present a proof for the finite set $E = \{c_k\}_{k=1}^m$ here, which is sufficient for applications in the proofs of Theorems 3.1 and 3.3. A proof of the general case will appear in a separate paper.

Let $M := \{z \in E : w(z) = \sup_E w = \max_E w\}$. Our first goal is to show that $d_E^w(z) = d_M(z) \max_E w$ in a neighborhood of infinity. Since E is finite, there exists $\varepsilon > 0$ such that

$$w(z) < \max_E w - \varepsilon, \quad z \in E \setminus M.$$

Suppose that there is a sequence of points $\{z_i\}_{i=1}^\infty$ in the plane such that $\lim_{i\to\infty} z_i = \infty$, and the weighted distance function $d_E^w(z_i)$ is attained at the points of $E \setminus M$ for each $i \in \mathbb{N}$. It follows that

$$d_E^w(z_i) = w(t_i)|z_i - t_i| < \left(\max_E w - \varepsilon\right)\left(|z_i| + \max_{1\leq k\leq m} |c_k|\right),$$

where $t_i \in E \setminus M$. Since $M \subset E$, we have that $d_M^w(z) \le d_E^w(z),\ z \in \mathbb{C}$. Hence

$$d_M^w(z_i) = \max_{t\in M} w(t)|z_i - t| = \max_E w \max_{t\in M} |z_i - t| = d_M(z_i) \max_E w$$
$$\le d_E^w(z_i) < \left(\max_E w - \varepsilon\right)\left(|z_i| + \max_{1\le k\le m} |c_k|\right).$$

If we divide the above inequality by $|z_i|$ and let $|z_i| \to \infty$, then we come to the obvious contradiction $\max_E w \le \max_E w - \varepsilon$. Thus there exists $R > 0$ such that

$$d_E^w(z) = \max_{t\in M} w(t)|z - t| = d_M(z) \max_E w, \quad |z| > R. \tag{4.6}$$

Since $\log w(t) + \log |z - t|$ is a subharmonic function of z in $\mathbb{C}$, it follows that

$$\log d_E^w(z) = \max_{t\in E} (\log w(t) + \log |z - t|), \quad z \in \mathbb{C},$$

is also subharmonic in the plane, cf. [18, p. 38]. Let $D_r := \{z \in \mathbb{C} : |z| < r\}$, and write by the Riesz Decomposition Theorem [18, p. 76]

$$\log d_E^w(z) = \int \log |z - t|\, d\sigma_r^w(t) + h_r(z), \quad z \in D_r,$$

where σ_r^w is a positive Borel measure on D_r, and where h_r is harmonic in D_r. Considering a sequence of disks D_r with $r \to \infty$, we extend σ_r^w to the measure σ_E^w on the whole plane. It is known [5, 14, 10] that

$$\log d_M(z) = \int \log |z - t|\, d\sigma_M(t), \quad z \in \mathbb{C},$$

where σ_M is a probability measure on $\mathbb{C}$. Therefore,

$$\log d_E^w(z) = \log \max_E w + \log d_M(z) = \log \max_E w + \int \log |z - t|\, d\sigma_M(t), \quad |z| > R,$$

by (4.6). For any function u that is subharmonic in $\mathbb{C}$, one can find the Riesz measure of D_r from the formula

$$\mu(D_r) = r \frac{d}{dr} L(u; r)$$

except for at most countably many r, where

$$L(u; r) := \frac{1}{2\pi} \int_0^{2\pi} u(re^{i\theta})\, d\theta,$$

see Theorem 1.2 of [19, p. 84]. We remark that Theorem 1.2 is stated in [19, p. 84] for potentials of compactly supported measures, but the more general version we use here follows immediately by writing the Riesz decomposition of u on any disk into the sum of a potential and a harmonic function. It is clear that

$$L(\log d_E^w; r) = L(\log d_M; r) + \log \max_E w, \quad r > R,$$

so that

$$\sigma_E^w(D_r) = \sigma_M(D_r), \quad r > R,$$

except for at most countably many r. Consequently, $\sigma_E^w(\mathbb{C}) = \sigma_M(\mathbb{C}) = 1$.

