Convergence of Julia polynomials

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Abstract

We study the approximation of conformal mappings with the polynomials defined by Keldysh and Lavrentiev from an extremal problem considered by Julia. These polynomials converge uniformly on the closure of any Smirnov domain to the conformal mapping of this domain onto a disk. We prove estimates for the rate of such convergence on domains with piecewise analytic boundaries, expressed through the smallest exterior angles at the boundary.

Keywords. Conformal mapping, extremal problems, Smirnov domains, Smirnov spaces.

2000 MCS. Primary 30C40, 30E10; Secondary 41A10, 30C30.

1 Extremal problems in Smirnov spaces and the associated polynomials

Let G be a Jordan domain in the complex plane, bounded by a rectifiable curve L of length l. We consider the Smirnov spaces $E_p(G),\ 0 < p < \infty$, of analytic functions in G, whose boundary values satisfy

$$\|f\|_p := \left(\int_L |f(z)|^p |dz| \right)^{1/p} < \infty$$

*Research supported in part by the National Security Agency under Grant No. MDA904-03-1-0081.

(see Duren [8], Smirnov and Lebedev [23]). Julia [12] studied the extremal problem

$$\inf\{\|f\|_p : f \in E_p(G),\ f(\zeta) = 1)\}, \tag{1.1}$$

where $\zeta \in G$ is a fixed point. He showed that the above infimum is attained by the function $(\phi')^{1/p}$, where ϕ is the conformal mapping of G onto a disk $D_R := \{w : |w| < R\}$, normalized by the conditions $\phi(\zeta) = 0,\ \phi'(\zeta) = 1$. It is known that one can define sequences of polynomials associated with many extremal problems. Thus Keldysh and Lavrentiev (cf. [14], [13] and [15]) considered the polynomials $Q_{n,p}(z)$ that minimize (1.1) among all polynomials $P_n(z)$ of degree n, such that $P(\zeta) = 1$. These extremal polynomials were attributed to Julia by Keldysh and Lavrentiev in [14]. Perhaps, it is more appropriate to call them Julia-Keldysh-Lavrentiev polynomials. The goal of such construction is that, provided polynomials are dense in $E_p(G)$, it would furnish the approximation to $(\phi')^{1/p}$ by $Q_{n,p}$ as $n \to \infty$. Keldysh and Lavrentiev [15] characterized a class of domains for which these desired properties hold true. Their results are summarized below.

Theorem KL *Let $\psi = \phi^{-1}$. The following conditions are equivalent:*

1. $\log|\psi'|$ *is representable in D_R by the Poisson integral of its boundary values*
2. $\lim_{n\to\infty} \|(\phi')^{1/p} - Q_{n,p}\|_p = 0$
3. $Q_{n,p}$ *converge to $(\phi')^{1/p}$ locally uniformly in G*
4. *Polynomials are dense in norm in $E_p(G)$*

The domains with property *4* were later named after Smirnov. Although there is no complete geometric description of such domains, many standard classes of domains possess the Smirnov polynomial density property. The widest currently known class is probably that of Ahlfors-regular domains (cf. Pommerenke [18, Chap. 7]). They are defined by the condition that there exists a constant $C > 0$ such that

$$|L \cap D_r| \leq Cr,$$

for any disk D_r of radius r, where $|\cdot|$ denotes the arclength measure on $L = \partial G$.

Since $Q_{n,p}$ converge locally uniformly to $(\phi')^{1/p}$ in Smirnov domains, they have no zeros in any fixed compact subset of G for large n, by Hurwitz's theorem. Hence the following functions

$$J_{n,p}(z) := \int_{\zeta}^{z} (Q_{n,p}(t))^p \, dt, \qquad z \in G, \tag{1.2}$$

are well defined for $p \in (0, \infty)$, when n is large. Of course, they are well defined polynomials for any $n \in \mathbb{N}$, if $p \in \mathbb{N}$. Let $\| \cdot \|_\infty$ denote the uniform (sup) norm on $\overline{G}$.

Theorem 1.1 *If $p \in \mathbb{N}$ then*

$$\|\phi - J_{n,p}\|_\infty \le \frac{1}{2} \left((2\pi R)^{1/p} + l^{1/p} \right)^{p-1} \left\| (\phi')^{1/p} - Q_{n,p} \right\|_p .$$

Hence $\|\phi - J_{n,p}\|_\infty \to 0$ as $n \to \infty$, when G is a Smirnov domain.

For $p = 1$, the uniform convergence of $J_{n,1}$ on $\overline{G}$ was already observed by Keldysh and Lavrentiev in [15], but without any estimate. The case $p = 2$ is of special interest because of its close connections with the Szegő kernel and orthogonal polynomials. Ahlfors [1] considered these polynomials for the numerical approximation of conformal mappings, and developed interesting representations for $Q_{n,2}$ via iterated integrals of Vandermonde determinants. For $p = 2$, a similar result to Theorem 1.1 was proved by Warschawski [26]. Gaier [9, pp. 130-131] gave the estimates of the uniform convergence rates for some smooth domains, based on the results of Rosenbloom and Warschawski [22]. The explicit rates of convergence for domains with corners are stated in the following section. Note that the case $p = 2$ was already considered in [20], but with a somewhat different normalization for the polynomials.

A natural analogue of the polynomials $Q_{n,p}$ is given by the best approximating polynomials $\tilde{Q}_{n,p}$ to $(\phi')^{1/p}$ in $E_p(G)$:

$$\left\| (\phi')^{1/p} - \tilde{Q}_{n,p} \right\|_p = \inf \left\{ \left\| (\phi')^{1/p} - P_n \right\|_p : P_n(\zeta) = 1 \right\}, \tag{1.3}$$

where the inf is taken over the polynomials P_n of degree n. Following the same convention as for (1.2), we define the functions

$$\tilde{J}_{n,p}(z) := \int_{\zeta}^{z} (\tilde{Q}_{n,p}(t))^p \, dt. \tag{1.4}$$

It can be readily seen that $\tilde{Q}_{n,2} \equiv Q_{n,2}$ and $\tilde{J}_{n,2} \equiv J_{n,2}$ (for $p = 2$), because

$$\left\|(\phi')^{1/2} - P_n\right\|_2 = \|P_n\|_2 - \left\|(\phi')^{1/2}\right\|_2$$

for any polynomial P_n of degree at most n, $P_n(\zeta) = 1$, see [9, p. 128] and [1]. The explicit representation of $Q_{n,2}$ via the contour orthonormal polynomials $\{p_n\}_{n=0}^{\infty}$ in $E_2(G)$ follows from the standard Hilbert space theory (cf. [9, Chap. III] and [23, Chap. 4]):

$$Q_{n,2}(z) = \frac{\sum_{k=0}^{n} \overline{p_k(\zeta)} p_k(z)}{\sum_{k=0}^{n} |p_k(\zeta)|^2}, \quad n \in \mathbb{N}.$$

Thus these polynomials coincide (up to a constant factor) with the partial sums of the Szegő kernel $K(z,\zeta) = \sum_{k=0}^{\infty} \overline{p_k(\zeta)} p_k(z)$ (cf. Szegő [24]). They can be used for the constructive approximation of the conformal mapping ϕ, see [9], [23] and [20] for the details.

