

# How to find a measure from its potential

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*Dedicated to Professor W. K. Hayman on his 80th birthday*

**Abstract.** We consider the problem of finding a measure from the given values of its logarithmic potential on the support. It is well known that a solution to this problem is given by the generalized Laplacian. The case of our main interest is when the support is contained in a rectifiable curve, and the measure is absolutely continuous with respect to the arclength on this curve. Then the generalized Laplacian is expressed by a sum of normal derivatives of the potential. Such representation was available for smooth curves, and we show it holds for any rectifiable curve in the plane. We also relax the assumptions imposed on the potential.

Finding a measure from its potential often leads to another closely related problem of solving a singular integral equation with Cauchy kernel. The theory of such equations is well developed for smooth curves. We generalize this theory to the class of Ahlfors regular curves and arcs, and characterize the bounded solutions on arcs.

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## 1. Logarithmic potentials in the complex plane

Let  $\mu$  be a positive Borel measure with the compact support  $\text{supp } \mu$  in the complex plane  $\mathbb{C}$ . Consider its logarithmic potential

$$(1.1) \quad u(z) := \int \log |z - t| d\mu(t),$$

which is a subharmonic function in  $\mathbb{C}$ , see, e.g., Hayman and Kennedy [13] and Ransford [25]. Furthermore,  $u$  is harmonic in  $\mathbb{C} \setminus \text{supp } \mu$ . Suppose that we know the values of the potential function  $u$  on a certain set. How can one recover the measure from those values? A general answer to this question is well known

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[13, 25] in terms of the generalized (distributional) Laplacian, and is a part of the Riesz Representation Theorem for subharmonic functions (cf. Chapter 3 of [13] and Section 3.7 of [25]).

**Theorem A.** *Suppose that  $\mu$  is a positive Borel measure with compact support  $\text{supp } \mu \subset \mathbb{C}$ , whose potential  $u$  is defined by (1.1). Then the measure is represented by*

$$(1.2) \quad d\mu = \frac{1}{2\pi} \Delta u,$$

where  $\Delta$  is the generalized Laplacian.

While this theorem gives a very general answer, it is not always easy to apply in specific problems. The generalized Laplacian may, in particular, take quite different forms. The most known and natural case is when the generalized Laplacian reduces to the regular one, under the additional  $C^2$  smoothness assumptions on the potential  $u$  (see Theorem 1.3 in Saff and Totik [28, p. 85]).

**Theorem B.** *If  $u$  has continuous second partial derivatives in a domain  $D \subset \mathbb{C}$ , then  $\mu$  is absolutely continuous with respect to the area Lebesgue measure  $dxdy$  in  $D$ , and*

$$(1.3) \quad d\mu(x, y) = \frac{1}{2\pi} \Delta u(z) dxdy, \quad z = x + iy \in D,$$

where  $\Delta$  is the regular Laplacian.

We remark that the  $C^2$  assumption on  $u$  may be relaxed if one follows a standard proof as given in [28], but uses a more general version of Green's theorem found in Shapiro [29] or Cohen [5] (see also Bochner [1]). In particular, it is sufficient to assume in Theorem B that the second partial derivatives of  $u$  exist and are integrable in  $D$  with respect to  $dxdy$ .

Another well known example of the generalized Laplacian is given by a sum of points masses. If  $p(z) = \prod_{j=1}^n (z - a_j)$  is any polynomial and  $u(z) = \log |p(z)|$ , then  $\mu = \sum_{j=1}^n \delta_{a_j}$ , where  $\delta_{a_j}$  is a unit point mass at  $a_j \in \mathbb{C}$  (cf. Theorem 3.7.8 in [25, p. 76]).

We are mostly interested in the case when  $\mu$  is supported on a rectifiable curve (or arc), and is absolutely continuous with respect to the arclength measure on this curve. Then the form of the result expressed through the normal derivatives of potential is also classical, while it is certainly difficult to find the original source. We follow the statement of Theorem 1.5 in [28, p. 92].

**Theorem C.** *Suppose that the intersection of  $\text{supp } \mu$  with a domain  $D \subset \mathbb{C}$  is a simple  $C^{1+\delta}$  arc,  $\delta > 0$ , with the left normal  $n_+$  and the right normal  $n_-$ . If the potential  $u$  is Lip 1 in a neighborhood of this arc then  $\mu$  is absolutely continuous with respect to the arclength  $ds$  on this arc and*

$$(1.4) \quad d\mu(s) = \frac{1}{2\pi} \left( \frac{\partial u}{\partial n_+}(s) + \frac{\partial u}{\partial n_-}(s) \right) ds.$$

Attempts to remove smoothness assumptions imposed on the curve go back to Plemelj [18] and Radon [24], where the idea of flux of a potential along a curve was introduced. This approach was further developed by Burago, Maz'ya and Sapozhnikova [4], and by Burago and Maz'ya [3]. We show that (1.4) remains valid for an arbitrary rectifiable curve, and also relax assumptions on the potential  $u$ . In order to state our assumptions, we need to introduce the Smirnov spaces of analytic and harmonic functions.

Let  $\Gamma \subset \mathbb{C}$  be a closed Jordan rectifiable curve. The complement of  $\Gamma$  in  $\overline{\mathbb{C}}$  is the union of a bounded Jordan domain  $D_+$  and an unbounded Jordan domain  $D_-$ . Consider a conformal mapping  $\psi : \mathbb{D} \rightarrow D_+$ , where  $\mathbb{D} := \{w : |w| < 1\}$ , and define the level curves  $\Gamma_r := \{z = \psi(w) \in D_+ : |w| = r\}$ ,  $0 \leq r < 1$ . A function  $f$  analytic in  $D_+$  is said to belong to the Smirnov space  $E^p(D_+)$ ,  $1 \leq p < \infty$ , if

$$(1.5) \quad \sup_{0 \leq r < 1} \int_{\Gamma_r} |f(z)|^p |dz| < \infty,$$

see Chapter 10 of Duren [9]. These spaces are natural analogs of the Hardy spaces  $H^p$  on the unit disk  $\mathbb{D}$ . Furthermore,  $f \in E^p(D_+)$  if and only if  $f(\psi(w))(\psi'(w))^{1/p} \in H^p$ , cf. [9, p. 169]. A function from  $E^p(D_+)$  has nontangential limit values almost everywhere on  $\Gamma$  with respect to the arclength measure by Theorem 10.3 of [9, p. 170], and these values are in  $L^p(\Gamma, ds)$ . We similarly introduce the harmonic Smirnov space  $e^p(D_+)$  that consists of harmonic functions in  $D_+$  satisfying (1.5). Using a conformal mapping  $\Psi : \mathbb{D} \rightarrow D_-$ , we repeat the same steps to define the corresponding spaces  $E^p(D_-)$  and  $e^p(D_-)$  for the unbounded domain  $D_-$ .

