AN AREAL ANALOG OF MAHLER'S MEASURE

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Abstract. We consider a version of height on polynomial spaces defined by the integral over the normalized area measure on the unit disk. This natural analog of Mahler's measure arises in connection with extremal problems for Bergman spaces. It inherits many nice properties such as the multiplicative one. However, this height is a lower bound for Mahler's measure, and it can be substantially lower. We discuss some similarities and differences between the two.

1. Definition and main properties

Let $\mathbb{C}_n[z]$ and $\mathbb{Z}_n[z]$ be the sets of all polynomials of degree at most n with complex and integer coefficients respectively. Mahler's measure of a polynomial $P_n \in \mathbb{C}_n[z]$ is defined by

$$M(P_n) := \|P_n\|_{H^0} = \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \log|P_n(e^{i\theta})|\, d\theta\right).$$

It is also known as the H^0 Hardy space norm or the contour geometric mean. An application of Jensen's inequality immediately gives that

$$M(P_n) = |a_n| \prod_{|z_j|>1} |z_j|$$

for $P_n(z) = a_n \prod_{j=1}^n (z - z_j) \in \mathbb{C}_n[z]$. This height on the space of polynomials is extensively used in number theory. Recall that a cyclotomic (circle dividing) polynomial is defined as an irreducible factor of $z^n - 1,\ n \in \mathbb{N}$. Clearly, if Q_n is cyclotomic, then $M(Q_n) = 1$. A well known and difficult open problem related to Mahler's measure is the Lehmer conjecture on the lower bound for the measure of irreducible non-cyclotomic polynomials from $\mathbb{Z}_n[z]$. Lehmer [26] carried out extensive computations of the values of $M(P_n),\ P_n \in \mathbb{Z}_n[z]$, but found no non-cyclotomic polynomial with Mahler's measure smaller than that of the polynomial $L(z) := z^{10} + z^9 - z^7 - z^6 - z^5 - z^4 - z^3 + z + 1$, which

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he conjectured to be always true. Eight zeros of L lie on the unit circle, one inside and one outside. The latter zero ζ is believed to be the smallest Salem number, with the value $M(L) = \zeta = 1.1762808\ldots$, which is the smallest Mahler's measure according to the Lehmer conjecture. More history of this conjecture may be found in [13], [7], [18] and [6].

A natural counterpart of Mahler's measure is obtained by replacing the normalized arclength measure on the unit circumference $\mathbb{T}$ by the normalized area measure on the open unit disk $\mathbb{D}$. Namely, we define the A^0 Bergman space norm by

$$\|P_n\|_0 := \exp\left(\frac{1}{\pi}\iint_{\mathbb{D}} \log|P_n(z)|\,dA\right).$$

This also gives a multiplicative height of the polynomial P_n. Furthermore, it has the same relation to Bergman spaces as Mahler's measure to Hardy spaces:

$$\|P_n\|_0 = \lim_{p\to 0+} \|P_n\|_p,$$

see [19], where

$$\|P_n\|_p := \left(\frac{1}{\pi}\iint_{\mathbb{D}} |P_n(z)|^p\,dA\right)^{1/p}, \quad 0 < p < \infty.$$

In addition, it arises in the following version of the extremal problem considered by Szegő [37] for the Hardy space H^2:

$$\inf_{Q(0)=0} \frac{1}{\pi}\iint_{\mathbb{D}} |1 - Q(z)|^p\,|P_n(z)|\,dA(z) = \|P_n\|_0, \quad 0 < p < \infty,$$

where Q is any polynomial vanishing at 0, see [17, p. 136].

Using the fact that the integral means of $\log|P_n(re^{it})|$ over $|z| = r$ are increasing with r [11], we immediately obtain that

$$\|P_n\|_0 \le M(P_n). \tag{1.1}$$

Also, if $P_n(z) = \sum_{k=0}^n a_k z^k$ then

$$\|P_n\|_0 \ge |a_0|, \tag{1.2}$$

which follows from the area mean value inequality for the subharmonic function $\log|P_n|$ (cf. [11]). Hence

$$\|P_n\|_0 \ge 1 \quad \text{for all } P_n \in \mathbb{Z}_n[z],\ P_n(0) \ne 0. \tag{1.3}$$

In fact, there is a direct relation between Mahler's measure and its areal analog, given below.

