INEQUALITIES FOR PRODUCTS OF POLYNOMIALS I

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Abstract. We study inequalities connecting the product of uniform norms of polynomials with the norm of their product. This circle of problems include the Gelfond-Mahler inequality for the unit disk and the Kneser-Borwein inequality for the segment $[-1, 1]$. Furthermore, the asymptotically sharp constants are known for such inequalities over arbitrary compact sets in the complex plane. It is shown here that this best constant is smallest (namely: 2) for a disk. We also conjecture that it takes its largest value for a segment, among all compact connected sets in the plane.

1. The problem and its history

Let E be a compact set in the complex plane $\mathbb{C}$. For a function $f : E \to \mathbb{C}$ define the uniform (sup) norm as follows:

$$\|f\|_E = \sup_{z\in E} |f(z)|.$$

Clearly $\|f_1 f_2\|_E \le \|f_1\|_E \|f_2\|_E$, but this inequality is not reversible, in general, not even with a constant factor in front of the right hand side. Indeed, $\|f_1\|_E \|f_2\|_E \le C \|f_1 f_2\|_E$ does not hold for functions with disjoint supports in E, for example. However, the situation is quite different for algebraic polynomials $\{p_k(z)\}_{k=1}^m$ and their product $p(z) := \prod_{k=1}^m p_k(z)$. Polynomial inequalities of the form

$$\prod_{k=1}^{m} \|p_k\|_E \le C\|p\|_E, \tag{1.1}$$

exist and are readily available. One of the first results in this direction is due to Kneser [19], for $E = [-1, 1]$ and $m = 2$ (see also Aumann [1]), who proved that

$$\|p_1\|_{[-1,1]} \|p_2\|_{[-1,1]} \le K_{\ell,n} \|p_1 p_2\|_{[-1,1]}, \quad \deg p_1 = \ell,\ \deg p_2 = n - \ell, \tag{1.2}$$

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where

$$K_{\ell,n} := 2^{n-1} \prod_{k=1}^{\ell} \left(1 + \cos \frac{2k-1}{2n}\pi\right) \prod_{k=1}^{n-\ell} \left(1 + \cos \frac{2k-1}{2n}\pi\right). \tag{1.3}$$

Note that equality holds in (1.2) for the Chebyshev polynomial $t(z) = \cos n \arccos z = p_1(z)p_2(z)$, with a proper choice of the factors $p_1(z)$ and $p_2(z)$. P. B. Borwein [7] generalized this to the multifactor inequality

$$\prod_{k=1}^{m} \|p_k\|_{[-1,1]} \le 2^{n-1} \prod_{k=1}^{[\frac{n}{2}]} \left(1 + \cos \frac{2k-1}{2n}\pi\right)^2 \|p\|_{[-1,1]}. \tag{1.4}$$

He also showed that

$$2^{n-1} \prod_{k=1}^{[\frac{n}{2}]} \left(1 + \cos \frac{2k-1}{2n}\pi\right)^2 \sim (3.20991\ldots)^n \text{ as } n \to \infty. \tag{1.5}$$

A different version of inequality (1.1) for $E = D$, where $D := \{w : |w| \le 1\}$ is the closed unit disk, was considered by Gelfond [15, p. 135] in connection with the theory of transcendental numbers:

$$\prod_{k=1}^{m} \|p_k\|_D \le e^n \|p\|_D. \tag{1.6}$$

The latter inequality was improved by Mahler [23], who replaced e by 2:

$$\prod_{k=1}^{m} \|p_k\|_D \le 2^n \|p\|_D. \tag{1.7}$$

It is easy to see that the base 2 cannot be decreased, if $m = n$ and $n \to \infty$. However, (1.7) has recently been further improved in two directions. D. W. Boyd [9, 10] showed that, given the number of factors m in (1.7), one has

$$\prod_{k=1}^{m} \|p_k\|_D \le (C_m)^n \|p\|_D, \tag{1.8}$$

where

$$C_m := \exp\left(\frac{m}{\pi} \int_0^{\pi/m} \log\left(2\cos\frac{t}{2}\right) dt\right) \tag{1.9}$$

is asymptotically best possible for *each fixed* m, as $n \to \infty$. Kroó and Pritsker [20] showed that, for any $m \le n$,

$$\prod_{k=1}^{m} \|p_k\|_D \le 2^{n-1} \|p\|_D, \tag{1.10}$$

where equality holds in (1.10) for *each* $n \in \mathbb{N}$, with $m = n$ and $p(z) = z^n - 1$.