Our interest in $\tilde{Q}_{n,p}$ and $\tilde{J}_{n,p}$ is explained by the fact that one can produce estimates of the convergence rates for these polynomials. We first state the analogue of Theorem 1.1 in this case.

Theorem 1.2 *If $p \in \mathbb{N}$ then*

$$\left\|\phi - \tilde{J}_{n,p}\right\|_\infty \le \frac{1}{2}\left(3(2\pi R)^{1/p} + l^{1/p}\right)^{p-1} \left\|(\phi')^{1/p} - \tilde{Q}_{n,p}\right\|_p.$$

Hence $\left\|\phi - \tilde{J}_{n,p}\right\|_\infty \to 0$ *as* $n \to \infty$, *when* G *is a Smirnov domain.*

The rates of convergence quantifying this result are studied in the next section.

We conclude this section by showing that the zeros of the polynomials $\tilde{Q}_{n,p}$ are typically dense in L. This indicates that the definition $\tilde{J}_{n,p}$ in (1.4) cannot be extended to $\overline{G}$ for $p \notin \mathbb{N}$, as every zero generates a branch point. Let $\nu_{n,p}$ and $\tilde{\nu}_{n,p}$ be the normalized counting measures for the zeros of $Q_{n,p}$ and $\tilde{Q}_{n,p}$ respectively. They are obtained by placing the point mass $1/n$ at each zero of $Q_{n,p}$ and $\tilde{Q}_{n,p}$, according to multiplicities. Denote the equilibrium measure of $\overline{G}$ (in the sense of logarithmic potential theory) by μ [21]. The following Jentzsch-Szegő type theorem on the asymptotic zero distribution is stated in terms of the weak* convergence for measures.

Theorem 1.3 *Let G be a Smirnov domain. Suppose that ϕ cannot be continued as analytic on $\overline{G}$ function. Then for any $p \in (0, \infty)$ there exists an infinite subsequence $\tilde{N} \subset \mathbb{N}$ such that*

$$\tilde{\nu}_{n,p} \xrightarrow{*} \mu \qquad \text{as } n \to \infty,\ n \in \tilde{N}.$$

We conjecture that this theorem also holds for the zeros of $Q_{n,p}$, i.e.,

$$\nu_{n,p} \xrightarrow{*} \mu \qquad \text{as } n \to \infty,\ n \in N,$$

where $N \subset \mathbb{N}$ is an infinite subsequence. Such asymptotic behavior is generic for zeros of many extremal polynomials, see [3]. It is possible to prove a converse of Theorem 1.3, i.e., if there is a subsequence of $\tilde{\nu}_{n,p}$ weakly convergent to μ, then ϕ is not analytic on $\overline{G}$ (cf. [7]).

2 Convergence in domains with piecewise analytic boundaries

We consider domains with piecewise analytic boundaries in this section, which are important in applications. An analytic arc is defined as the image of a segment under a mapping that is conformal in an open neighborhood of the segment. Thus a domain has piecewise analytic boundary if it is bounded by a Jordan curve consisting of a finite number of analytic arcs. Let L be piecewise analytic, with the smallest exterior angle $\lambda\pi,\ 0 < \lambda \leq 2$, at the junction points of the analytic arcs. The following results contain estimates for the rates of convergence of $\tilde{Q}_{n,p}$ in terms of geometric properties of domains.

Theorem 2.1 *If $0 < \lambda < 2$ then*

$$\left\| (\phi')^{1/p} - \tilde{Q}_{n,p} \right\|_p \leq C_1 \begin{cases} n^{-\frac{\lambda}{p(2-\lambda)}}, & 1 < p < \infty, \\ n^{-\frac{\lambda}{2-\lambda}} \log n, & p = 1, \\ n^{-\frac{\lambda(\lambda-1)}{p(2-\lambda)} - \lambda}, & \frac{1-\lambda}{2-\lambda} < p < 1. \end{cases} \tag{2.1}$$

Note that $p \geq 1/2$ works for all $\lambda \in (0, 2)$. For $p \in \mathbb{N}$, we also have

$$\|\phi - \tilde{J}_{n,p}\|_\infty \leq C_2 \begin{cases} n^{-\frac{\lambda}{2-\lambda}} \log n, & p = 1, \\ n^{-\frac{\lambda}{p(2-\lambda)}}, & p = 2, 3, \ldots. \end{cases} \tag{2.2}$$

The constants $C_1 > 0$ and $C_2 > 0$ are independent of $n \geq 2$.

The rates of convergence were previously known only in the case of $p = 2$ for some smooth domains, see [9, p. 131]. This theorem is new for any $p \in (0, \infty)$. It is worth noting that the exponent $\lambda/(2-\lambda)$ for $p = 1$ in (2.1) and (2.2) is best possible. Indeed, it is known that this exponent cannot be improved, in general, for approximation of ϕ in the uniform norm by *any* sequence of polynomials (cf. [10] and [11]). We believe that the exponents of n are also sharp in (2.1) and (2.2) for any $p \in (1, \infty)$.

When all angles at the boundary are the outward pointing cusps, we can make an even stronger conclusion.

Theorem 2.2 *If $\lambda = 2$ then for any $p \in (0, \infty)$ there exist $q, r \in (0, 1)$ such that*

$$\left\| (\phi')^{1/p} - \tilde{Q}_{n,p} \right\|_p \le C_3\, q^{n^r}, \qquad n \in \mathbb{N}. \tag{2.3}$$

Furthermore, if $p \in \mathbb{N}$ then

$$\|\phi - \tilde{J}_{n,p}\|_\infty \le C_4\, q^{n^r}, \qquad n \in \mathbb{N}. \tag{2.4}$$

Here, $C_3 > 0$ and $C_4 > 0$ are independent of n.

It should be mentioned that r cannot be equal to 1 in the above theorem, because the geometric rate of convergence implies that ϕ is analytic on $\overline{G}$ [25]. However, this is clearly not possible when G has an outward pointing cusp, see [5]. A convergence result of similar form was proved in [5] for Bieberbach polynomials in the Bergman kernel method.