Assume that  $\text{supp } \mu \subset \Gamma$ . Recall that the potential  $u$  is harmonic in  $\mathbb{C} \setminus \text{supp } \mu$ . Hence its partial derivatives  $u_x$  and  $u_y$  are harmonic in  $\mathbb{C} \setminus \text{supp } \mu$  too. We say that  $\nabla u \in e^p(D_\pm)$  if both  $u_x, u_y \in e^p(D_+)$  and  $u_x, u_y \in e^p(D_-)$  hold. Let  $n_+$  be the inner normal vector (pointing into  $D_+$ ), and  $n_-$  be the outer normal vector (pointing into  $D_-$ ) on  $\Gamma$ . The normal direction is well defined on  $\Gamma$  for almost every point with respect to the arclength measure. Hence we can define the directional derivatives  $\partial u / \partial n_+$  in  $D_+$  and  $\partial u / \partial n_-$  in  $D_-$ . The corresponding boundary limit values for these normal derivatives exist a.e. on  $\Gamma$ , as we show in the proof of the theorem stated below. Thus (1.4) is understood in the sense of such boundary values.

**Theorem 1.1.** *Let  $\Gamma \subset \mathbb{C}$  be an arbitrary rectifiable Jordan curve. Suppose that  $\text{supp } \mu \subset \Gamma$  and  $u$  is the potential of  $\mu$  defined by (1.1). If  $\nabla u \in e^1(D_\pm)$  then  $\mu$  is absolutely continuous with respect to the arclength  $ds$  on  $\Gamma$  and (1.4) holds.*

We present a ‘‘complex function theory’’ proof of Theorem 1.1 in Section 3, which is based on the Cauchy transform of  $\mu$ . It is possible to extend Theorem C by following the conventional proof of [28], and by employing a version of Green’s theorem from [29]. However, this gives a less general result than the one in

Theorem 1.1. Note that if  $u$  is Lipschitz continuous in an open neighborhood  $G$  of  $\text{supp } \mu$ , then the partial derivatives of  $u$  are bounded on  $G \setminus \Gamma$  by the Lipschitz constant. But they are also bounded in a neighborhood of  $\Gamma \setminus G$ , being harmonic outside  $\text{supp } \mu$ . Hence the assumptions of Theorem C imply that  $u_x$  and  $u_y$  are bounded in both domains  $D_+$  and  $D_-$  by the maximum principle. It immediately follows that  $\nabla u \in e^1(D_\pm)$ , i.e. our assumption on  $u$  is indeed weaker.

We also remark that Theorem 1.1 has local nature, in fact. If the intersection of  $\text{supp } \mu$  with a domain  $G \subset \mathbb{C}$  is a Jordan arc  $\gamma$ , then we write  $\mu_1 := \mu|_\gamma$  and  $\mu_2 := \mu|_{\text{supp } \mu \setminus \gamma}$ . Hence

$$u(z) = \int \log |z - t| d\mu_1(t) + \int \log |z - t| d\mu_2(t),$$

and Theorem 1.1 is applicable to the first potential in the above equation for recovering the measure on  $\gamma$ , provided  $u$  satisfies the assumptions. Note that the second potential is harmonic in  $G$ , so that it gives no contribution to the measure on  $\gamma$  by Theorem A (or B, or C).

One of the most natural applications for Theorem 1.1 is to the equilibrium potential of a compact set  $E \subset \mathbb{C}$ . If  $E$  is not polar, then the equilibrium measure  $\mu_E$  exists and is a unique positive Borel measure of mass one, whose potential  $u_E$  is equal to a constant  $V_E$  everywhere on  $E$ , with a possible exception of a polar subset (cf. [13, 25]). Furthermore, if  $E := D_+ \cup \Gamma$  is the closure of a Jordan domain, then  $u_E(z) = V_E$  for all  $z \in E$ , and  $\text{supp } \mu_E = \Gamma$ , see [13, 25] and Theorem B. Let  $g_E$  be the Green function of  $D_-$ , with pole at infinity. Then  $u_E(z) = V_E + g_E(z)$  and  $g_E(z) = \log |\Phi(z)|$ ,  $z \in D_-$ , where  $\Phi : D_- \rightarrow \{w : |w| > 1\}$  is a conformal map satisfying  $\Phi(\infty) = \infty$ . Since  $\Phi \in E^1(D_-)$  for a rectifiable  $\Gamma$ , see [9, 11, 20], we have that  $\nabla u_E = \nabla g_E \in e^1(D_-)$ . Obviously,  $\nabla u_E = (0, 0) \in e^1(D_+)$ , because  $u_E$  is constant in this domain. Hence all assumptions of Theorem 1.1 are satisfied. Also,  $\partial u_E / \partial n_+$  has zero boundary values a.e. on  $\Gamma$  and (1.4) takes the following familiar form.

**Example 1.2.** *Let  $E \subset \mathbb{C}$  be the closure of a Jordan domain  $D_+$  bounded by a rectifiable Jordan curve  $\Gamma$ . The equilibrium measure  $\mu_E$  for  $E$  is absolutely continuous with respect to the arclength measure  $ds$  on  $\Gamma$ , and*

$$d\mu_E = \frac{1}{2\pi} \frac{\partial g_E}{\partial n_-} ds.$$

More examples of applications to the equilibrium measures for energy problems with external fields may be found in [28, 8, 21, 22] (see also references therein).

The problem of finding a measure from its potential is of interest in higher dimensions too. For example, consider a positive Borel measure  $\sigma$  compactly supported in  $\mathbb{R}^3$ , and define its Newtonian potential by

$$U(x) := \int \frac{d\sigma(y)}{|x - y|}, \quad x \in \mathbb{R}^3,$$

where  $|x - y|$  is the Euclidean distance between  $x, y \in \mathbb{R}^3$ . Clearly,  $U$  is superharmonic in  $\mathbb{R}^3$  and harmonic in  $\mathbb{R}^3 \setminus \text{supp } \sigma$ , see [13] and [15]. If  $U \in C^2(D)$  for a domain  $D$  in  $\mathbb{R}^3$ , then an analog of Theorem B gives [15, p. 156]

$$d\sigma = -\frac{1}{4\pi} \Delta U dV,$$

where  $dV$  is the volume measure. When  $\sigma$  is supported on a sufficiently smooth surface  $S$ , we have an analog of Theorem C [15, p. 164]

$$d\sigma = -\frac{1}{4\pi} \left( \frac{\partial U}{\partial n_+} + \frac{\partial U}{\partial n_-} \right) dS,$$

where  $dS$  is the surface area measure, and  $n_+, n_-$  are the inner and the outer normals to  $S$ . Clearly, one should be able to relax smoothness assumptions for the surface  $S$ , but a natural analog of rectifiable curve (cf. Theorem 1.1) in this setting is not obvious at all.