Theorem 1.1. *Let $P_n(z) = a_n \prod_{j=1}^n (z - z_j) = \sum_{k=0}^n a_k z^k \in \mathbb{C}_n[z]$. If P_n has no roots in $\mathbb{D}$, then $\|P_n\|_0 = M(P_n) = |a_0|$. Otherwise,*

$$\|P_n\|_0 = M(P_n) \exp\left(\frac{1}{2} \sum_{|z_j|<1} (|z_j|^2 - 1)\right). \tag{1.4}$$

This shows that the value of $\|P_n\|_0$ is influenced by the zeros inside the unit disk more than that of $M(P_n)$. We immediately obtain the following comparison result from Theorem 1.1.

Corollary 1.2. *For any $P_n \in \mathbb{C}_n[z]$, we have*

$$e^{-n/2} M(P_n) \le \|P_n\|_0 \le M(P_n). \tag{1.5}$$

Equality holds in the lower estimate if and only if $P_n(z) = a_n z^n$. The upper estimate turns into equality for any polynomial without zeros in the unit disk.

A well known theorem of Kronecker [22] states that any monic irreducible polynomial $P_n \in \mathbb{Z}_n[z],\ P_n(0) \neq 0$, with all zeros in the closed unit disk, must be cyclotomic. One can write that statement in the form: $M(P_n) = 1$ for such P_n if and only if P_n is cyclotomic. A direct analog of this result exists for $\|P_n\|_0$.

Theorem 1.3. *Suppose that $P_n \in \mathbb{Z}_n[z],\ P_n(0) \neq 0$, is an irreducible polynomial with all zeros in the closed unit disk. It is cyclotomic if and only if $\|P_n\|_0 = 1$.*

The next natural question is whether one can find a uniform lower bound $\|P_n\|_0 \ge c > 1$ for all non-cyclotomic $P_n \in \mathbb{Z}_n[z],\ P_n(0) \neq 0$. It is especially interesting in view of Lehmer's conjecture, because of (1.1). However, the answer to the question is negative, as we show with the following example.

Example 1.4. *Consider $P_n(z) = nz^n - 1$. It has zeros $z_j,\ j = 1, \ldots, n$, that are equally spaced on the circumference $|z| = n^{-1/n}$. Note that $M(P_n) = n$ and*

$$\|P_n\|_0 = n \exp\left(\frac{n(n^{-2/n} - 1)}{2}\right),$$

by (1.4). *Since*

$$n^{-2/n} = \exp\left(\frac{-2\log n}{n}\right) = 1 - \frac{2\log n}{n} + O\left(\frac{\log^2 n}{n^2}\right),$$

we obtain that

$$\|P_n\|_0 = \exp\left(O\left(\frac{\log^2 n}{n}\right)\right) \to 1 \quad \text{as } n \to \infty.$$

Similarly, we have for the reciprocal polynomial $P_{2n}(z) = z^{2n} + nz^n + 1$ *that*

$$M(P_n) = \frac{n + \sqrt{n^2 - 4}}{2} \sim n \quad \text{as } n \to \infty,$$

and

$$\|P_n\|_0 = \frac{n + \sqrt{n^2 - 4}}{2} \exp\left(\frac{n}{2}\left(\left(\frac{n - \sqrt{n^2 - 4}}{2}\right)^{2/n} - 1\right)\right) \to 1 \quad \text{as } n \to \infty.$$

One may notice that for both sequences of polynomials in this example the zeros are asymptotically equidistributed near the unit circumference. This is a part of a more general phenomenon discussed in the next section.

We conclude this section with a remark on the arithmetic nature of $\|P_n\|_0$.

Proposition 1.5. *If* $P_n(z) = \sum_{k=0}^n a_k z^k \in \mathbb{Z}_n[z]$ *has at least one zero in* $\mathbb{D}$, *then* $\|P_n\|_0$ *is a transcendental number. Otherwise,* $\|P_n\|_0 = M(P_n) = |a_0|$ *is an integer.*

2. Asymptotic zero distribution

Consider a polynomial $P_n(z) = a_n \prod_{j=1}^n (z - z_j) \in \mathbb{C}_n[z]$, and define its normalized zero counting measure by

$$\nu_n := \frac{1}{n} \sum_{j=1}^n \delta_{z_j},$$

where δ_{z_j} is the unit pointmass at z_j. Our main result on the asymptotic zero distribution is as follows.