3 Proofs

3.1 Proofs of the results from Section 1

Proof of Theorem 1.1. Using the mapping $\psi := \phi^{-1}$, we obtain for any $z \in G$ that

$$\begin{aligned} |\phi(z) - J_{n,p}(z)| &= \left| \int_\zeta^z \big(\phi'(t) - J'_{n,p}(t)\big)\, dt \right| \\ &= \left| \int_0^{\phi(z)} \big(\phi'(\psi(u)) - J'_{n,p}(\psi(u))\big)\, \psi'(u) du \right| \\ &\le \int_0^{\phi(z)} \left|\phi'(\psi(u)) - J'_{n,p}(\psi(u))\right| |\psi'(u)||du|, \end{aligned}$$

where the integration is carried over the segment connecting 0 and $\phi(z)$ in D_R. Since L is rectifiable, the function under the latter integral belongs to the Hardy class $H^1(D_R)$. Hence we obtain by the Fejér-Riesz inequality (cf. [8, Theorem 3.13]) that

$$\begin{aligned}|\phi(z) - J_{n,p}(z)| &\le \frac{1}{2}\int_{|u|=R} \left|\phi'(\psi(u)) - J'_{n,p}(\psi(u)\right| |\psi'(u)||du| \\ &= \frac{1}{2}\int_L \left|\phi'(t) - Q^p_{n,p}(t)\right| |dt|.\end{aligned}$$

If $p = 1$ then we are done. Applying Hölder's inequality for $p \ge 2$, we have

$$\begin{aligned}|\phi(z) - J_{n,p}(z)| &\le \frac{1}{2}\int_L \left|(\phi'(t))^{1/p} - Q_{n,p}(t)\right| \left|\sum_{k=0}^{p-1} (\phi'(t))^{k/p} (Q_{n,p}(t))^{p-k-1}\right| |dt| \\ &\le \frac{1}{2}\ \left\|(\phi')^{1/p} - Q_{n,p}\right\|_p \left\|\sum_{k=0}^{p-1} (\phi')^{k/p} (Q_{n,p})^{p-k-1}\right\|_q,\end{aligned} \tag{3.1}$$

where $q = p/(p-1)$. Observe that

$$\begin{aligned}\left|\sum_{k=0}^{p-1} (\phi'(t))^{k/p} (Q_{n,p}(t))^{p-k-1}\right| &\le \sum_{k=0}^{p-1} |\phi'(t)|^{k/p} |Q_{n,p}(t)|^{p-k-1} \\ &\le \left(|\phi'(t)|^{1/p} + |Q_{n,p}(t)|\right)^{p-1},\end{aligned}$$

so that

$$\begin{aligned}\left\|\sum_{k=0}^{p-1} (\phi')^{k/p} (Q_{n,p})^{p-k-1}\right\|_q &\le \left(\int_L \left(|\phi'(t)|^{1/p} + |Q_{n,p}(t)|\right)^p |dt|\right)^{\frac{p-1}{p}} \\ &\le \left(\left\|(\phi')^{1/p}\right\|_p + \|Q_{n,p}\|_p\right)^{p-1},\end{aligned}$$

by Minkowski's inequality. Since $\left\|(\phi')^{1/p}\right\|_p = \left(\int_L |\phi'(t)|\ |dt|\right)^{1/p} = (2\pi R)^{1/p}$, it follows from (3.1) that

$$\|\phi - J_{n,p}\|_\infty \le \frac{1}{2}\left((2\pi R)^{1/p} + \|Q_{n,p}\|_p\right)^{p-1} \left\|(\phi')^{1/p} - Q_{n,p}\right\|_p. \tag{3.2}$$

Recall that $\|Q_{n,p}\|_p \le \|1\|_p = l^{1/p}$, by the definition of $Q_{n,p}$, so that the inequality is proved. The second statement now follows from Part *2* of Theorem KL.
■

Proof of Theorem 1.2. Repeating all steps of the proof of Theorem 1.1 up to (3.2), but with $\tilde{J}_{n,p}$ and $\tilde{Q}_{n,p}$ instead of $J_{n,p}$ and $Q_{n,p}$, we obtain the inequality

$$\|\phi - \tilde{J}_{n,p}\|_\infty \le \frac{1}{2}\left((2\pi R)^{1/p} + \|\tilde{Q}_{n,p}\|_p\right)^{p-1} \left\|(\phi')^{1/p} - \tilde{Q}_{n,p}\right\|_p .$$

The proof of the desired inequality is finished by estimating

$$\begin{aligned}\|\tilde{Q}_{n,p}\|_p &\le \left\|\tilde{Q}_{n,p} - (\phi')^{1/p}\right\|_p + \left\|(\phi')^{1/p}\right\|_p \le \left\|1 - (\phi')^{1/p}\right\|_p + \left\|(\phi')^{1/p}\right\|_p \\ &\le \|1\|_p + 2\left\|(\phi')^{1/p}\right\|_p = l^{1/p} + 2(2\pi R)^{1/p}.\end{aligned}$$

We also have from the definition of $\tilde{Q}_{n,p}$ in (1.3) that

$$\left\|(\phi')^{1/p} - \tilde{Q}_{n,p}\right\|_p \le \left\|(\phi')^{1/p} - Q_{n,p}\right\|_p ,$$

which tends to 0 as $n \to \infty$ in Smirnov domains, by Part *2* of Theorem KL.
■

We connect the analyticity of ϕ on $\overline{G}$ and the asymptotics for the leading coefficients of $\tilde{Q}_{n,p}$ in the following lemma.

Lemma 3.1 *Let G be a Smirnov domain. Set $\tilde{Q}_{n,p}(z) = \tilde{a}_{n,p} z^n + \ldots + \tilde{a}_{0,p}$, $n \in \mathbb{N}$. If ϕ is not analytic on $\overline{G}$, then*

$$\limsup_{n\to\infty} |\tilde{a}_{n,p}|^{1/n} = \frac{1}{\operatorname{cap}(\overline{G})}, \tag{3.3}$$

where $\operatorname{cap}(\overline{G})$ *is the logarithmic capacity of* $\overline{G}$.