## 2. Singular integral equations with Cauchy kernel

Define the Cauchy transform of the measure  $\mu$  by

$$C\mu(z) := \frac{1}{2\pi i} \int \frac{d\mu(t)}{t - z}, \quad z \in \mathbb{C} \setminus \text{supp } \mu,$$

which is an analytic in  $\overline{\mathbb{C}} \setminus \text{supp } \mu$  function such that  $C\mu(\infty) = 0$ . It is well known that  $C\mu$  is closely related to the potential  $u$  of (1.1). Indeed, if we consider a multivalued analytic function  $F(z) := \int \log(z - t) d\mu(t)$ , then  $u = \Re(F)$  and  $C\mu = -F'$  in  $\mathbb{C} \setminus \text{supp } \mu$ . In fact, these ideas are used in the proof of Theorem 1.1, see the argument beginning with (3.1). More discussion and history of such relations may be found in Muskhelishvili [17] and Danilyuk [6], see also the work of Plemelj [18, 19], Radon [24] and Bertrand [2].

Suppose that  $\text{supp } \mu \subset \Gamma$ , where  $\Gamma$  is a Jordan rectifiable curve of length  $l$ . We keep the same notation  $D_+$  for the bounded component of the complement of  $\Gamma$ , and  $D_-$  for the unbounded one. Let  $\Gamma$  be parametrized by  $z = z(s)$ , where  $s \in [0, l]$  is the arclength parameter and  $z'(s)$  is the unit tangent vector to the curve. Recall that the tangent and normal vectors exist almost everywhere on  $\Gamma$  with respect to the arclength measure. Assume further that  $\mu$  is absolutely continuous with respect to  $ds$  with the density  $f(z(s))z'(s)$ , where  $f \in L^1(\Gamma, ds)$ . Then we can consider the singular Cauchy integral of  $f$  defined by

$$(2.1) \quad Sf(z) := \frac{1}{\pi i} \int_{\Gamma} \frac{f(t)dt}{t - z} = \lim_{\varepsilon \rightarrow 0} \frac{1}{\pi i} \int_{\Gamma_{\varepsilon}(z)} \frac{f(t)dt}{t - z}, \quad z \in \Gamma,$$

where  $\Gamma_{\varepsilon}(z) := \{t \in \Gamma : |t - z| \geq \varepsilon\}$ , i.e. the integral is understood as the Cauchy principal value, cf. [17], [11] and [6]. Existence of  $Sf$  is subject to appropriate

conditions on the function  $f$  and the curve  $\Gamma$ . For the Cauchy transform

$$(2.2) \quad Cf(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(t) dt}{t - z}, \quad z \in \mathbb{C} \setminus \Gamma,$$

let  $Cf_+(\zeta)$  (respectively  $Cf_-(\zeta)$ ) denote the nontangential limit value from  $D_+$  (respectively from  $D_-$ ) at a point  $\zeta \in \Gamma$ . A classical and important result of Privalov [23] gives connections between  $Sf$  and the boundary values of  $Cf$ .

**Privalov's Fundamental Lemma.** *The nontangential boundary limit values  $Cf_+$  (or  $Cf_-$ ) exist a.e. on  $\Gamma$  if and only if  $Sf$  exists a.e. on  $\Gamma$ . Furthermore, in the case of a.e. existence, the Plemelj-Sokhotski formulas*

$$(2.3) \quad Cf_+(z) - Cf_-(z) = f(z) \quad \text{and} \quad Cf_+(z) + Cf_-(z) = Sf(z)$$

hold for a.e.  $z \in \Gamma$ .

For the proof and thorough discussion of this result, we refer to [23, 11, 6]. If  $1 < p < \infty$ , then another fundamental result of David [7] states that  $S : L^p(\Gamma, ds) \rightarrow L^p(\Gamma, ds)$  is a bounded operator if and only if  $\Gamma$  is Ahlfors regular. The Ahlfors regularity condition means that there is a constant  $A > 0$  such that for any disk  $D_r$  of radius  $r$  we have

$$|\Gamma \cap D_r| \leq Ar,$$

where  $|\Gamma \cap D_r|$  is the length of the intersection. This class of curves is sufficiently wide as it allows any angles (even cusps), see Chapter 7 of Pommerenke [20] for more on geometry. A Jordan arc is said to be Ahlfors regular if it is a subarc of an Ahlfors regular curve. In the sequel, we shall always make a natural assumption that  $\Gamma$  is Ahlfors regular, to insure the a.e. existence of  $Sf \in L^p(\Gamma, ds)$  for  $f \in L^p(\Gamma, ds)$ , and the validity of (2.3). If  $f$  belongs to the class  $H_\alpha(\Gamma)$ ,  $0 < \alpha < 1$ , of Hölder continuous functions on  $\Gamma$ , then  $Sf \in H_\alpha(\Gamma)$  for Ahlfors regular  $\Gamma$ . This generalization of the Plemelj-Privalov theorem was proved by Salaev [26]. A complete description of curves that allow the Cauchy singular operator to preserve moduli of continuity is contained in Guseinov [12].

Singular integral equations with Cauchy kernel arise naturally in the problem of finding a measure from its potential. For example, the equation

$$(2.4) \quad Sf(z) = \frac{1}{\pi i} \int_{\Gamma} \frac{f(t) dt}{t - z} = g(z)$$

was used repeatedly to find the weighted equilibrium measures in [28, 8, 16, 21, 22] and many other papers. Here, the function  $g$  is either obtained by differentiation of the known values for the potential  $u$  on the support of  $\mu$ , or found from the Plemelj-Sokhotski formula (2.3) as  $g(z) = Cf_+(z) + Cf_-(z)$ . The solution of (2.4) for a closed contour  $\Gamma$  is well known (see, e.g., [17, §27]), and represents the self-inversive property of the operator  $S$ .

**Proposition 2.1.** *Suppose that  $\Gamma$  is an Ahlfors regular closed Jordan curve, and that  $g \in L^p(\Gamma, ds)$ ,  $1 < p < \infty$ . The equation  $Sf = g$  on  $\Gamma$  has the unique solution  $f = Sg \in L^p(\Gamma, ds)$ .*

In many applications to recovering a measure from its potential, (2.4) holds on the support of the measure, which may be different from a closed curve. A rather common situation is when the support consists of several arcs, see [28, 8, 16, 21, 22]. Let  $L := \cup_{j=1}^N \gamma(a_j, b_j)$  be the union of  $N$  disjoint Ahlfors regular arcs  $\gamma(a_j, b_j)$  with endpoints  $a_j$  and  $b_j$ . For a function  $f \in L^1(L, ds)$ , we define the Cauchy singular integral operator  $S_L f$  on  $L$  similarly to (2.1). We can assume that  $L \subset \Gamma$ , where  $\Gamma$  is an Ahlfors regular closed Jordan curve with interior  $D_+$  and exterior  $D_-$ . It is always possible to extend  $f$  from  $L$  to  $\Gamma$  by letting  $f(z) = 0$ ,  $z \in \Gamma \setminus L$ , and view the operator  $S_L$  as a restriction of  $S$ , by setting  $S_L f = Sf$  for  $f \in L^1(\Gamma, ds)$ ,  $f|_{\Gamma \setminus L} \equiv 0$ . Hence Privalov's Fundamental Lemma and many other facts easily carry over to the case of  $S_L$ .