Inequalities (1.2)-(1.10) clearly indicate that the constant C in (1.1) grows exponentially fast with n, with the base for the exponential depending on the set E. A natural general problem arising here is to find the *smallest* constant $M_E > 0$, such that

$$\prod_{k=1}^{m} \|p_k\|_E \le M_E^n \|p\|_E \tag{1.11}$$

for arbitrary algebraic polynomials $\{p_k(z)\}_{k=1}^m$ with complex coefficients, where $p(z) = \prod_{k=1}^m p_k(z)$ and $n = \deg p$. The solution of this problem is based on the logarithmic potential theory (cf. [36] and [35]). Let $\mathrm{cap}(E)$ be the *logarithmic capacity* of a compact set $E \subset \mathbb{C}$. For E with $\mathrm{cap}(E) > 0$, denote the *equilibrium measure* of E by μ_E. We remark that μ_E is a positive unit Borel measure supported on ∂E (see [36, p. 55]). Define

$$d_E(z) := \max_{t \in E} |z - t|, \qquad z \in \mathbb{C}, \tag{1.12}$$

which is clearly a positive and continuous function in $\mathbb{C}$. It is easy to see that the logarithm of this distance function is subharmonic in $\mathbb{C}$. Furthermore, it has the following integral representation

$$\log d_E(z) = \int \log |z - t| d\sigma_E(t), \quad z \in \mathbb{C},$$

where σ_E is a positive unit Borel measure in $\mathbb{C}$ with unbounded support, see Lemma 5.1 of [31] and [22]. For further in-depth analysis of the representing measure σ_E, we refer to the recent paper of Gardiner and Netuka [14]. This integral representation is the key fact used by the first author to prove the following result [31].

Theorem 1.1. *Let $E \subset \mathbb{C}$ be a compact set,* $\mathrm{cap}(E) > 0$. *Then the best constant M_E in* (1.11) *is given by*

$$M_E = \frac{\exp\left(\int \log d_E(z) d\mu_E(z)\right)}{\mathrm{cap}(E)}. \tag{1.13}$$

Theorem 1.1 is applicable to any compact set with a connected component consisting of more than one point (cf. [36, p. 56]). In particular, if E is a continuum, i.e., a connected set, then we obtain a simple universal bound for M_E [31]:

Corollary 1.2. *Let $E \subset \mathbb{C}$ be a bounded continuum (not a single point). Then we have*

$$M_E \le \frac{\mathrm{diam}(E)}{\mathrm{cap}(E)} \le 4, \tag{1.14}$$

where $\mathrm{diam}(E)$ *is the Euclidean diameter of the set E.*

On the other hand, for non-connected sets E the constants M_E can be arbitrarily large. For example, consider $E_k = [-\sqrt{k+4}, -\sqrt{k}] \cup [\sqrt{k}, \sqrt{k+4}]$, so that $\mathrm{cap}(E_k) = 1$ [35] and

$$M_E = \exp\left(\int \log d_{E_k}(z)\, d\mu_{E_k}(z)\right) \geq e^{\log(2\sqrt{k})} \to \infty \quad \text{as } k \to \infty.$$

For the closed unit disk D, we have that $\mathrm{cap}(D) = 1$ [36, p. 84] and that

$$d\mu_D = \frac{d\theta}{2\pi}, \tag{1.15}$$

where $d\theta$ is the arclength on ∂D. Thus Theorem 1.1 yields

$$M_D = \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \log d_D(e^{i\theta})\ d\theta\right) = \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \log 2\ d\theta\right) = 2, \tag{1.16}$$

so that we immediately obtain Mahler's inequality (1.7).

If $E = [-1,1]$ then $\mathrm{cap}([-1,1]) = 1/2$ and

$$d\mu_{[-1,1]} = \frac{dx}{\pi\sqrt{1-x^2}}, \quad x \in [-1,1], \tag{1.17}$$

which is the Chebyshev (or arcsin) distribution (see [36, p. 84]). Using Theorem 1.1, we obtain

$$\begin{aligned} M_{[-1,1]} &= 2\exp\left(\frac{1}{\pi}\int_{-1}^{1} \frac{\log d_{[-1,1]}(x)}{\sqrt{1-x^2}} dx\right) = 2\exp\left(\frac{2}{\pi}\int_0^1 \frac{\log(1+x)}{\sqrt{1-x^2}} dx\right) \\ &= 2\exp\left(\frac{2}{\pi}\int_0^{\pi/2} \log(1+\sin t)dt\right) \approx 3.2099123, \end{aligned} \tag{1.18}$$

which gives the asymptotic version of Borwein's inequality (1.4)-(1.5).