Proof. The idea of this proof is suggested by Blatt and Saff [6]. We first note that, for any polynomial $P_n(z) = a_n z^n + \ldots$, the following holds true

$$|a_n| \le \left(\operatorname{cap}(\overline{G})\right)^{-n} \|P_n\|_\infty, \tag{3.4}$$

see Lemma 4.1 in [6]. Indeed, if Φ is the conformal mapping of $\Omega := \overline{\mathbb{C}}\backslash\overline{G}$ onto the exterior of the unit disk, normalized by $\Phi(\infty) = \infty$ and $\lim_{z\to\infty} \Phi(z)/z = 1/\text{cap}(\overline{G})$, then

$$\left|\frac{P_n(z)}{\Phi^n(z)}\right| \le \|P_n\|_\infty, \qquad z \in \Omega,$$

by the maximum modulus principle for $P_n(z)/\Phi^n(z)$ in Ω. Now let $z \to \infty$ to obtain (3.4). It follows from Theorem 1.1 of [19] that

$$\|P_n\|_\infty \le c_1 n^{2/p} \|P_n\|_p, \tag{3.5}$$

where $c_1 > 0$ is independent of n. Therefore,

$$\limsup_{n\to\infty} |\tilde{a}_{n,p}|^{1/n} \le \frac{\limsup_{n\to\infty} \|\tilde{Q}_{n,p}\|_\infty^{1/n}}{\text{cap}(\overline{G})} \le \frac{\limsup_{n\to\infty} \|\tilde{Q}_{n,p}\|_p^{1/n}}{\text{cap}(\overline{G})} = \frac{1}{\text{cap}(\overline{G})},$$

because of (3.4), (3.5) and $\lim_{n\to\infty} \|\tilde{Q}_{n,p}\|_p = \| (\phi')^{1/p} \|_p$. We assume that

$$\limsup_{n\to\infty} |\tilde{a}_{n,p}|^{1/n} < \frac{1}{\text{cap}(\overline{G})},$$

and show this leads to a contradiction. Consider a sequence of Fekete polynomials $F_n,\ n \in \mathbb{N}$, for $\overline{G}$, so that

$$\lim_{n\to\infty} \|F_n\|_\infty^{1/n} = \text{cap}(\overline{G}),$$

see [21, Sect. 5.5]. We define a new sequence $q_n(z) := \tilde{a}_{n,p}(z-\zeta)F_{n-1}(z) = \tilde{a}_{n,p}z^n + \dots,\ n \in \mathbb{N}$. It follows from the extremal property (1.3) that

$$\left\|(\phi')^{1/p} - \tilde{Q}_{n-1,p}\right\|_p \le \left\|(\phi')^{1/p} - \left(\tilde{Q}_{n,p} - q_n\right)\right\|_p \le \left\|(\phi')^{1/p} - \tilde{Q}_{n,p}\right\|_p + \|q_n\|_p,$$

for $p \in [1,\infty)$. Thus we obtain from the above that

$$\begin{aligned}\limsup_{n\to\infty} \left(\left\|(\phi')^{1/p} - \tilde{Q}_{n-1,p}\right\|_p - \left\|(\phi')^{1/p} - \tilde{Q}_{n,p}\right\|_p\right)^{1/n} &\le \limsup_{n\to\infty} \|q_n\|_p^{1/n} \\ &\le \limsup_{n\to\infty} |\tilde{a}_{n,p}|^{1/n} \lim_{n\to\infty} \|F_{n-1}\|_\infty^{1/n} < 1.\end{aligned}$$

Consequently,

$$d := \limsup_{n\to\infty} \left\|(\phi')^{1/p} - \tilde{Q}_{n,p}\right\|_p^{1/n} < 1, \tag{3.6}$$

as $\lim_{n\to\infty}\left\|(\phi')^{1/p}-\tilde{Q}_{n,p}\right\|_p=0$ by Theorem KL. If $p\in(0,1)$ then we have that

$$\left\|(\phi')^{1/p}-\tilde{Q}_{n-1,p}\right\|_p^p \le \left\|(\phi')^{1/p}-\tilde{Q}_{n,p}\right\|_p^p + \|q_n\|_p^p,$$

and (3.6) follows by a similar argument.

Since $\tilde{Q}_{n,p}$ converges to $(\phi')^{1/p}$ locally uniformly in G by Theorem KL, we can write

$$(\phi'(z))^{1/p}-\tilde{Q}_{n,p}(z)=\sum_{k=1}^{\infty}\left(\tilde{Q}_{(k+1)n,p}(z)-\tilde{Q}_{kn,p}(z)\right), \qquad z\in G.$$

Thus

$$\begin{aligned}\left\|(\phi')^{1/p}-\tilde{Q}_{n,p}\right\|_\infty &\le \sum_{k=1}^{\infty}\left\|\tilde{Q}_{(k+1)n,p}-\tilde{Q}_{kn,p}\right\|_\infty \\ &\le \sum_{k=1}^{\infty} c_1\left((k+1)n\right)^{2/p}\left\|\tilde{Q}_{(k+1)n,p}-\tilde{Q}_{kn,p}\right\|_p,\end{aligned}$$

by (3.5). It is clear from (3.6) that there exist $c_2,\varepsilon>0$ such that $d+\varepsilon<1$ and

$$\left\|(\phi')^{1/p}-\tilde{Q}_{n,p}\right\|_p < c_2(d+\varepsilon/2)^n, \qquad n\in\mathbb{N}.$$

Hence for $p\in[1,\infty)$

$$\begin{aligned}\left\|\tilde{Q}_{(k+1)n,p}-\tilde{Q}_{kn,p}\right\|_p &\le \left\|(\phi')^{1/p}-\tilde{Q}_{(k+1)n,p}\right\|_p+\left\|(\phi')^{1/p}-\tilde{Q}_{kn,p}\right\|_p \\ &\le 2\left\|(\phi')^{1/p}-\tilde{Q}_{kn,p}\right\|_p \le 2c_2(d+\varepsilon/2)^{kn},\end{aligned}$$

and for $p\in(0,1)$

$$\begin{aligned}\left\|\tilde{Q}_{(k+1)n,p}-\tilde{Q}_{kn,p}\right\|_p^p &\le \left\|(\phi')^{1/p}-\tilde{Q}_{(k+1)n,p}\right\|_p^p+\left\|(\phi')^{1/p}-\tilde{Q}_{kn,p}\right\|_p^p \\ &\le 2\left\|(\phi')^{1/p}-\tilde{Q}_{kn,p}\right\|_p^p \le 2c_2^p(d+\varepsilon/2)^{knp}.\end{aligned}$$

It now follows that

$$\begin{aligned}\left\|(\phi')^{1/p}-\tilde{Q}_{n,p}\right\|_\infty &\le c_3\sum_{k=1}^{\infty}\left((k+1)n\right)^{2/p}(d+\varepsilon/2)^{kn} \le c_4\sum_{k=1}^{\infty}(d+\varepsilon)^{kn} \\ &\le c_5(d+\varepsilon)^n, \quad n\in\mathbb{N}.\end{aligned}$$