For  $R(z) := \prod_{j=1}^N (z - a_j)(z - b_j)$ , we consider the branch of  $\sqrt{R(z)}$  defined in the domain  $\mathbb{C} \setminus L$  by  $\lim_{z \rightarrow \infty} \sqrt{R(z)}/z^N = 1$ . By the values of  $\sqrt{R(z)}$  on  $L$ , we understand the boundary limit values from  $D_+$ . A general solution of the equation  $S_L f = g$  for Hölder continuous  $g$  on smooth arcs was first found by Muskhelishvili [17, Chap. 11]. The case of  $L^p$  solutions was later considered by Hvedelidze [14]. We generalize their ideas to prove the following.

**Theorem 2.2.** *Let  $L := \cup_{j=1}^N \gamma(a_j, b_j)$  be a union of disjoint Ahlfors regular arcs, and let  $R(z) := \prod_{j=1}^N (z - a_j)(z - b_j)$ . If  $g \in L^p(L, ds)$ ,  $2 < p < \infty$ , then any solution of the equation  $S_L f = g$  in  $L^1(L, ds)$  has the form*

$$(2.5) \quad f(z) = \frac{1}{\pi i \sqrt{R(z)}} \int_L \frac{g(t) \sqrt{R(t)} dt}{t - z} + \frac{P_{N-1}(z)}{\sqrt{R(z)}} \quad \text{a.e. on } L,$$

where  $P_{N-1} \in \mathbb{C}_{N-1}[z]$ .

Here,  $\mathbb{C}_{N-1}[z]$  denotes the set of polynomials with complex coefficients of degree at most  $N - 1$ .

It is of interest that certain solutions may also be written in a different form.

**Corollary 2.3.** *Let  $L$  and  $g$  be as in Theorem 2.2. The function*

$$(2.6) \quad f_0(z) = \frac{\sqrt{R(z)}}{\pi i} \int_L \frac{g(t) dt}{\sqrt{R(t)}(t - z)}, \quad z \in L,$$

is a solution of  $S_L f = g$  in  $L^1(L, ds)$  if and only if

$$(2.7) \quad \int_L \frac{t^k g(t) dt}{\sqrt{R(t)}} = 0, \quad k = 0, \dots, N - 1.$$

If the right hand side of the equation  $S_L f = g$  is Hölder (or Dini) continuous, then the continuity properties are also preserved for the solutions, provided that we stay away from the endpoints of  $L$ . This follows from the corresponding results for the operator  $S$  on closed contours, see [26, 12].

**Corollary 2.4.** *Let  $L$  be as in Theorem 2.2. If  $g \in H_\alpha(L)$ ,  $0 < \alpha < 1$ , then for any compact set  $E \subset L \setminus \{a_j, b_j\}_{j=1}^N$  the general solution (2.5) belongs to  $H_\alpha(E)$ .*

It is important for applications to find the bounded solution of the equation  $S_L f = g$ , and describe conditions for its existence. For example, bounded solutions play a special role in finding the weighted equilibrium measures for “good” weights [28, 8, 16, 21, 22].

**Corollary 2.5.** *Assume that  $L$  satisfies the conditions of Theorem 2.2, and that  $g \in H_\alpha(L)$ ,  $\alpha > 0$ . A bounded solution  $f_0 \in L^\infty(L, ds)$  for  $S_L f = g$  exists if and only if (2.7) holds true.*

Furthermore, if (2.7) is satisfied, then

$$(2.8) \quad f_0(z) = \frac{\sqrt{R(z)}}{\pi i} \int_L \frac{g(t) dt}{\sqrt{R(t)}(t-z)}, \quad z \in L,$$

where  $f_0 \in C(L)$  and  $f_0(a_j) = f_0(b_j) = 0$ ,  $j = 1, \dots, N$ .

Several applications of Corollary 2.5 on arcs of the unit circle may be found in [22], see Theorems 1.5, 2.1 and 2.2. In particular, those results describe the explicit forms of equilibrium measures with external fields defined by the exponential and polynomial weights.

### 3. Proofs

**Proof of Theorem 1.1.** Let  $v^+$  be a harmonic conjugate of  $u$  in  $D_+$ , so that  $F_+ := u + iv^+$  is analytic in  $D_+$ . By the Cauchy-Riemann equations, we have  $u_x(z) = v_y^+(z)$  and  $u_y(z) = -v_x^+(z)$  for all  $z \in D_+$ . Hence  $v_x^+, v_y^+ \in E^1(D_+)$  and  $F'_+ = u_x + iv_x^+ \in E^1(D_+)$ . It immediately follows that the nontangential limit values of  $F'_+$  exist almost everywhere on  $\Gamma$  with respect to the arclength measure, and the same is true for  $u_x, u_y, v_x^+, v_y^+$ . Since  $\Gamma$  is rectifiable, the tangent and normal vectors exist at almost every  $z \in \Gamma$ . Therefore, for almost every  $z \in \Gamma$ , we *simultaneously* have the normal vector and the nontangential limit values for  $u_x$  and  $u_y$ . If  $n_+ = (-\sin \theta, \cos \theta)$  is the inner unit normal (pointing inside  $D_+$ ) at such a point  $z$ , then the derivative in the direction  $n_+$  at any point  $\zeta \in D_+$  is  $\partial u / \partial n_+(\zeta) = -u_x \sin \theta + u_y \cos \theta$ . Thus we can define the limit boundary values of  $\partial u / \partial n_+(z)$  for a.e.  $z \in \Gamma$  and

$$\int_\Gamma \left| \frac{\partial u}{\partial n_+}(z(s)) \right| ds \leq \int_\Gamma |u_x(z(s))| ds + \int_\Gamma |u_y(z(s))| ds < \infty.$$



Applying the same argument to a conjugate  $v^-$  and the analytic completion  $F_- := u + iv_-$  for  $u$  in  $D_-$ , we obtain that  $F'_- \in E^1(D_-)$  and  $\partial u / \partial n_- \in L^1(\Gamma, ds)$ . Note that  $F_-$  is multi-valued, in general, but  $F'_-$  is single-valued.

The Cauchy-Riemann equations imply, after passing to the boundary values, that

$$\frac{\partial u}{\partial n_+}(z) = -\frac{\partial v^+}{\partial s}(z) \quad \text{and} \quad \frac{\partial u}{\partial n_-}(z) = \frac{\partial v^-}{\partial s}(z) \quad \text{for a.e. } z \in \Gamma.$$

Therefore, we have a.e. on  $\Gamma$  that

$$(3.1) \quad \begin{aligned} \frac{\partial u}{\partial n_+}(z) + \frac{\partial u}{\partial n_-}(z) &= \frac{\partial v^-}{\partial s}(z) - \frac{\partial v^+}{\partial s}(z) = \Im \left( \frac{\partial F_-}{\partial s}(z) - \frac{\partial F_+}{\partial s}(z) \right) \\ &= \Im ([F'_-(z) - F'_+(z)]z'(s)), \end{aligned}$$

where  $z'(s)$  is the unit tangent vector to  $\Gamma$  at  $z(s)$ .