Considering the above analysis of Theorem 1.1, it is natural to conjecture that the sharp universal bounds for M_E are given by

$$2 = M_D \leq M_E \leq M_{[-1,1]} \approx 3.2099123, \tag{1.19}$$

for any bounded non-degenerate continuum E, see [33].

It follows directly from the definition that M_E is invariant with respect to the similarity transformations of the plane. Thus we can normalize the problem by setting $\mathrm{cap}(E) = 1$. Thus, equivalently, we want to find the maximum and the minimum of the functional

$$\tau(E) := \int \log d_E(z) d\mu_E(z) \tag{1.20}$$

over all compact connected sets E in the plane satisfying the above normalization. These questions are addressed in Section 2 of the paper. Section 3 discusses a more refined version of our problem on the best constant in (1.1). All proofs are given in Section 4.

In the forthcoming paper [34], we consider various improved bounds of the constant M_E, e.g., bounds for rotationally symmetric sets. From a different perspective, the results of Boyd (1.8)-(1.9) suggest that for some sets the constant M_E can be replaced by a smaller one, if the number of factors is fixed. We characterize such sets in [34], and find the improved constant.

The problems considered in this paper have many applications in analysis, number theory and computational mathematics. We mention specifically applications in transcendence theory (see Gelfond [15]), and in designing algorithms for factoring polynomials (see Boyd [11] and Landau [21]). A survey of the results involving norms different from the sup norm (e.g., Bombieri norms) can be found in [11]. For polynomials in several variables, see the results of Mahler [24] for the polydisk, of Avanissian and Mignotte [2] for the unit ball in $\mathbb{C}^k$. Also, see Beauzamy and Enflo [5], and Beauzamy, Bombieri, Enflo and Montgomery [4] for multivariate polynomials in different norms.

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2. Sharp bounds for the constant M_E

We study bounds for the constant M_E in this section, where $E \subset \mathbb{C}$ is a compact set satisfying $\operatorname{cap}(E) > 0$. Our main goal here is to prove (1.19). It is convenient to first give some general observations on the properties of M_E.

Theorem 2.1. *Let $I \subset E$ be compact sets in $\mathbb{C}$, $\operatorname{cap}(I) > 0$. Denote the unbounded components of $\overline{\mathbb{C}} \setminus E$ and $\overline{\mathbb{C}} \setminus I$ by Ω_E and Ω_I. If $d_E(z) = d_I(z)$ for all $z \in \partial\Omega_I$ then $M_E \leq M_I$, with equality holding only when $\operatorname{cap}(\Omega_I \setminus \Omega_E) = 0$.*

This theorem gives several interesting consequences. In particular, we show that if the set E is contained in a disk whose diameter coincides with the diameter of E then its constant M_E does not exceed that of a segment. Thus segments indeed maximize M_E among such sets. Denote the closed disk of radius r centered at z by $D(z, r)$.

Corollary 2.2. *Let $z, w \in E$ satisfy $\operatorname{diam} E = |z - w|$ and $[z, w] \subset E$. If $E \subset D\left(\frac{z+w}{2}, \frac{\operatorname{diam} E}{2}\right)$ then $M_E \leq M_{[z,w]} = M_{[-2,2]}$.*

The next results shows that the constant decreases when the set is enlarged in a certain way.

Corollary 2.3. *Let $E^* := \bigcap_{z\in\partial\Omega_E} D(z, d_E(z))$, where $E \subset \mathbb{C}$ is compact, $\operatorname{cap}(E) > 0$. If H is a compact set such that $E \subset H \subset E^*$, then $M_H \leq M_E$. Equality holds if and only if $\operatorname{cap}(\Omega_E \setminus \Omega_H) = 0$.*

Let conv(H) be the convex hull of H. The operation of taking the convex hull of a set satisfies the assumption of Corollary 2.3 (or Theorem 2.1), which gives

Corollary 2.4. *Let $V \subset \mathbb{C}$ be a compact set,* $\mathrm{cap}(V) > 0$. *If $H := \overline{\mathbb{C}} \setminus \Omega_V$ is not convex, then $M_{\mathrm{conv}(H)} < M_H$.*

The above results help us to show that the minimum of M_E is attained for the closed unit disk D, among all sets of positive capacity (connected or otherwise).