We also have for any $r > R$ that

$$\int \log |z-t|\, d\sigma_r^w(t) + h_r(z) = \log \max_E w + \int \log |z-t|\, d\sigma_M(t), \quad R < |z| < r.$$

Applying the Unicity Theorem [19, p. 97], we conclude that the two measures coincide in $R < |z| < r$ for any $r > R$, which gives

$$\sigma_E^w|_{|z|>R} = \sigma_M|_{|z|>R}.$$

This implies that

$$h_r(z) = \log \max_E w + \int_{|t|\leq 2R} \log |z-t|\, d\sigma_M(t) - \int_{|t|\leq 2R} \log |z-t|\, d\sigma_E^w(t), \quad R < |z| < r,$$

for all $r > R$. But the right-hand side of this equation is harmonic and bounded for $|z| > 2R$, with the limit value $\log \max_E w$ at ∞. Thus h_r is continued to a harmonic and bounded function in $\mathbb{C}$, and it must be identically equal to the constant $\log \max_E w$ by Liouville's theorem.

■

References

[1] G. Aumann, *Satz über das Verhalten von Polynomen auf Kontinuen*, Sitz. Preuss. Akad. Wiss. Phys.-Math. Kl. (1933), 926-931.

[2] A. Baernstein II, R. S. Laugesen, and I. E. Pritsker, *Moment inequalities for equilibrium measures*, Pure Appl. Math. Q. (to appear)

[3] P. B. Borwein, *Exact inequalities for the norms of factors of polynomials*, Can. J. Math. *46* (1994), 687-698.

[4] D. W. Boyd, *Two sharp inequalities for the norm of a factor of a polynomial*, Mathematika *39* (1992), 341-349.

[5] D. W. Boyd, *Sharp inequalities for the product of polynomials*, Bull. London Math. Soc. *26* (1994), 449-454.

[6] S. J. Gardiner and I. Netuka, *Potential theory of the farthest-point distance function*, J. Anal. Math. *101* (2006), 163-177.

[7] A. O. Gelfond, *Transcendental and Algebraic Numbers*, Dover, New York, 1960.

[8] H. Kneser, *Das Maximum des Produkts zweies Polynome*, Sitz. Preuss. Akad. Wiss. Phys.-Math. Kl. (1934), 429-431.

[9] A. Kroó and I. E. Pritsker, *A sharp version of Mahler's inequality for products of polynomials*, Bull. London Math. Soc. *31* (1999), 269-278.

[10] R. S. Laugesen and I. E. Pritsker, *Potential theory of the farthest-point distance function*, Can. Math. Bull. *46* (2003), 373-387.

[11] G. G. Lorentz, *Approximation by incomplete polynomials (problems and results)*, in "Padé and Rational Approximation: Theory and Applications," E. B. Saff and R. S. Varga (eds.), pp. 289-302, Academic Press, New York, 1977.

[12] K. Mahler, *An application of Jensen's formula to polynomials*, Mathematika *7* (1960), 98-100.

[13] H. Mhaskar and E. B. Saff, *Where does the sup norm of a weighted polynomial live? (A generalization of incomplete polynomials)*, Constr. Approx. *1* (1985), 71-91.

[14] I. E. Pritsker, *Products of polynomials in uniform norms*, Trans. Amer. Math. Soc. *353* (2001), 3971-3993.

[15] I. E. Pritsker, *Norms of products and factors of polynomials*, in "Number Theory for the Millennium III," M. A. Bennett, B. C. Berndt, N. Boston, H. Diamond, A. J. Hildebrand and W. Philipp (eds.), pp. 173-189, A K Peters, Ltd., Natick, 2002.

[16] I. E. Pritsker and S. Ruscheweyh, *Inequalities for products of polynomials I*, Math. Scand. (to appear)

[17] I. E. Pritsker and S. Ruscheweyh, *Inequalities for products of polynomials II*, Aequationes Math. (submitted)

[18] T. Ransford, *Potential Theory in the Complex Plane*, Cambridge University Press, Cambridge, 1995.

[19] E. B. Saff and V. Totik, *Logarithmic Potentials with External Fields*, Springer-Verlag, Berlin, 1997.

[20] M. Tsuji, *Potential Theory in Modern Function Theory*, Chelsea Publ. Co., New York, 1975.

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