Introducing the multivalued function  $\int \log(z-t) d\mu(t)$  with the real part  $u(z)$ , we observe that  $F_+$  and  $F_-$  are branches of this function and

$$F'_\pm(z) = \int \frac{d\mu(t)}{z-t}, \quad z \in D_\pm.$$

Consider the Cauchy transform of  $\mu$

$$C\mu(z) = \frac{1}{2\pi i} \int \frac{d\mu(t)}{t-z}, \quad z \in \overline{\mathbb{C}} \setminus \text{supp } \mu.$$

Since  $2\pi i C\mu(z) = -F'_\pm(z)$ ,  $z \in D_\pm$ , the nontangential limit values  $C\mu_+$  from  $D_+$  and  $C\mu_-$  from  $D_-$  exist a.e. on  $\Gamma$ . The Fundamental Lemma of Privalov on the Cauchy singular integral for measures (cf. Privalov [23, pp. 183-189] and Danilyuk [6, pp. 118-125]) gives that the Plemelj-Sokhotski formula

$$[C\mu_+(z(s)) - C\mu_-(z(s))]z'(s) = \frac{d\mu}{ds}(s)$$

holds a.e. on  $\Gamma$ , where  $d\mu/ds$  is the density of the absolutely continuous part of  $\mu$ . Note that the right hand side is real. It now follows from (3.1) that

$$\frac{1}{2\pi} \left( \frac{\partial u}{\partial n_+}(z(s)) + \frac{\partial u}{\partial n_-}(z(s)) \right) = [C\mu_+(z(s)) - C\mu_-(z(s))]z'(s) \quad \text{a.e. on } \Gamma.$$

Let  $\nu$  be the measure supported on  $\Gamma$ , which is absolutely continuous with respect to  $ds$ , and whose density is defined by the left hand side of the above equation (cf. (1.4)). We shall show that  $\mu = \nu$ . Recall that  $F'_+ \in E^1(D_+)$  and  $F'_- \in E^1(D_-)$  with  $F'_-(\infty) = 0$ . Using Cauchy's integral formula, we obtain by Theorem 10.4 of [9, p. 170] that

$$\begin{aligned} \int \frac{d\nu(t)}{t-z} &= \int_\Gamma \frac{(C\mu_+(t) - C\mu_-(t))dt}{t-z} = \frac{1}{2\pi i} \int_\Gamma \frac{(F'_-(t) - F'_+(t))dt}{t-z} \\ &= -F'_\pm(z) = 2\pi i C\mu(z), \quad z \in D_\pm. \end{aligned}$$

Hence the Cauchy transforms of  $\nu$  and  $\mu$  coincide, i.e.,

$$\int \frac{d(\nu - \mu)(t)}{t - z} = 0, \quad z \in D_{\pm}.$$

Expanding  $1/(t - z)$  in a series of negative powers of  $z$  around  $z = \infty$ , we obtain in a standard way that

$$\int t^n d(\nu - \mu)(t) = 0, \quad n = 0, 1, 2, \dots$$

Similarly, expanding the kernel  $1/(t - z)$  near a fixed point  $z_0 \in D_+$ , we have

$$\int \frac{d(\nu - \mu)(t)}{(t - z_0)^n} = 0, \quad n \in \mathbb{N}.$$

But the span of the function system  $\{t^n\}_{n=0}^{\infty} \cup \{(t - z_0)^{-n}\}_{n=1}^{\infty}$  is dense in  $C(\Gamma)$ , see Chapter 3, §1 of Gaier [10]. Hence

$$\int f(t) d(\nu - \mu)(t) = 0,$$

for any continuous function  $f$  on  $\Gamma$ . Since  $\nu - \mu$  is orthogonal to all continuous functions on its support, it must vanish identically. ■

**Proof of Proposition 2.1.** Observe that for  $g \in L^p(\Gamma, ds)$  we also have that  $Sg \in L^p(\Gamma, ds)$  by [7]. Hence we have from (2.3) that  $Cg_+ - Cg_- = g$  and  $Cg_+ + Cg_- = Sg$  a.e. on  $\Gamma$ . Note that the function  $H_+(z) := Cg(z)$ ,  $z \in D_+$ , is analytic in  $D_+$  and has the boundary values  $Cg_+ \in L^p(\Gamma, ds)$ . It follows that  $H_+ \in E^p(D_+)$  and we obtain by Theorem 10.4 of [9] that

$$(3.2) \quad C(Cg_+)(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{Cg_+(t) dt}{t - z} = \begin{cases} H_+(z), & z \in D_+, \\ 0, & z \in D_-. \end{cases}$$

Thus we also have for  $H_-(z) := Cg(z)$ ,  $z \in D_-$ , that  $H_- \in E^p(D_-)$  and  $H_-(\infty) = 0$ . Hence

$$(3.3) \quad C(Cg_-)(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{Cg_-(t) dt}{t - z} = \begin{cases} -H_-(z), & z \in D_-, \\ 0, & z \in D_+, \end{cases}$$

where the integral is taken in the positive direction with respect to  $D_+$ .

Applying (2.3) to  $h := Sg$ , we have that  $Ch_+ - Ch_- = h$  and  $Ch_+ + Ch_- = Sh$  a.e. on  $\Gamma$ , where  $Sh \in L^p(\Gamma, ds)$ . Consider the Cauchy transform  $Ch(z)$  for  $z \in D_+$  and use (3.2)-(3.3) to evaluate

$$Ch(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{Sg(t) dt}{t - z} = \frac{1}{2\pi i} \int_{\Gamma} \frac{(Cg_+(t) + Cg_-(t)) dt}{t - z} = H_+(z).$$

Similarly, we obtain for  $z \in D_-$ :

$$Ch(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{Sg(t) dt}{t - z} = \frac{1}{2\pi i} \int_{\Gamma} \frac{(Cg_+(t) + Cg_-(t)) dt}{t - z} = -H_-(z).$$

Hence  $Sh = Ch_+ + Ch_- = Cg_+ - Cg_- = g$  a.e. on  $\Gamma$ , so that  $h = Sg$  is a solution. This also implies that  $S^2g = S(Sg) = g$  for any  $g \in L^p(\Gamma, ds)$ . If we assume that there are two solutions  $f_1$  and  $f_2$ , then applying  $S$  to  $Sf_1 = Sf_2 = g$  gives  $f_1 = f_2 = Sg$ . ■

For the proof of Theorem 2.2, we need a lemma that describes all solutions of the homogeneous equation  $S_L f = 0$  on  $L$ . It follows from the lemma below that the kernel of the operator  $S_L$  has dimension  $N$ , while the kernel of  $S$  is trivial in the case of a closed curve, by the previous proof.