The latter estimate is well known to imply that $(\phi')^{1/p}$ is analytic on $\overline{G}$, see [25]. Contradiction.
■

Proof of Theorem 1.3. Consider the monic polynomials $\tilde{Q}_{n,p}(z)/\tilde{a}_{n,p},\ n \in \mathbb{N}$. Let $\tilde{N}$ be a subsequence such that (3.3) holds along $\tilde{N}$ as a regular limit. Then we obtain with the help of (3.5) that

$$\limsup_{\substack{n\to\infty \\ n\in\tilde{N}}} \left\|\tilde{Q}_{n,p}/\tilde{a}_{n,p}\right\|_\infty^{1/n} = \lim_{\substack{n\to\infty \\ n\in\tilde{N}}} |\tilde{a}_{n,p}|^{-1/n} \limsup_{\substack{n\to\infty \\ n\in\tilde{N}}} \left\|\tilde{Q}_{n,p}\right\|_\infty^{1/n} \le \mathrm{cap}(\overline{G}),$$

as $\lim_{n\to\infty} \|\tilde{Q}_{n,p}\|_p = \|\,(\phi')^{1/p}\,\|_p$. Since $\tilde{Q}_{n,p}$ converge to $(\phi')^{1/p}$ locally uniformly in G, we have

$$\lim_{n\to\infty} \tilde{\nu}_{n,p}(E) = 0$$

for any compact $E \subset G$, by Hurwitz's theorem. Theorem 1.3 now follows from Theorem 2.1 of [7] (or Theorem 2.1.7 of [3]).
■

3.2 Proofs of the results from Section 2

Proof of Theorem 2.1. We extend the ideas of [20] for $p = 2$ to the general case. Let us continue the mapping ϕ conformally beyond the boundary L, by using reflections across the analytic arcs $L_i,\ L = \cup_{i=1}^m L_i$. Suppose that τ_i is a mapping such that $L_i = \tau_i([0,1])$, which is conformal in an open neighborhood of $[0,1]$. Then we can find a symmetric lens shaped domain S_i, bounded by two circular arcs subtended by $[0,1]$, whose closure is contained in this open neighborhood of $[0,1]$. Defining

$$\tilde{G} := G \cup \left(\cup_{i=1}^m \tau_i(S_i)\right),$$

we extend ϕ into $\tilde{G}$ as follows:

$$\phi(z) := \frac{R^2}{\overline{\phi\left[\tau_i\left(\overline{\tau_i^{-1}(z)}\right)\right]}}, \qquad z \in \tau_i(S_i)\setminus\overline{G},$$

where $i = 1, \ldots, m$. The boundary $\partial\tilde{G}$ consists of m analytic arcs Γ_i that share endpoints with the arcs L_i of ∂G:

$$\partial\tilde{G} \cap \partial G = \{z_i\}_{i=1}^m,$$

which are clearly the corner points of ∂G. Since each $\tau_i,\ i=1,\dots,m$, is conformal and has bounded derivative (together with its inverse) on S_i, we obtain the inequalities

$$\operatorname{dist}(z,\partial G) \geq c_1 \min_{1\leq i\leq m} |z-z_i|, \qquad z\in\partial\tilde{G}, \tag{3.7}$$

where $\operatorname{dist}(z,\partial G)$ is the distance from z to ∂G, and

$$|\gamma| \leq c_2|z-t|, \qquad z,t\in\partial\tilde{G}, \tag{3.8}$$

where $|\gamma|$ is the length of the shorter arc $\gamma\subset\partial\tilde{G}$, connecting z and t. We denote various positive constants by c_1, c_2, etc.

Let Γ_j be an arc of $\partial\tilde{G}$ with the endpoints z_j and z_{j+1}, and let $\zeta_j\in\Gamma_j$ be a fixed point, $j=1,\dots,m$. Note that ζ_j divides Γ_j into Γ_j^1 and Γ_j^2, so that $\partial\tilde{G}=\bigcup_{j=1}^m\bigcup_{i=1}^2\Gamma_j^i$. We obtain from Cauchy's integral formula for the continuation of $(\phi')^{1/p}$ into $\tilde{G}$ that

$$(\phi'(z))^{1/p} = \frac{1}{2\pi i}\int_{\partial\tilde{G}} \frac{(\phi'(t))^{1/p}}{t-z}\,dt = \frac{1}{2\pi i}\sum_{j=1}^m\sum_{i=1}^2\int_{\Gamma_j^i}\frac{(\phi'(t))^{1/p}}{t-z}\,dt, \quad z\in\tilde{G}. \tag{3.9}$$

Hence we need to approximate the functions of the form

$$g(z) := \int_\gamma \frac{(\phi'(t))^{1/p}}{t-z}\,dt \tag{3.10}$$

in $E_p(G)$ norm, where γ is any of the arcs Γ_j^i, with $i=1,2$ and $j=1,\dots,m$.

Let $\Omega := \overline{\mathbb{C}}\setminus\overline{G}$. Consider the standard conformal mapping $\Phi:\Omega\to\Delta$, where $\Delta:=\{w:|w|>1\}$, normalized by $\Phi(\infty)=\infty$ and $\Phi'(\infty)>0$. We define the level curves of Φ by

$$L_n := \{z : |\Phi(z)| = 1+1/n\}, \qquad n\in\mathbb{N}.$$

Denote by γ_2 the part of γ from its endpoint $\zeta_j\in\Gamma_j$ to the first point ξ of intersection with L_n, so that $\gamma_2\subset\{z:|\Phi(z)|>1+1/n\}$. Then $\gamma_1:=\gamma\setminus\gamma_2$ connects ξ with the corner point z_j of L. Write

$$g(z) := \int_{\gamma_1}\frac{(\varphi'(t))^{1/p}}{t-z}\,dt + \int_{\gamma_2}\frac{(\varphi'(t))^{1/p}}{t-z}\,dt =: g_1(z)+g_2(z). \tag{3.11}$$