Theorem 2.5. *Let $E \subset \mathbb{C}$ be an arbitrary compact set,* $\mathrm{cap}(E) > 0$. *Then $M_E \geq 2$, where equality holds if and only if $\overline{\mathbb{C}} \setminus \Omega_E$ is a closed disk.*

In other words, $M_E = 2$ only for sets whose polynomial convex hull is a disk. This may also be described by saying that $M_E = 2$ if and only if $\partial U \subset E \subset U$, where U is a closed disk.

Proving that the maximum of M_E for *arbitrary* continua is attained for a segment is a more difficult problem. In fact, it is related to some old open problems on the moments of the equilibrium measure (or circular means of conformal maps), see Pólya and Schiffer [27], and Pommerenke [28]. In particular, we use the results of [27] and [28] to show that

Theorem 2.6. *Let $E \subset \mathbb{C}$ be a connected compact set,* $\mathrm{cap}(E) > 0$.
(i) *If the center of mass $c := \int z\, d\mu_E(z)$ for μ_E belongs to E, then*

$$M_E < 2 + 4.02/\pi \approx 3.279606. \tag{2.1}$$

(ii) *If E is convex then*

$$M_E < 2 + 4/\pi \approx 3.27324. \tag{2.2}$$

This should be compared with $M_{[-2,2]} = M_{[-1,1]} \approx 3.2099123$.

After this paper had been written, a new related manuscript [3] appeared. That manuscript contains a proof of our conjecture $M_E \leq M_{[-2,2]}$ for centrally symmetric continua, as well as another quite general conjecture (if true) implying $M_E \leq M_{[-2,2]}$ holds for all continua.

3. Refined problem

The constant M_E represents the base of rather crude exponential asymptotic for the constant in inequality (1.1). A more refined question is to find the sharp constant attained with equality. Such constants are known in the case of a segment, see (1.4) and [7]; and in the case of a disk, see (1.10) and [20]. Let E be any compact set in the plane, and let $\prod_{k=1}^m p_k(z) = \prod_{j=1}^n (z - z_j)$, where $p_k(z)$ are arbitrary monic polynomials with complex coefficients. Define the constant

$$C_E(n) := \sup_{p_k} \frac{\prod_{k=1}^{m} \|p_k\|_E}{\left\| \prod_{k=1}^{m} p_k \right\|_E} = \sup_{z_j \in \mathbb{C}} \frac{\prod_{j=1}^{n} \|z - z_j\|_E}{\left\| \prod_{j=1}^{n} (z - z_j) \right\|_E}. \tag{3.1}$$

If $\operatorname{cap}(E) > 0$ then it follows from Theorem 1.1 that $1 \leq C_E(n) \leq M_E^n$. The refined version of our conjecture in (1.19) is as follows:

$$2^{n-1} = C_D(n) \leq C_E(n) \leq C_{[-2,2]}(n) = 2^{n-1} \prod_{k=1}^{[n/2]} \left(1 + \cos \frac{2k-1}{2n} \pi\right)^2 \tag{3.2}$$

for any connected compact set E of positive capacity.

4. Proofs

Proof of Theorem 2.1. Since $I \subset E$, we have that $\operatorname{cap}(E) \geq \operatorname{cap}(I) > 0$. Let $g_E(z,\infty)$ and $g_I(z,\infty)$ be the Green's functions for Ω_E and Ω_I, with poles in infinity. We follow the standard convention by setting $g_E(z,\infty) = 0,\ z \notin \overline{\Omega}_E$ and $g_I(z,\infty) = 0,\ z \notin \overline{\Omega}_I$. It follows from the maximum principle that $g_E(z,\infty) \leq g_I(z,\infty)$ for all $z \in \mathbb{C}$. Furthermore, this inequality is strict in Ω_E, unless $\operatorname{cap}(\Omega_I \setminus \Omega_E) = 0$.