**Lemma 3.1.** *Let  $L := \cup_{j=1}^N \gamma(a_j, b_j)$  be a union of disjoint Ahlfors regular arcs, and let  $R(z) := \prod_{j=1}^N (z - a_j)(z - b_j)$ . All solution of the equation  $S_L f = 0$  in  $L^1(L, ds)$  are given by the functions  $f = P_{N-1}/\sqrt{R}$ , where  $P_{N-1} \in \mathbb{C}_{N-1}[z]$ .*

**Proof.** For any  $P_{N-1} \in \mathbb{C}_{N-1}[z]$ , we first show that

$$(3.4) \quad \frac{1}{\pi i} \int_L \frac{P_{N-1}(t) dt}{(t-z)\sqrt{R(t)}} = \begin{cases} 0 & \text{for a.e. } z \in L, \\ P_{N-1}(z)/\sqrt{R(z)}, & z \in \mathbb{C} \setminus L, \end{cases}$$

where the integral is understood in the Cauchy principal value sense for  $z \in L$ . Let  $h(z) = P_{N-1}(z)/\sqrt{R(z)}$ ,  $z \in \mathbb{C} \setminus L$ . It is clear that the limit values of  $\sqrt{R(z)}$  as  $z$  tends to  $\zeta \in L$  from  $D_+$  and from  $D_-$  are negatives of each other. Hence the boundary limit values of  $h$  on  $L$  satisfy

$$(3.5) \quad h_+(\zeta) = h(\zeta) = -h_-(\zeta) \quad \text{for a.e. } \zeta \in L.$$

Consider a contour  $\Lambda$  which consists of  $N$  simple closed curves, one around each of the arcs  $\gamma(a_j, b_j)$ . Then

$$\frac{1}{2\pi i} \int_{\Lambda} \frac{h(t)}{t-z} dt = h(z)$$

for  $z$  in the exterior of  $\Lambda$ , and the integral equals zero for  $z \in L$ . If we take  $z \in \mathbb{C} \setminus L$  and shrink  $\Lambda$  to  $L$ , then

$$h(z) = \frac{1}{\pi i} \int_L \frac{h(t)}{t-z} dt, \quad z \in \mathbb{C} \setminus L,$$

by (3.5). Using Privalov's Fundamental Lemma, we obtain that

$$S_L h(z) = \frac{1}{\pi i} \int_L \frac{h(t)}{t-z} dt = \frac{h_+(z)}{2} + \frac{h_-(z)}{2} \quad \text{for a.e. } z \in L.$$

But the right hand side is zero for a.e.  $z \in L$  by (3.5), and (3.4) is proved. Thus we showed that every function  $h = P_{N-1}/\sqrt{R}$  is a solution of the equation  $S_L f = 0$ .

Suppose now that  $f$  satisfies  $S_L f(z) = 0$  for a.e.  $z \in L$ . Then (2.3) gives that

$$(3.6) \quad C_L f_+ + C_L f_- = 0 \quad \text{and} \quad C_L f_+ - C_L f_- = f \quad \text{a.e. on } L,$$

where

$$C_L f(z) := \frac{1}{2\pi i} \int_L \frac{f(t)}{t-z} dt, \quad z \in \mathbb{C} \setminus L.$$

It follows from the first equation of (3.6) that  $(\sqrt{R}C_L f)_+ - (\sqrt{R}C_L f)_- = 0$  and  $(\sqrt{R}C_L f)_+ = (\sqrt{R}C_L f)_-$  a.e. on  $L$ . Clearly, these boundary values belong to  $L^1(L, ds)$  by (3.6). This allows us to show in the usual way that the integral of  $\sqrt{R}C_L f$  over any closed contour (even intersecting  $L$ ) is zero. Hence the analytic in  $\mathbb{C} \setminus L$  function  $\sqrt{R}C_L f$  can be continued analytically to the whole  $\mathbb{C}$  by a Morera-type theorem of Zalcman [30, Th. 1]. Observe that  $\sqrt{R(z)}C_L f(z) = O(z^{N-1})$  as  $z \rightarrow \infty$ , which implies that this function is a polynomial  $P_{N-1} \in \mathbb{C}_{N-1}[z]$ . Finally, we apply the second equations of (3.6) and of (3.4) to find that

$$f(z) = C_L f_+(z) - C_L f_-(z) = \frac{1}{2} \frac{P_{N-1}(z)}{\sqrt{R(z)}} + \frac{1}{2} \frac{P_{N-1}(z)}{\sqrt{R(z)}} = \frac{P_{N-1}(z)}{\sqrt{R(z)}}$$

for a.e.  $z \in L$ . ■

**Proof of Theorem 2.2.** Consider the function

$$f_0(z) = \frac{1}{\pi i \sqrt{R(z)}} \int_L \frac{g(t) \sqrt{R(t)} dt}{t-z} = \frac{S_L(g\sqrt{R})(z)}{\sqrt{R(z)}}, \quad z \in L,$$

which is obtained from (2.5) by setting  $P_{N-1} \equiv 0$ . We proceed by first proving that  $f_0$  is a solution of  $S_L f = g$ . Since  $g\sqrt{R} \in L^p(L, ds)$ ,  $2 < p < \infty$ , we also have that  $S_L(g\sqrt{R}) \in L^p(L, ds)$ , see [7]. It is not difficult to see that  $1/\sqrt{R} \in L^q(L, ds)$  for all  $q < 2$  (cf. [27], for example). Using Hölder's inequality, we immediately conclude that  $f_0 \in L^r(L, ds)$  for some  $r > 1$ . Hence  $S_L f_0 \in L^r(L, ds)$ . Consider the analytic function

$$F(z) := \frac{1}{2\pi i} \int_L \frac{g(t) \sqrt{R(t)} dt}{t-z} = C_L(g\sqrt{R})(z), \quad z \in \mathbb{C} \setminus L.$$

The Plemelj-Sokhotski formulas (2.3) read in this case

$$F_+ - F_- = g\sqrt{R} \quad \text{and} \quad F_+ + F_- = f_0\sqrt{R} \quad \text{a.e. on } L.$$

If we define  $\Phi(z) := F(z)/\sqrt{R(z)}$ ,  $z \in \mathbb{C} \setminus L$ , and use  $(\sqrt{R})_+(z) = \sqrt{R(z)} = -(\sqrt{R})_-(z)$ ,  $z \in L$ , then we obtain

$$(3.7) \quad \Phi_+ + \Phi_- = g \quad \text{and} \quad \Phi_+ - \Phi_- = f_0 \quad \text{a.e. on } L.$$

But we also have  $(C_L f_0)_+ - (C_L f_0)_- = f_0$  a.e. on  $L$  for the Cauchy transform of  $f_0$ , see (2.3). It follows that the function  $H := \Phi - C_L f_0$  is analytic in  $\overline{\mathbb{C}} \setminus L$ , with  $H(\infty) = 0$ , and satisfies  $H_+ - H_- = 0$  a.e. on  $L$ . We can now argue in the same way as in Lemma 3.1, and use a Morera-type theorem [30] to deduce that  $H$  can be continued to an entire function. Thus this function is identically 0 in

