Please give complete and clearly written solutions.

1. (20 points) Determine the second-order Taylor formula for

\[ f(x, y) = \sin(xy) + \cos(xy) \]

near \((0, 0)\).

Recall that the second-order Taylor formula is given by

\[
f(x, y) = f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0) \]

\[
+ \frac{1}{2} \left( \frac{\partial^2 f}{\partial x^2}(x_0, y_0)(x - x_0)^2 + \frac{\partial^2 f}{\partial y^2}(x_0, y_0)(y - y_0)^2 \right) \]

\[
+ \frac{\partial^2 f}{\partial x \partial y}(x_0, y_0)(x - x_0)(y - y_0) + \frac{\partial^2 f}{\partial y \partial x}(x_0, y_0)(y - y_0)(x - x_0) \]

\[ + R_2(x, y), \]

where \((x_0, y_0) = (0, 0)\). Computing the necessary partial derivatives, we obtain

\[
\frac{\partial f}{\partial x} = y \cos(xy) - y \sin(xy), \quad \frac{\partial f}{\partial y} = x \cos(xy) - x \sin(xy) \]

\[
\frac{\partial^2 f}{\partial x^2} = -y^2 \sin(xy) - y^2 \cos(xy), \quad \frac{\partial^2 f}{\partial y^2} = -x^2 \sin(xy) - x^2 \cos(xy) \]

\[
\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = \cos(xy) - x y \sin(xy) - \sin(xy) - x y \cos(xy) \]

(Note that \(f \in C^2\), which implies that mixed partials are equal). It follows that

\[
f(x, y) = 1 + \frac{1}{2} \cdot 2 \cdot (x - 0)(y - 0) + R_2(x, y) = 1 + xy + R_2(x, y). \]
2. (20 points) Find all critical points of

\[ f(x, y) = 3x^2 + 2xy + 2x + y^2 + y + 2, \]

and classify them as local minima, maxima or saddle points. State the theorem that is used in this problem.

In order to find the critical points of \( f \), we need to solve the system of equations:

\[
\frac{\partial f}{\partial x} = 6x + 2y + 2 = 0 \quad \frac{\partial f}{\partial y} = 2x + 2y + 1 = 0
\]

Subtracting the second equation from the first one, we obtain that \( 4x + 1 = 0 \), i.e., \( x = -1/4 \) and \( y = -1/4 \). Thus there exists just one critical point for this function. We analyze it by Theorem 6 on page 198 of your textbook:

\[
(i) \quad \frac{\partial f}{\partial x}(-1/4, -1/4) = \frac{\partial f}{\partial y}(-1/4, -1/4) = 0
\]

\[
(ii) \quad \frac{\partial^2 f}{\partial x^2} = 6 > 0
\]

\[
(iii) \quad D = \frac{\partial^2 f}{\partial x^2} \frac{\partial^2 f}{\partial y^2} - \left( \frac{\partial^2 f}{\partial x \partial y} \right)^2 = 6 \cdot 2 - 2^2 > 0
\]

We conclude that \((-1/4, -1/4)\) is the strict local minimum, with \( f(-1/4, -1/4) = 13/8 \).
3. (20 points) Find the absolute maximum and minimum of the function

\[ f(x, y) = x^2 - xy + y^2 \]

on the unit disk \( D = \{ (x, y) | x^2 + y^2 \leq 1 \} \). Use the method of Lagrange multipliers to locate the maximum and minimum points of \( f \) on the boundary of \( D \).

We first look for the critical points of \( f \) inside of \( D \):

\[
\frac{\partial f}{\partial x} = 2x - y = 0 \quad \frac{\partial f}{\partial y} = -x + 2y = 0.
\]

Solving this system, one finds that \( x = 0 \) and \( y = 0 \). Hence \((0,0)\) is the only critical point inside.