Using the integral representation for $d_E(z)$ from Lemma 5.1 of [31] (see also [22] and [14]) and the Fubini theorem, we obtain that

$$\begin{aligned}
\log M_E &= \int \log d_E(z)\, d\mu_E(z) - \log \operatorname{cap}(E) \\
&= \int\int \log|z-t|\, d\sigma_E(t) d\mu_E(z) - \log \operatorname{cap}(E) \\
&= \int \left(\int \log|z-t|\, d\mu_E(z) - \log \operatorname{cap}(E) \right) d\sigma_E(t) = \int g_E(t,\infty)\, d\sigma_E(t),
\end{aligned}$$

where the last equality follows from the well known identity
$g_E(t,\infty) = \int \log|z-t|\, d\mu_E(z) - \log \operatorname{cap}(E)$ [35]. It is clear that

$$\int g_E(t,\infty)\, d\sigma_E(t) \leq \int g_I(t,\infty)\, d\sigma_E(t),$$

with equality possible if and only if $\operatorname{cap}(\Omega_I \setminus \Omega_E) = 0$. Indeed, if we have equality in the above inequality, then $g_E(z,\infty) = g_I(z,\infty)$ for all $z \in \operatorname{supp} \sigma_E$. But $\operatorname{supp} \sigma_E$ is unbounded, so that $g_E(z,\infty) = g_I(z,\infty)$ in Ω_E by the maximum principle. Hence we obtain that

$$\begin{aligned}\log M_E &\le \int g_I(t,\infty)\,d\sigma_E(t) = \int \left(\int \log|z-t|\,d\mu_I(z) - \log \operatorname{cap}(I)\right) d\sigma_E(t)\\ &= \int \log d_E(z)\,d\mu_I(z) - \log \operatorname{cap}(I) = \int \log d_I(z)\,d\mu_I(z) - \log \operatorname{cap}(I)\\ &= \log M_I,\end{aligned}$$

with equality if and only if $\operatorname{cap}(\Omega_I \backslash \Omega_E) = 0$. Note that we used $\operatorname{supp} \mu_I \subset \partial\Omega_I$, so that $d_E(z) = d_I(z)$ for $z \in \operatorname{supp} \mu_I$.

□

Proof of Corollary 2.2. Let $I = [z, w]$ be the segment connecting the points z and w, i.e., the common diameter of E and the disk containing it. Observe that we have $d_E(t) = d_I(t)$ for all $t \in \partial\Omega_I = I$ under the stated geometric conditions. Since all assumptions of Theorem 2.1 are satisfied, we obtain that $M_E \le M_{[z,w]} = M_{[-2,2]}$, where the last equality follows from the invariance with respect to the similarity transformations of the plane.

□

Proof of Corollary 2.3. Observe that $E \subset D(z, d_E(z))$ for any $z \in \mathbb{C}$. Hence $E \subset E^*$. Since $E \subset H \subset E^*$, we immediately obtain that $d_E(z) \le d_H(z) \le d_{E^*}(z),\ z \in \mathbb{C}$. On the other hand, the definition of E^* gives that $d_E(z) = d_{E^*}(z)$ for all $z \in \partial\Omega_E$. Therefore $d_E(z) = d_H(z)$ for all $z \in \partial\Omega_E$, and the result follows from Theorem 2.1.

□

Proof of Corollary 2.4. We apply Theorem 2.1 again, with $I = H$ and $E = \operatorname{conv}(H)$. It was shown in [22] that $d_H(z) = d_{\operatorname{conv}(H)}(z)$ for all $z \in \mathbb{C}$, where H is an arbitrary compact set. Since H is not convex in our case, we obtain that $\operatorname{cap}(\Omega_I \setminus \Omega_E) > 0$ and $M_E < M_I$.

□

For the proof of Theorem 2.5 we need a special case of the following lemma, which may be of some independent interest. Let $\Delta := \{w : |w| > 1\}$, and $\mathbb{D} := \{z : |z| < 1\}$ the unit disk.

Lemma 4.1. *Let Γ be a Jordan domain and let $\Psi(z) := cw + \sum_{k=0}^{\infty} a_k w^{-k}$ be a conformal map of Δ onto Ω_Γ. Furthermore assume that*

$$\forall x, z \in \partial\Delta : \quad |\Psi(z) - \Psi(x)| \le |\Psi(z) - \Psi(-z)|. \tag{4.1}$$

Then Γ is a disk.