$\overline{\mathbb{C}}$  by Liouville's theorem. Since  $\Phi = C_L f_0$  in  $\mathbb{C} \setminus L$ , we have by (2.3) and (3.7) that

$$S_L f_0 = (C_L f_0)_+ + (C_L f_0)_- = \Phi_+ + \Phi_- = g \quad \text{a.e. on } L.$$

If  $f$  is any solution of  $S_L f = g$ , then  $h = f - f_0$  is a solution of the homogeneous equation  $S_L h = 0$ . Hence it has the form  $h = P_{N-1}/\sqrt{R}$  by Lemma 3.1. ■

**Proof of Corollary 2.3.** We shall compute the Cauchy transform of  $f_0$  to show that  $f_0$  always satisfies a certain modified integral equation, which is found below. Consider

$$C_L f_0(z) = \frac{1}{2\pi i} \int_L \frac{f_0(t) dt}{t - z}, \quad z \in \mathbb{C} \setminus L,$$

and define

$$\Psi(z) := \frac{\sqrt{R(z)}}{2\pi i} \int_L \frac{g(t) dt}{\sqrt{R(t)}(t - z)} = \sqrt{R(z)} C_L(g/\sqrt{R})(z), \quad z \in \mathbb{C} \setminus L.$$

Since  $(\sqrt{R})_+ = \sqrt{R} = -(\sqrt{R})_-$  on  $L$  and  $S_L(g/\sqrt{R}) = C_L(g/\sqrt{R})_+ + C_L(g/\sqrt{R})_-$  a.e. on  $L$  by (2.3), we have that

$$(3.8) \quad \begin{aligned} f_0 &= \sqrt{R} S_L(g/\sqrt{R}) = \sqrt{R} C_L(g/\sqrt{R})_+ + \sqrt{R} C_L(g/\sqrt{R})_- \\ &= \Psi_+ - \Psi_- \end{aligned}$$

holds a.e. on  $L$ . Passing to the contour integral over both sides of the cut  $L$  in the plane, we obtain

$$C_L f_0(z) = \frac{1}{2\pi i} \int_L \frac{(\Psi_+(t) - \Psi_-(t)) dt}{t - z} = \frac{1}{2\pi i} \oint_L \frac{\Psi(t) dt}{t - z}, \quad z \in \mathbb{C} \setminus L.$$

Let  $\Lambda$  be a contour consisting of  $N$  simple closed curves, one around each of the arcs of  $L$ , such that  $z$  is outside  $\Lambda$ . Cauchy's integral theorem and the definition of  $\Psi$  give that

$$\begin{aligned} C_L f_0(z) &= \frac{1}{2\pi i} \int_\Lambda \frac{\Psi(t) dt}{t - z} = \frac{1}{2\pi i} \int_\Lambda \frac{\sqrt{R(t)}}{t - z} \left( \frac{1}{2\pi i} \int_L \frac{g(w) dw}{\sqrt{R(w)}(w - t)} \right) dt \\ &= \frac{1}{2\pi i} \int_L \frac{g(w)}{\sqrt{R(w)}} \left( \frac{1}{2\pi i} \int_\Lambda \frac{\sqrt{R(t)} dt}{(t - z)(w - t)} \right) dw. \end{aligned}$$

Next, we use residues at  $z$  and at  $\infty$  to evaluate the inner integral. For the residue at  $z$ , we immediately obtain  $\sqrt{R(z)}/(w - z)$ . Writing  $\sqrt{R(t)} = Q(t) + O(1/t)$  near infinity, where  $Q(t)$  is a polynomial of degree  $N$ , it follows that

$$\frac{\sqrt{R(t)}}{t - z} = Q(z, t) + O\left(\frac{1}{t}\right) \quad \text{as } t \rightarrow \infty,$$

with

$$(3.9) \quad Q(z, t) := \frac{Q(z) - Q(t)}{z - t}.$$

Note that  $Q(z, t)$  is a polynomial in both variables  $z$  and  $t$ , of degree  $N - 1$ . Hence the residue at infinity for the inner integral is equal to  $Q(z, w)$ , and we obtain that

$$\frac{1}{2\pi i} \int_{\Lambda} \frac{\sqrt{R(t)} dt}{(t - z)(w - t)} = \frac{\sqrt{R(z)}}{w - z} + Q(z, w).$$

Finally,

$$C_L f_0(z) = \frac{\sqrt{R(z)}}{2\pi i} \int_L \frac{g(w) dw}{\sqrt{R(w)}(w - z)} + \frac{1}{2}P(z), \quad z \in \mathbb{C} \setminus L,$$

where

$$(3.10) \quad P(z) := \frac{1}{\pi i} \int_L \frac{g(w)Q(z, w) dw}{\sqrt{R(w)}}$$

is a polynomial of degree at most  $N - 1$ . Recall that  $(\sqrt{R})_+ = \sqrt{R} = -(\sqrt{R})_-$  on  $L$  and  $g/\sqrt{R} = C_L(g/\sqrt{R})_+ - C_L(g/\sqrt{R})_-$  a.e. on  $L$  by (2.3). Thus

$$(C_L f_0)_+ + (C_L f_0)_- = \sqrt{R} \left( C_L(g/\sqrt{R})_+ - C_L(g/\sqrt{R})_- \right) + P = g + P$$

a.e. on  $L$ . On the other hand, (2.3) directly gives that

$$(C_L f_0)_+ + (C_L f_0)_- = S_L f_0$$

a.e. on  $L$ . We established in this manner that  $f_0$  satisfies the modified integral equation

$$(3.11) \quad S_L f_0 = g + P,$$

where  $P$  is defined by (3.10). Obviously,  $f_0$  is a solution of the original equation  $S_L f = g$  if and only if  $P \equiv 0$ . Thus it remains to show that the vanishing of  $P$  is equivalent to (2.7). Observe from (3.9) that

$$Q(z, t) = q_0(t)z^{N-1} + q_1(t)z^{N-2} + \dots + q_{N-1}(t),$$

where each  $q_k$  is a monic polynomial of degree  $k$ . Then  $P \equiv 0$  is equivalent to the system

$$\frac{1}{2\pi i} \int_L \frac{g(w)q_k(w) dw}{\sqrt{R(w)}} = 0, \quad k = 0, \dots, N - 1,$$

by (3.10). Noting that the polynomials  $q_k$  are linearly independent, we conclude that the above system is equivalent to (2.7). ■