We show that $\|g_1\|_p \to 0$ sufficiently fast as $n \to \infty$, while g_2 is well approximated by polynomials of degree n. To estimate the norm of g_1, we need to know the behavior of $(\phi')^{1/p}$ near the corner point $z_j \in L$. This is found from the asymptotic expansion of Lehman [16]. Assume that $z_j = 0$ and that $\lambda_j \pi,\ 0 < \lambda_j < 2$, is the exterior angle formed by L at this point. Then we have in a neighborhood of $z_j = 0$ that

$$\phi(z) - \phi(0) = b\, z^{\frac{1}{2-\lambda_j}} + o\left(z^{\frac{1}{2-\lambda_j}}\right) \qquad \text{as } z \to 0,$$

where $b \neq 0$, and

$$\phi'(z) = \frac{b}{2-\lambda_j}\, z^{\frac{1}{2-\lambda_j}-1} + o\left(z^{\frac{1}{2-\lambda_j}-1}\right) \qquad \text{as } z \to 0.$$

Hence there exists a constant $c_3 > 0$ such that

$$|\phi'(z)|^{1/p} \le c_3\, |z|^{\alpha}, \qquad z \in \tilde{G} \cup \partial \tilde{G}, \tag{3.12}$$

where we set

$$\alpha := \frac{1}{p(2-\lambda_j)} - \frac{1}{p}.$$

For the endpoints $\xi \in L_n$ and 0 of γ_1, we let

$$d_n := |\xi - 0| = |\xi|.$$

It follows from (3.8) that

$$|\gamma_1| \le c_2 d_n.$$

We now estimate that

$$\|g_1\|_p^p = \int_L \left| \int_{\gamma_1} \frac{(\varphi'(t))^{1/p}}{t-z}\, dt \right|^p |dz| \le c_4 \int_L \left(\int_{\gamma_1} \frac{|t|^{\alpha} |dt|}{|t-z|} \right)^p |dz|, \tag{3.13}$$

by (3.11) and (3.12). Note that if $z \in L$ satisfies $|z| \ge d_n$, then $|t-z| \sim |z|$ by (3.7). Consequently,

$$\begin{aligned} \int_{L \cap \{|z| \ge d_n\}} \left(\int_{\gamma_1} \frac{|t|^{\alpha} |dt|}{|t-z|} \right)^p |dz| &\le c_5 \int_{L \cap \{|z| \ge d_n\}} \left(\frac{d_n^{\alpha+1}}{|z|} \right)^p |dz| \qquad (3.14) \\ &\le c_6 \begin{cases} d_n^{p\alpha+1}, & 1 < p < \infty, \\ d_n^{\alpha+1} |\log d_n|, & p = 1, \\ d_n^{p\alpha+p}, & \frac{1-\lambda}{2-\lambda} < p < 1, \end{cases} \end{aligned}$$

because $\alpha+1>0$ (which defines the latter range for p). On the other hand, if $z\in L$ satisfies $|z|\le d_n$, then $|t-z|\sim|t|+|z|$ by (3.7), and we obtain by using (3.8) that

$$\begin{aligned}\int_{L\cap\{|z|\le d_n\}}\left(\int_{\gamma_1}\frac{|t|^\alpha|dt|}{|t-z|}\right)^p|dz| &\le c_7\int_0^{c_8d_n}\left(\int_0^{c_9d_n}\frac{s^\alpha ds}{s+r}\right)^p dr \qquad (3.15)\\ &\le c_7\int_0^{c_8d_n}\left(\int_0^r\frac{s^\alpha}{r}\,ds+\int_r^{c_9d_n}s^{\alpha-1}ds\right)^p dr\\ &= c_7\int_0^{c_8d_n}\left(\frac{r^\alpha}{\alpha+1}+\frac{(c_9d_n)^\alpha-r^\alpha}{\alpha}\right)^p dr\\ &\le c_{10}\,d_n^{p\alpha+1},\end{aligned}$$

for $\alpha\ne 0$. If $\alpha=0$ then we estimate

$$\begin{aligned}\int_{L\cap\{|z|\le d_n\}}\left(\int_{\gamma_1}\frac{|dt|}{|t-z|}\right)^p|dz| &= \int_{L\cap\{|z|\le d_n\}}\left(\int_{\gamma_1}\frac{|t|^{1/2}\,|t|^{-1/2}\,|dt|}{|t-z|}\right)^p|dz|\\ &\le (c_2\,d_n)^{p/2}\int_{L\cap\{|z|\le d_n\}}\left(\int_{\gamma_1}\frac{|t|^{-1/2}\,|dt|}{|t-z|}\right)^p|dz|\\ &\le c_2^{p/2}\,d_n^{p/2}\,c_{10}\,d_n^{-p/2+1}=c_2^{p/2}\,c_{10}\,d_n,\end{aligned}$$

as above. Combining (3.13)-(3.15), we have that

$$\begin{aligned}\|g_1\|_p &\le c_{11}\begin{cases} d_n^{\alpha+1/p}, & 1<p<\infty,\\ d_n^{\alpha+1}|\log d_n|, & p=1,\\ d_n^{\alpha+1}, & \frac{1-\lambda}{2-\lambda}<p<1.\end{cases}\\ &\le c_{11}\begin{cases} d_n^{\frac{1}{p(2-\lambda)}}, & 1<p<\infty,\\ d_n^{\frac{1}{2-\lambda}}|\log d_n|, & p=1,\\ d_n^{\frac{\lambda-1}{p(2-\lambda)}+1}, & \frac{1-\lambda}{2-\lambda}<p<1,\end{cases} \qquad (3.16)\end{aligned}$$

where $\lambda=\min_{1\le j\le m}\lambda_j$.

The next step is the construction of approximating polynomials P_n for g_2. This is accomplished by using Dzjadyk's kernels (see, e.g., [2]) of the form

$$K_n(t,z)=\sum_{i=0}^n a_i(t)z^i,\qquad n\in\mathbb{N},$$

which approximate the Cauchy kernel. It was proved in Lemma 5 of [4] that a sequence of such kernels can be selected, so that for any fixed $k \in \mathbb{N}$, and for all $t \in \gamma$ with $|\Phi(t)| \geq 1 + 1/n$, we have

$$\left| \frac{1}{t-z} - K_n(t,z) \right| \leq c_{12} \frac{d_n^k}{|t-z|^{k+1}}, \qquad z \in L, \tag{3.17}$$

for all sufficiently large $n \in \mathbb{N}$. In particular, (3.17) holds for $t \in \gamma_2$. Define the polynomials

$$P_n(z) := \int_{\gamma_2} (\phi'(t))^{1/p} \; K_n(t,z)\, dt,$$

and estimate

$$\begin{aligned} \|g_2 - P_n\|_p^p &= \int_L \left| \int_{\gamma_2} \left(\frac{1}{t-z} - K_n(t,z) \right) (\phi'(t))^{1/p} \; dt \right|^p \; |dz| \\ &\leq c_{13} d_n^{kp} \int_L \left(\int_{\gamma_2} \frac{|t|^\alpha |dt|}{|t-z|^{k+1}} \right)^p \; |dz|, \end{aligned}$$