Next we locate the extrema of \( f \) on the boundary of \( D \), which is defined by the constraint \( g(x, y) = x^2 + y^2 = 1 \). Using the method of Lagrange multipliers, we obtain

\[ \nabla f = \lambda \nabla g \quad \text{or} \quad (2x - y, -x + 2y) = \lambda (2x, 2y). \]

Consequently,

\[ 2x - y = \lambda 2x \quad \text{and} \quad -x + 2y = \lambda 2y. \]

Adding the equations, we have \( x + y = 2\lambda (x + y) \), i.e., \((x + y)(2\lambda - 1) = 0\). Thus either

(a) \( \lambda = 1/2 \) or (b) \( y = -x \).

(a) If \( \lambda = 1/2 \) then \( y = x \), so that \( x^2 + y^2 = 2x^2 = 1 \). This gives the points \((1/\sqrt{2}, 1/\sqrt{2})\) and \((-1/\sqrt{2}, -1/\sqrt{2})\).

(b) If \( y = -x \) then \( x^2 + y^2 = 2x^2 = 1 \) again, which gives two additional points \((1/\sqrt{2}, -1/\sqrt{2})\) and \((-1/\sqrt{2}, 1/\sqrt{2})\). Evaluating \( f \) at all five found points, we choose the largest value for the absolute maximum and the smallest value for the absolute minimum on \( D \):

\[
\begin{align*}
  f(1/\sqrt{2}, 1/\sqrt{2}) &= f(-1/\sqrt{2}, -1/\sqrt{2}) = 1/2, \\
  f(-1/\sqrt{2}, 1/\sqrt{2}) &= f(1/\sqrt{2}, -1/\sqrt{2}) = 3/2 \quad (\text{abs. max.}), \\
  f(0, 0) &= 0 \quad (\text{abs. min.})
\end{align*}
\]
4. (20 points)
(a) Find the length of the path \( c(t) = (\sqrt{3} \cos t, \sqrt{3} \sin t, t), \) where \( \pi \leq t \leq 3\pi. \)
(b) Find the path integral 
\[
\int_{c} e^{z} ds
\]
along the path \( c(t) = (5, -9, t^2), \quad t \in [0, \sqrt{\ln 4}]. \)

(a) Recall that the length of a path \( c(t) = (x(t), y(t), z(t)), \quad t \in [t_0, t_1], \) is given by
\[
L = \int_{t_0}^{t_1} \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2} \, dt.
\]
Thus
\[
L = \int_{\pi}^{3\pi} \sqrt{(-\sqrt{3} \sin t)^2 + (\sqrt{3} \cos t)^2 + 1^2} \, dt = \int_{\pi}^{3\pi} \sqrt{3(\sin^2 t + \cos^2 t) + 1} \, dt
\]
\[
= \int_{\pi}^{3\pi} 2 \, dt = 4\pi.
\]
(b)
\[
\int_{c} e^{z} ds = \int_{0}^{\sqrt{\ln 4}} e^{t^2} \| c'(t) \| \, dt = \int_{0}^{\sqrt{\ln 4}} e^{t^2} \sqrt{0^2 + 0^2 + (2t)^2} \, dt
\]
\[
= \int_{0}^{\sqrt{\ln 4}} e^{t^2} 2t \, dt = e^{\sqrt{\ln 4}} - e^{0} = e^{\ln 4} - e^{0} = 4 - 1 = 3.
\]
5. (20 points) Calculate the divergence and the curl of the vector field

\[ \mathbf{F}(x, y, z) = xy\mathbf{i} + yz\mathbf{j} + xz\mathbf{k}. \]

Is this a gradient vector field? Is this an irrotational vector field? Explain.

We only need to remember the formulas:

\[
\text{div} \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = y + z + x.
\]

\[
\text{curl} \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
F_1 & F_2 & F_3
\end{vmatrix}
= \left( \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) \mathbf{i} + \left( \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \right) \mathbf{j} + \left( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \mathbf{k}
= (0 - y)\mathbf{i} + (0 - z)\mathbf{j} + (0 - x)\mathbf{k} = -y\mathbf{i} - z\mathbf{j} - x\mathbf{k}.
\]

Since \( \text{curl} \mathbf{F} \neq 0 \), this field is neither gradient (see Theorem 1 on page 280) nor irrotational.