Selected Theorems for Test 2

We use the standard notation \( s = \sum_{n=1}^{\infty} a_n, \) \( s_n = \sum_{k=1}^{n} a_k \) and \( R_n = s - s_n = \sum_{k=n+1}^{\infty} a_k. \)

**The Integral Test** Suppose that \( f \) is a continuous positive decreasing function on \([1, \infty)\), and let \( a_n = f(n) \). The series \( \sum_{n=1}^{\infty} a_n \) is convergent if and only if the improper integral \( \int_{1}^{\infty} f(x) \, dx \) is convergent. Furthermore, if the integral is convergent, then

\[
\int_{n+1}^{\infty} f(x) \, dx \leq R_n \leq \int_{n}^{\infty} f(x) \, dx.
\]

**The Comparison Test** Assume that \( \sum_{n=1}^{\infty} a_n \) and \( \sum_{n=1}^{\infty} b_n \) are series with positive terms.

(i) If \( \sum_{n=1}^{\infty} b_n \) is convergent and \( a_n \leq b_n \) for all \( n \), then \( \sum_{n=1}^{\infty} a_n \) is also convergent.

(ii) If \( \sum_{n=1}^{\infty} b_n \) is divergent and \( a_n \geq b_n \) for all \( n \), then \( \sum_{n=1}^{\infty} a_n \) is also divergent.

**The Limit Comparison Test** Suppose that \( \sum_{n=1}^{\infty} a_n \) and \( \sum_{n=1}^{\infty} b_n \) are series with positive terms. If the limit

\[
\lim_{n \to \infty} \frac{a_n}{b_n} = c
\]

exists, where \( c \) is a finite positive number, then either both series converge or both diverge.

**The Alternating Series Test** If the alternating series \( \sum_{n=1}^{\infty} (-1)^{n-1} b_n \) satisfies

(i) \( b_n \geq 0 \) for all \( n \)

(ii) \( b_{n+1} \leq b_n \) for all \( n \)

(iii) \( \lim_{n \to \infty} b_n = 0 \)

then the series converges. Furthermore, we have \( |R_n| \leq b_{n+1} \) in this case.

**The Ratio Test** For a series \( \sum_{n=1}^{\infty} a_n \), assume that the following limit exists:

\[
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = L.
\]

(i) If \( L < 1 \) then the series is absolutely convergent.

(ii) If \( L > 1 \) (or \( L = \infty \)) then the series is divergent.

(iii) If \( L = 1 \) then the test is inconclusive.

**The Root Test** For a series \( \sum_{n=1}^{\infty} a_n \), assume that the following limit exists:

\[
\lim_{n \to \infty} |a_n|^{1/n} = L.
\]

(i) If \( L < 1 \) then the series is absolutely convergent.

(ii) If \( L > 1 \) (or \( L = \infty \)) then the series is divergent.

(iii) If \( L = 1 \) then the test is inconclusive.