Homework 11 (Extra Credit)
MATH 5293

1. Give an example of $f \in H(\mathbb{C})$ such that f is not a polynomial and $f(\mathbb{C}) = \mathbb{C}$.

2. Let f and g be entire functions satisfying the equation $f^n(z) + g^n(z) = 1$ for all $z \in \mathbb{C}$, where $n \geq 2,\ n \in \mathbb{N}$. Prove that if g has no zeros, then both f and g are constant. Is this conclusion true if f and g are allowed to have zeros?

3. Let
$$f(z) = \sum_{n=1}^{\infty} z^{n!}.$$
Show that $f \in H(\mathbb{D})$. Prove that f cannot be continued analytically through $\partial\mathbb{D}$, i.e., it is not possible to find $g \in H(\mathbb{D} \cup D(z, r))$, where $z \in \partial\mathbb{D}$ and $r > 0$, such that $g|_{\mathbb{D}} = f$.

4. Let
$$f(z) = \sum_{n=1}^{\infty} \frac{z^{n!}}{n!}.$$
Show that f is analytic in $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$, but has no analytic continuation through $\partial\mathbb{D}$.