by (3.17) and (3.12). Observe that $|t-z| \sim |t| + |z|$ for $t \in \gamma_2$. Therefore, we have for $k > \alpha + 1/p$ that

$$\begin{aligned} \int_L \left(\int_{\gamma_2} \frac{|t|^\alpha |dt|}{|t-z|^{k+1}} \right)^p |dz| &\leq c_{14} \int_0^{c_{15}} \left(\int_{c_{16} d_n}^{c_{17}} \frac{s^\alpha ds}{(s+r)^{k+1}} \right)^p dr \\ &\leq c_{14} \int_0^{c_{16} d_n} \left(\int_{c_{16} d_n}^{c_{17}} s^{\alpha-k-1} \, ds \right)^p dr \\ &+ c_{14} \int_{c_{16} d_n}^{c_{15}} \left(r^{-k-1} \int_{c_{16} d_n}^{r} s^\alpha \, ds + \int_r^{c_{17}} s^{\alpha-k-1} \, ds \right)^p dr \\ &\leq c_{18} \, d_n^{p(\alpha-k)+1} + c_{19} \int_{c_{16} d_n}^{c_{15}} r^{p(\alpha-k)} \, dr \\ &\leq c_{20} \, d_n^{p(\alpha-k)+1}. \end{aligned}$$

It follows that

$$\|g_2 - P_n\|_p \leq c_{21} \, d_n^{\alpha+1/p} \leq c_{21} \, d_n^{\frac{1}{p(2-\lambda)}}. \tag{3.18}$$

Combining (3.16) and (3.18), we obtain

$$\|g - P_n\|_p \leq \|g_1\|_p + \|g_2 - P_n\|_p \leq c_{22} \begin{cases} d_n^{\frac{1}{p(2-\lambda)}}, & 1 < p < \infty, \\ d_n^{\frac{1}{2-\lambda}} |\log d_n|, & p = 1, \end{cases} \tag{3.19}$$

and

$$\|g-P_n\|_p^p \le \|g_1\|_p^p + \|g_2-P_n\|_p^p \le c_{22}\, d_n^{\frac{\lambda-1}{2-\lambda}+p}, \quad \frac{1-\lambda}{2-\lambda} < p < 1. \tag{3.20}$$

Recall that $d_n = |\xi|$, where $\xi \in L_n \cap \gamma_1$. Applying the results of [16] to the conformal mapping $\Psi := \Phi^{-1}$, we obtain

$$z = \Psi(\Phi(z)) - \Psi(\Phi(0)) = a\,(\Phi(z)-\Phi(0))^{\lambda_j} + o\left((\Phi(z)-\Phi(0))^{\lambda_j}\right) \quad \text{as } z \to 0,$$

where $\lambda_j \pi$ is the exterior angle at $z_j = 0$, and $a \neq 0$. Thus

$$d_n = |\xi| \le c_{23} \min_{z\in L_n} |z| \le c_{24}\, n^{-\lambda_j} \le c_{24}\, n^{-\lambda}, \qquad n \in \mathbb{N},$$

and

$$\|g-P_n\|_p \le c_{25} \begin{cases} n^{-\frac{\lambda}{p(2-\lambda)}}, & 1<p<\infty, \\ n^{-\frac{\lambda}{2-\lambda}} \log n, & p=1, \\ n^{-\frac{\lambda(\lambda-1)}{p(2-\lambda)}-\lambda}, & \frac{1-\lambda}{2-\lambda} < p < 1, \end{cases} \tag{3.21}$$

where $n \ge 2$, by (3.19)-(3.20). Hence there exists a sequence of polynomials Q_n such that

$$\|(\phi')^{1/p} - Q_n\|_p \le c_{26} \begin{cases} n^{-\frac{\lambda}{p(2-\lambda)}}, & 1<p<\infty, \\ n^{-\frac{\lambda}{2-\lambda}} \log n, & p=1, \\ n^{-\frac{\lambda(\lambda-1)}{p(2-\lambda)}-\lambda}, & \frac{1-\lambda}{2-\lambda} < p < 1, \end{cases} \tag{3.22}$$

for $n \ge 2$. Since $\left|\left((\phi')^{1/p} - Q_n\right)\circ\psi\right|^p |\psi'|$ is subharmonic in D_R, we have

$$\begin{aligned} |1-Q_n(\zeta)|^p &= \left|(\phi'(\zeta))^{1/p} - Q_n(\zeta)\right|^p = \left|(\phi'(\psi(0)))^{1/p} - Q_n(\psi(0))\right|^p |\psi'(0)| \\ &\le \frac{1}{2\pi R} \left\|(\phi')^{1/p} - Q_n\right\|_p^p . \end{aligned}$$

Thus (2.1) follows from (3.22) and the extremal property (1.3) of $\tilde{Q}_{n,p}$, as

$$\left\|(\phi')^{1/p} - \tilde{Q}_{n,p}\right\|_p \le \left\|(\phi')^{1/p} - (Q_n - Q_n(\zeta)+1)\right\|_p .$$

The second part of the theorem, stated in (2.2), is a direct consequence of (2.1) and Theorem 1.2.
■

Proof of Theorem 2.2. We use a combination of methods employed in the previous proof and in the proof of Theorem 2.1 of [5]. Note that the analytic arcs can only have a polynomial order of contact at the junction points $z_j,\ j = 1, \dots, m$, as explained in Remark 2.3 of [5] and its proof. Thus we have x^a-type outward pointing cusps with some finite $a > 1$. Applying analytic continuation via reflection to ϕ, we write the Cauchy integral formula (3.9) for $(\phi')^{1/p}$, and again reduce the problem to approximation of the function g in (3.10). The only difference from the proof of Theorem 2.1 is that instead of the lens shaped domain S_i one has to use a symmetric in real axes domain, bounded by the arcs of $y = \pm Ax^a$ and $y = \pm A(1-x)^a$, where $A > 0$ is sufficiently small (see [5] for the details). In this case, we have

$$\operatorname{dist}(z, \partial G) \geq c_1 \min_{1 \leq i \leq m} |z - z_i|^a, \qquad z \in \partial \tilde{G}, \tag{3.23}$$

instead of (3.7), by Lemma 4.2 of [5].

