LECTURE 22

Integrals over Curves

Recall that the arc length of a parameterized curve $\sigma : [a, b] \to \mathbb{R}^n$ is given by

$$L = \int_a^b \left\| \frac{d\sigma}{dt} \right\| \, dt$$

This integral should be thought of as the limit of a Riemann sum of the form

$$\sum L_i = \sum_i \left\| \frac{d\sigma}{dt}(t_i) \right\| \Delta t$$

where the $L_i$ is the length of the curve segment between $\sigma(t_i)$ and $\sigma(t_i + \Delta t)$.

It is sometimes useful to consider “weighted” Riemann sums of the form

$$\sum f(\sigma(t_i)) L_i = \sum_i f(\sigma(t_i)) \left\| \frac{d\sigma}{dt}(t_i) \right\| \Delta t$$

In which case we are lead to consider integrals of the form

$$\int_a^b f(\sigma(t)) \left\| \frac{d\sigma}{dt} \right\| \, dt$$

Such integrals are called path integrals and are commonly presented via the notation

$$\int_C f \, ds$$

where $C = \{ \sigma(t) \in \mathbb{R}^n \mid t \in [a, b] \}$ denotes the corresponding curve.

For example, if we wished to calculate the moment of inertia about the $y$-axis of a wire winding through the $xy$-plane we might consider a Riemann sum of the form

$$\sum_i x(\sigma(t_i)) \rho \left\| \frac{d\sigma}{dt}(t_i) \right\| \Delta t$$

Here $\rho$ is the density of the wire, so that

$$\rho \left\| \frac{d\sigma}{dt}(t_i) \right\| \Delta t$$

is interpretable as the mass of the wire lying between $\sigma(t_i)$ and $\sigma(t_i + \Delta t)$; and $x(\sigma(t))$ is the distance of that segment from the $y$-axis. Passing from the Riemann sum to an integral expression in the usual fashion yields an integral of the form

$$\int_a^b x(\sigma(t)) \rho \left\| \frac{d\sigma}{dt}(t_i) \right\| \, dt \equiv \int_C x \, ds$$
0.1. Line Integrals. Another kind of integral that arises frequently in applications is the so-called line integral. This is defined as follows.

**Definition 22.1.** Let \( \mathbf{F} \) be a vector field on \( \mathbb{R}^n \) and let \( \sigma : [a, b] \to \mathbb{R}^n \) be a parameterized path in \( \mathbb{R}^n \). The **line integral** of \( \mathbf{F} \) along the corresponding curve \( C = \{ \sigma(t) \in \mathbb{R}^n \mid t \in [a, b] \} \) is the integral

\[
\int_C \mathbf{F} \cdot ds = \int_a^b \mathbf{F}(\sigma(t)) \cdot \frac{d\sigma}{dt} \, dt
\]

**Example 22.2.** Let \( \mathbf{F}(x) \) be a vector field describing the total force acting on a particle at position \( x \). The work done in moving the particle a small displacement \( \Delta x \) is given by

\[
\Delta W = \mathbf{F} \cdot \Delta x
\]

If we seek to estimate the work done in moving a particle along a path \( \sigma : [a, b] \to \mathbb{R}^n \) we are then led to a Riemann sum of the form

\[
W = \sum \Delta W = \sum \mathbf{F}(x_i) \cdot \Delta x = \sum \mathbf{F}(x_i) \cdot \frac{d\sigma}{dt} \Delta t
\]

and hence to an integral of the form

\[
\int_C \mathbf{F} \cdot ds \equiv \int_a^b \mathbf{F}(\sigma(t)) \cdot \frac{d\sigma}{dt} \, dt
\]

0.2. Properties of Path Integrals and Line Integrals.

**Definition 22.3.** Let \( h(t) \) be a differentiable real-valued function mapping an interval \( [c, d] \) on the real line to another interval \( [a, b] \). Assume moreover that \( h(t) \) is 1:1 and increasing. Let \( \sigma : [a, b] \to \mathbb{R}^n \) be a piecewise differentiable path. Then the path

\[
\gamma = \sigma \circ h : [c, d] \to \mathbb{R}^n
\]

is called a **reparameterization** of \( \sigma \).

**Theorem 22.4.** If \( \gamma : [c, d] \to \mathbb{R}^n \) is a reparameterization of a path \( \sigma : [a, b] \to \mathbb{R}^n \) then

1. For any function \( f : \mathbb{R}^n \to \mathbb{R} \)

\[
\int_{\sigma} f \, ds = \int_{\gamma} f \, ds
\]

2. For any vector field \( \mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n \) A vector field \( \mathbf{F} \) is said to be **conservative** if there exists a function \( f : \mathbb{R}^n \to \mathbb{R} \) such that

\[
\int_{\sigma} \mathbf{F} \cdot ds = \int_{\gamma} \mathbf{F} \cdot ds
\]

**Definition 22.5.**

\[
\mathbf{F} = \nabla f
\]

**Theorem 22.6.** Suppose \( f : \mathbb{R}^n \to \mathbb{R} \) is differentiable and that \( \sigma : [a, b] \to \mathbb{R}^n \) be a piecewise differentiable path. Then

\[
\int_{\sigma} \nabla f \cdot ds = f(\sigma(b)) - f(\sigma(a))
\]

**Proof.** We have

\[
\int_{\sigma} \nabla f \cdot ds \equiv \int_a^b \nabla f \cdot \frac{d\sigma}{dt} \, dt
\]

\[
= \int_a^b \frac{d}{dt}(f \circ \sigma) \, dt \quad \text{(by the chain rule)}
\]

\[
= f(\sigma(b)) - f(\sigma(a)) \quad \text{(by the Fundamental Theorem of Calculus)}
\]
Definition 22.7. A vector field $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ is called conservative if $\mathbf{F} = \nabla f$ for some function $f : \mathbb{R}^n \to \mathbb{R}$.

Remark 22.8. When a force field is conservative, the work done in moving a body from one point to another depends only on the initial and final positions; independent of the path taken. For, in this case,

$$W = \int_{\sigma} \mathbf{F} \cdot d\mathbf{s} = \int_{\sigma} \nabla f \cdot d\mathbf{s} = f(\sigma(b)) - f(\sigma(a))$$