Proof. First note that by Carathéodory's theorem [30, p. 18] Ψ extends to a homeomorphism of $\overline{\Delta}$, so that (4.1) makes sense. Also there is no loss of generality in assuming $0 \in \Gamma$, so that $\Psi(z) \ne 0$ in $\overline{\Delta}$. Let

$$g(z) := \frac{1}{\Psi(1/z)}, \quad z \in \overline{\mathbb{D}}.$$

Then $g(z) = z/c + \sum_{k=2}^{\infty} b_k z^k$ is a homeomorphism of $\overline{\mathbb{D}}$ onto the closure of the Jordan domain Γ^*, the interior domain of the Jordan curve $1/\partial\Gamma$. Note that $g(0) = 0, g'(0) = 1/c \neq 0$.

Let $1/z \in \partial\mathbb{D}$, and in (4.1) we replace $1/x \in \partial\mathbb{D}$ by $-1/xz$ which is also in $\partial\mathbb{D}$. Condition (4.1) then becomes

$$1 \geq \left| \frac{\frac{1}{g(z)} - \frac{1}{g(-xz)}}{\frac{1}{g(z)} - \frac{1}{g(-z)}} \right| = \left| \frac{xg(-z)}{g(-xz)} \frac{g(-xz) - g(z)}{g(-z) - g(z)} \right|, \quad x, z \in \partial\mathbb{D}.$$

Note that the function

$$F(x,z) := \frac{xg(-z)}{g(-xz)} \frac{g(-xz) - g(z)}{g(-z) - g(z)}$$

is analytic in $(x,z) \in \mathbb{D}^2$, and by the maximum principle, applied to both variables separately, we find that

$$|F(x,z)| \leq 1, \quad x, z \in \overline{\mathbb{D}}.$$

Now fix z_0 with $0 < |z_0| < 1$. Then $x \mapsto F(x, z_0)$ is analytic in $\overline{\mathbb{D}}$, satisfies $|F(x, z_0)| \leq 1$ for $x \in \overline{\mathbb{D}}$, and, in addition, $F(1, z_0) = 1$. The Julia-Wolf Lemma [30, p. 82] then says that $F'(1, z_0) > 0$, or

$$1 + \frac{-z_0 g'(-z_0)}{g(-z_0)} \frac{g(z_0)}{g(-z_0) - g(z_0)} > 0.$$

Obviously this must be true for any z_0, and so, by the identity principle, we are left with the relation

$$\frac{-zg'(-z)}{g(-z)} \frac{g(z)}{g(-z) - g(z)} \equiv \alpha, \quad z \in \mathbb{D},$$

where $\alpha > -1$ is some real constant. Letting $z \to 0$, we find $\alpha = -\frac{1}{2}$. Hence we are left with the difference-differential equation

$$\frac{zg'(z)}{g(z)} \frac{g(-z)}{g(-z) - g(z)} = \frac{1}{2}, \quad z \in \mathbb{D}. \tag{4.2}$$

In terms of Ψ this reads

$$2w\Psi'(w) = \Psi(w) - \Psi(-w), \quad w \in \Omega_\Gamma.$$

From this we conclude that $w\Psi'(w)$ is an odd function, which, in turn, implies that $\Phi(w) := \Psi(w) - a_0$ is odd as well. For Φ we then get the equation $w\Phi'(w) = \Phi(w)$, or $\Phi(w) = cw$. This implies $\Psi(w) = cw + a_0$ and therefore that Γ is a disk. □

Proof of Theorem 2.5. Note that for any compact set E, we have $M_E = M_W$, where $W := \overline{\mathbb{C}} \setminus \Omega_E$. This follows because $\mu_E = \mu_W$ [35] and $d_E(z) = d_W(z), \ z \in \mathbb{C}$. Corollary 2.4 now implies that

$$\inf\{M_E : E \text{ is compact}\} = \inf\{M_H : H \text{ is convex and compact}\}.$$

Hence we can assume that E is convex from the start. We also set $\text{cap}(E) = 1$, because M_E is invariant under similarity transforms. Thus ∂E is a rectifiable Jordan curve (or a segment when $E = \partial E$). The following argument that shows $M_E \geq 2$ for all connected sets is due to A. Solynin. Let $\Psi : \Delta \to \Omega_E$ be the standard conformal map:

$$\Psi(w) = w + a_0 + \sum_{k=1}^{\infty} \frac{a_k}{w^k}, \qquad w \in \Delta.$$