Let $\Phi : \Omega \to \Delta$ be a conformal map of $\Omega := \overline{\mathbb{C}} \setminus \overline{G}$ onto $\Delta := \{w : |w| > 1\}$, satisfying the conditions $\Phi(\infty) = \infty$ and $\Phi'(\infty) > 0$. Define the level curves of Φ by

$$L_u := \{z \in \overline{\Omega} : |\Phi(z)| = u\}, \quad u > 1.$$

Let $G_u := \operatorname{Int} L_u,\ u > 1$, be the domain bounded by L_u. Denote $\gamma_1 := \gamma \cap \overline{G}_u$ and $\gamma_2 := \gamma \setminus \gamma_1$, so that γ_2 lies exterior to L_u. Hence the function g_2 of (3.11) is holomorphic in G_u, and is well approximable by polynomials. Namely, we obtain from Theorem 3 of [23, p. 145] that there exists a sequence of polynomials $\{p_n\}_{n=1}^{\infty}$ such that

$$\|g_2 - p_n\|_\infty \leq c_2 \frac{n}{(v-1)^2} \max_{z \in G_v} |g_2(z)|\ v^{-n}, \quad n \in \mathbb{N}, \tag{3.24}$$

where c_2 is an absolute constant and $1 < v < u$. On choosing $u = 1 + 2n^{-s}$ and $v = 1 + n^{-s}$, with $s \in (0,1)$, we estimate

$$\begin{aligned} \max_{z \in G_v} |g_2(z)| &\leq \int_{\gamma_2} \frac{|\phi'(t)|^{1/p}}{|t - z|} |dt| \leq \frac{c_3}{\displaystyle\min_{z \in G_v,\, t \in \gamma_2} |t - z|} \\ &\leq \frac{c_3}{\operatorname{dist}(L_u, L_v)}, \end{aligned}$$

where $\operatorname{dist}(L_u, L_v)$ is the distance between L_u and L_v. Note that

$$\operatorname{dist}(L_u, L_v) \geq c_4 (u - v)^2,$$

by a result of Loewner (see [2, p. 61]), which implies

$$\operatorname{dist}(L_u, L_v) \geq c_4 n^{-2s}.$$

We conclude that

$$\max_{z\in G_v} |g_2(z)| \leq c_5 n^{2s},$$

and, using (3.24), we obtain that

$$\|g_2 - p_n\|_\infty \leq c_6 n^{1+4s}(1+n^{-s})^{-n} \leq c_7 n^{1+4s} e^{-n^{1-s}}, \quad n \in \mathbb{N}. \tag{3.25}$$

For the companion function g_1 of (3.11), we estimate

$$\|g_1\|_\infty \leq \max_{z\in\overline{G}} \int_{\gamma_1} \frac{|\phi'(t)|^{1/p}|dt|}{|t-z|} \leq \max_{t\in\gamma_1} \frac{|\phi'(t)|^{1/p}}{\operatorname{dist}(t,L)}, \tag{3.26}$$

since $|\gamma_1| \to 0$ when $n \to \infty$. We now show that $\|g_1\|_\infty$ is sufficiently small. Indeed, we have by Corollary 1.4 of [18] that

$$|\phi'(t)| \leq c_8 \frac{R - |\phi(t)|}{\operatorname{dist}(t,L)}, \quad t \in G.$$

Hence

$$|\phi'(t)| \leq c_9 \frac{R - |\phi(t)|}{\operatorname{dist}(t,L)} \leq c_{10} \frac{|\phi(t) - \phi(z_j)|}{\operatorname{dist}(t,L)}, \quad t \in \gamma_1,$$

where z_j is the endpoint of γ_1 and the cusp point of L. It follows by Lemmas 4.4 and 4.2 of [5] that

$$\begin{aligned}|\phi(t) - \phi(z_j)| &\leq c_{11} \exp\left(-\frac{c_{12}}{|t-z_j|^{a-1}}\right) \\ &\leq c_{13} \exp\left(-c_{14} \left[\operatorname{dist}(t,L)\right]^b\right), \quad t \in \gamma_1,\end{aligned}$$

where $b < 0$. Applying these estimates in (3.26), we obtain that

$$\|g_1\|_\infty \leq c_{15} \max_{t\in\gamma_1} \frac{\exp\left(-c_{14} \left[\operatorname{dist}(t,L)\right]^b/p\right)}{\left[\operatorname{dist}(t,L)\right]^{1+1/p}}.$$

Since the function $x^{-1-1/p} \exp(-cx^b)$, where $c > 0$ and $b < 0$, is strictly increasing on an interval $(0, x_0)$, we deduce from the previous inequality that

$$\|g_1\|_\infty \leq c_{15} \frac{\exp\left(-c_{14} \left[\operatorname{dist}(t_u,L)\right]^b/p\right)}{\left[\operatorname{dist}(t_u,L)\right]^{1+1/p}}, \tag{3.27}$$

where $t_u \in L_u$ and $u = 1 + 2n^{-s}$ is sufficiently close to 1. It is known that $\Psi := \Phi^{-1}$ is Hölder continuous on $\overline{\Delta}$ (see Theorem 3 in [17]), so that

$$\operatorname{dist}(t_u, L) \le c_{16}(u-1)^{\beta} \le c_{17} n^{-s\beta},$$

for some $\beta > 0$. Hence we obtain from (3.27) that

$$\|g_1\|_\infty \le c_{18} n^{(1+1/p)s\beta} \exp\left(-c_{19}\, n^{-s\beta b}/p\right), \quad n \in \mathbb{N}. \tag{3.28}$$

Combining (3.25) and (3.28), we have from (3.11) that

$$\|g - p_n\|_\infty \le c_{20} \exp\left(-c_{21} n^r\right), \quad n \in \mathbb{N},$$

where $r \in (0,1)$ is any number satisfying $r < \min(1-s, -s\beta b)$. Furthermore, this immediately implies that there exists a sequence of polynomials $\{P_n(z)\}_{n=1}^{\infty}$ such that

$$\|(\phi')^{1/p} - P_n\|_\infty \le c_{22} \exp\left(-c_{21} n^r\right), \quad n \in \mathbb{N}, \tag{3.29}$$

by (3.9). That concludes the proof of (2.3), since by the extremal property (1.3)

$$\begin{aligned} \left\|(\phi')^{1/p} - \tilde{Q}_{n,p}\right\|_p &\le \left\|(\phi')^{1/p} - (P_n - P_n(\zeta) + 1)\right\|_p \\ &\le l^{\frac{1}{p}} \left\|(\phi')^{1/p} - (P_n - P_n(\zeta) + 1)\right\|_\infty \le 2l^{\frac{1}{p}} \left\|(\phi')^{1/p} - P_n\right\|_\infty . \end{aligned}$$

Equation (2.4) follows from Theorem 1.2 and (2.3).
■

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