**Proof of Corollary 2.4.** It is clear that (2.5) may be written in the form

$$f = \frac{S_L(g\sqrt{R})}{\sqrt{R}} + \frac{P_{N-1}}{\sqrt{R}},$$

where  $P_{N-1} \in \mathbb{C}_{N-1}[z]$ . Let  $E \subset L \setminus \{a_j, b_j\}_{j=1}^N$  be compact. In order to prove that  $f \in H_\alpha(E)$ , it is sufficient to show that  $S_L(g\sqrt{R}) \in H_\alpha(E)$ . The function  $G := g\sqrt{R} \in H_\alpha(L)$  can be continued to a function  $\tilde{G} \in H_\alpha(\Gamma)$ , where  $\Gamma$  is a closed Ahlfors regular curve, so that  $S\tilde{G} \in H_\alpha(\Gamma)$  by [26, 12]. But the Cauchy singular integral  $S$  of  $\tilde{G}|_{\Gamma \setminus L}$  is analytic at every  $z \in E$ . Thus it follows that  $S_L G \in H_\alpha(E)$ . ■

**Lemma 3.2.** *Let  $L = \cup_{j=1}^N \gamma(a_j, b_j)$  be a union of disjoint Ahlfors regular arcs. If  $g \in H_\alpha(L)$ ,  $\alpha > 0$ , then the function  $f_0$  defined in (2.6) is continuous on  $L$ , and  $f_0(a_j) = f_0(b_j) = 0$ ,  $j = 1, \dots, N$ .*

**Proof.** We observe that  $f_0 = \sqrt{R} S_L(g/\sqrt{R})$ , and that  $g/\sqrt{R}$  is Hölder continuous on any compact set  $E \subset L$  that does not include the endpoints of  $L$ . Using a similar argument with continuation to a Hölder continuous function on the closed curve  $\Gamma$ , such as in the above proof of Corollary 2.4, we conclude that  $S_L(g/\sqrt{R})$  is also Hölder continuous on  $E$ . Hence we now need to analyze the behavior of  $S_L(g/\sqrt{R})$  near the endpoints of  $L$ . This analysis was already carried out in Chapter 4 of [17] for smooth (or piecewise smooth)  $L$ . Since the argument is rather technical, and requires relatively small adjustments for the case of Ahlfors regular  $L$ , we do not reproduce it here. In particular, it is shown in [17] (see equations (29.8) and (29.9) on page 75) and in [27] that

$$S_L(g/\sqrt{R})(z) = G(z)/|z - c|^\beta, \quad \beta < 1/2,$$

for  $z \in L$  near any of the endpoints  $c$  of  $L$ , where  $G$  is Hölder continuous on  $L$ . It is immediate that

$$f_0(z) = G(z) |z - c|^{1/2-\beta}$$

for  $z \in L$  near  $c$ , so that  $f_0$  is Hölder continuous on  $L$  and  $f_0(a_j) = f_0(b_j) = 0$ ,  $j = 1, \dots, N$ . ■

**Proof of Corollary 2.5.** Suppose that there exists a bounded function  $f$  that satisfies  $S_L f = g$  on  $L$ . We know from the proof of Corollary 2.3 that  $f_0$  defined in (2.6) satisfies the equation  $S_L f_0 = g + P$  on  $L$ , where  $P$  is a polynomial of degree at most  $N - 1$ , see (3.10)-(3.11). Defining  $h := f_0 - f$ , we readily have that  $S_L h = P$ . It will be shown below that  $h \equiv 0$ , so that  $P \equiv 0$  and  $S_L f_0 = g$ . But then (2.7) holds by Corollary 2.3.

Consider the equation  $S_L h = P$ . All solutions of this equation are described by (2.5):

$$(3.12) \quad h(z) = \frac{1}{\pi i \sqrt{R(z)}} \int_L \frac{P(t) \sqrt{R(t)} dt}{t-z} + \frac{P_{N-1}(z)}{\sqrt{R(z)}} \quad \text{a.e. on } L,$$

where  $P_{N-1} \in \mathbb{C}_{N-1}[z]$  is arbitrary. We evaluate the integral  $S_L(P\sqrt{R})$  in the above formula by following an idea used in the proof of Corollary 2.3. Recall that  $(\sqrt{R})_+ = \sqrt{R} = -(\sqrt{R})_-$  on  $L$  and  $S_L(P\sqrt{R}) = C_L(P\sqrt{R})_+ + C_L(P\sqrt{R})_-$  a.e. on  $L$  by (2.3). We find the Cauchy transform  $C_L(P\sqrt{R})$  by passing to the contour integral over both sides of the cut  $L$  in the plane. This yields

$$C_L(P\sqrt{R})(z) = \frac{1}{2\pi i} \int_L \frac{P(t) \sqrt{R(t)} dt}{t-z} = \frac{1}{4\pi i} \oint_L \frac{P(t) \sqrt{R(t)} dt}{t-z}, \quad z \in \mathbb{C} \setminus L,$$

where in the second integral we have the *boundary limit values* of  $P(t)\sqrt{R(t)}$  on  $L$  (from  $\mathbb{C} \setminus L$ ). Let  $\Lambda$  be again a contour consisting of  $N$  simple closed curves, one around each of the arcs of  $L$ , such that  $z$  is outside  $\Lambda$ . Using Cauchy's integral theorem, we obtain that

$$C_L(P\sqrt{R})(z) = \frac{1}{4\pi i} \int_\Lambda \frac{P(t) \sqrt{R(t)} dt}{t-z}, \quad z \in \mathbb{C} \setminus L.$$

The latter integral is found by evaluating the residues of the integrand at  $z$  and at  $\infty$ . The residue at  $z$  is clearly equal to  $P(z)\sqrt{R(z)}/2$ . Writing  $P(t)\sqrt{R(t)} = T(t) + O(1/t)$  near infinity, where  $T(t)$  is a polynomial of degree at most  $2N-1$ , we find that the residue at  $\infty$  is equal to  $T(z)/2$ . Hence

$$C_L(P\sqrt{R})(z) = \frac{P(z)\sqrt{R(z)}}{2} + \frac{T(z)}{2}, \quad z \in \mathbb{C} \setminus L,$$

and (2.3) gives

$$S_L(P\sqrt{R})(z) = T(z), \quad z \in L,$$

because  $(P\sqrt{R})_+ = -(P\sqrt{R})_-$  on  $L$ . Returning to (3.12), we have

$$h(z) = \frac{T(z) + P_{N-1}(z)}{\sqrt{R(z)}} \quad \text{a.e. on } L.$$

Note that the numerator is a polynomial of degree  $2N-1$ , which has to vanish at  $2N$  endpoints of  $L$  in order for  $h$  to be bounded on  $L$ . Therefore, this polynomial is identically zero, with immediate implications that  $h \equiv 0$ ,  $f_0 \equiv f$ ,  $P \equiv 0$  and  $S_L f_0 = g$ . Thus (2.7) holds by Corollary 2.3, and we showed that  $f_0$  is the unique bounded solution of  $S_L f = g$ .

Conversely, assume that (2.7) is satisfied. Corollary 2.3 shows that  $f_0$  of (2.8) (or of (2.6)) is a solution of  $S_L f = g$ . Applying Lemma 3.2, we obtain that  $f_0$  is continuous on  $L$ , and it vanishes at the endpoints of  $L$ . Hence it is bounded on  $L$ , and the uniqueness of such solution follows from the first part of this proof. ■



## References

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