Recall that Ψ can be extended as a homeomorphism of $\overline{\Delta}$ onto $\overline{\Omega}_E$, with $\Psi(\mathbb{T}) = \partial E,\ \mathbb{T} := \partial\Delta$. It is clear that

$$d_E(\Psi(e^{it})) \geq |\Psi(e^{it}) - \Psi(-e^{it})|, \qquad t \in [0, 2\pi).$$

Since $\Psi(w)$ is univalent in Δ, the function

$$H(w) := \frac{\Psi(w) - \Psi(-w)}{w}$$

is analytic and non-vanishing in Δ, including $w = \infty$. Furthermore, $H(\infty) := \lim_{w\to\infty} H(w) = 2$. It follows that $h(w) := \log|H(w)|$ is harmonic in Δ. Recall that the equilibrium measure μ_E is the harmonic measure of Ω_E at ∞, which is invariant under the conformal transformation Ψ, see [35]. Hence

$$\begin{aligned}\log M_E &= \int \log d_E(z)\, d\mu_E(z) = \frac{1}{2\pi}\int_0^{2\pi} \log d_E(\Psi(e^{it}))\, dt \\ &\geq \frac{1}{2\pi}\int_0^{2\pi} \log\left|\frac{\Psi(e^{it}) - \Psi(-e^{it})}{e^{it}}\right| dt = \log 2,\end{aligned}$$

where we used the Mean Value Theorem for $h(w)$ on the last step. Thus we conclude that $M_E \geq 2 = M_D$ holds for all compact sets E.

Recall that $M_E = M_W$, where $W = \overline{\mathbb{C}}\setminus\Omega_E$. If $M_E = 2$ then $M_W = 2$, so that W must be convex by Corollary 2.4. Since $M_W > 3.2$ for any segment, we have that W is the closure of a convex domain. We can assume that $\text{cap}(W) = 1$ after a dilation. Repeating the above argument for W instead of E, we obtain that

$$\begin{aligned}\log 2 = \log M_W &= \frac{1}{2\pi}\int_0^{2\pi} \log d_W(\Psi(e^{it}))\, dt \\ &\geq \frac{1}{2\pi}\int_0^{2\pi} \log\left|\Psi(e^{it}) - \Psi(-e^{it})\right|\, dt = \log 2.\end{aligned}$$

It follows that

$$\int_0^{2\pi} \left(\log d_W(\Psi(e^{it})) - \log\left|\Psi(e^{it}) - \Psi(-e^{it})\right|\right)\, dt = 0,$$

and that $d_W(\Psi(e^{it})) = |\Psi(e^{it}) - \Psi(-e^{it})|$ a.e. on $[0, 2\pi)$. But these functions are clearly continuous, so that

$$d_W(\Psi(e^{it})) = \left|\Psi(e^{it}) - \Psi(-e^{it})\right| \quad \forall t \in \mathbb{R}.$$

An application of Lemma 4.1 with Γ the interior domain of W shows that W must be a disk. We would also like to mention that A. Solynin obtained a different proof of the fact that $M_E = 2$ for a connected set E implies W is a disk. $\square$

Proof of Theorem 2.6. Recall that M_E is invariant under similarity transformations. Hence we can assume again that $\mathrm{cap}(E) = 1$ and $\int z\, d\mu_E(z) = 0$. The latter condition means that the center of mass for the equilibrium measure is at the origin. If we introduce the conformal map $\Psi : \Delta \to \Omega_E$, as in the previous proof, then this condition translates into $a_0 = 0$, i.e.,

$$\Psi(w) = w + \sum_{k=1}^{\infty} \frac{a_k}{w^k}, \qquad w \in \Delta.$$

Theorem 1.4 of [29, p. 19] gives that $E \subset D(0,2)$, so that $d_E(z) \le 2 + |z|,\ z \in E$, by the triangle inequality. Note that this is sharp for $E = [-2,2]$. Applying Jensen's inequality, we have

$$\log M_E = \int \log d_E(z)\, d\mu_E(z) \le \int \log(2+|z|)\, d\mu_E(z) < \log\left(2 + \int |z|\, d\mu_E(z)\right).$$

Estimates (2.1) and (2.2) now follow from the results of Pommerenke [28], and of Pólya and Schiffer [27], who estimated the integral

$$\int |z|\, d\mu_E(z) = \frac{1}{2\pi}\int_0^{2\pi} |\Psi(e^{it})|\, dt < 4.02/\pi \quad (\text{or } \ \le 4/\pi),$$

under the corresponding assumptions.

$\square$

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