Theorem 2.1. *Suppose that* $P_n \in \mathbb{Z}_n[z]$, $\deg P_n = n$, *is a sequence of polynomials without multiple zeros. If* $\lim_{n\to\infty} \|P_n\|_0^{1/n} = 1$ *then* ν_n *converges to the normalized arclength measure* $d\theta/(2\pi)$ *on* $\mathbb{T}$ *in the weak* topology, as* $n \to \infty$.

This result extends a theorem of Bilu [4] for Mahler's measure, see also Bombieri [5] and Rumely [32]. From a more general point of view, Theorem 2.1 is a descendant of Jentzsch's result [21] on the asymptotic zero distribution of the partial sums of a power series, and its generalization by Szegő [38]. This area was further developed by Erdős and Turán [12], and by many others.

As an immediate application of Theorem 2.1 we obtain a result on the growth of $\|P_n\|_0$ for polynomials with restricted zeros.

Corollary 2.2. *Suppose that* $P_n \in \mathbb{Z}_n[z]$, $\deg P_n = n$, *is a sequence of polynomials with simple zeros contained in a closed set* $E \subset \mathbb{C}$. *If* $\mathbb{T} \not\subset E$ *then there exists a constant* $C = C(E) > 1$ *such that*

$$\liminf_{n\to\infty} \|P_n\|_0^{1/n} \ge C > 1.$$

This exhibits the geometric growth of $\|P_n\|_0$ for many families of polynomials such as polynomials with real zeros, polynomials with zeros in a sector, etc. Corresponding results with explicit bounds for Mahler's measure were obtained by Schinzel [33], Langevin [23, 24, 25], Mignotte [28], Rhin and Smyth [31], Dubickas and Smyth [10], and others.

In a somewhat different direction, we have the following result on the asymptotic behavior of zeros.

Theorem 2.3. *Suppose that* $P_n(z) = a_n z^n + \ldots + a_0 \in \mathbb{C}_n[z],\ |a_0| \ge 1,\ n \in \mathbb{N}$, *is a sequence of polynomials.*

(a) If $\lim_{n\to\infty} \|P_n\|_0 = 1$ *then*

$$\liminf_{n\to\infty} \min_{1\le j\le n} |z_j| \ge 1. \tag{2.1}$$

(b) If $|a_n| \ge 1$ *and* $\lim_{n\to\infty} M(P_n) = 1$, *then*

$$\lim_{n\to\infty} \min_{1\le j\le n} |z_j| = \lim_{n\to\infty} \max_{1\le j\le n} |z_j| = 1. \tag{2.2}$$

Thus part (a) of Theorem 2.3 indicates that all zeros of P_n are pushed out of $\mathbb{D}$ as $n \to \infty$, while in part (b) they all tend to the unit circumference.

3. Polynomial inequalities

We discuss some general polynomial inequalities related to $M(P_n)$ and $\|P_n\|_0$ in this section. For a polynomial $\Lambda_n(z) = \sum_{k=0}^n \lambda_k \binom{n}{k} z^k \in \mathbb{C}_n[z]$, consider the Szegő composition with $P_n(z) = \sum_{k=0}^n a_k z^k \in \mathbb{C}_n[z]$:

$$\Lambda P_n(z) := \sum_{k=0}^n \lambda_k a_k z^k. \tag{3.1}$$

If Λ_n is a fixed polynomial, then ΛP_n is a multiplier operator acting on P_n. More information on history and applications of this composition may be found in [9], [1], [2] and [29]. De Bruijn and Springer [9] proved a very interesting general inequality

$$M(\Lambda P_n) \le M(\Lambda_n) M(P_n), \tag{3.2}$$

which did not receive the attention it truly deserves. In particular, it contains the inequality

$$M(P_n') \le n M(P_n)$$

that is usually attributed to Mahler, who proved it later in [27]. To see this, just note that if $\Lambda_n(z) = nz(1+z)^{n-1} = \sum_{k=0}^n k\binom{n}{k} z^k$, then $\Lambda P_n(z) = zP_n'(z)$ and $M(\Lambda_n) = n$. Furthermore, (3.2) immediately answers a question about a lower bound for Mahler's measure of derivative raised in [13, pp. 12 and 194], see [36]. For $P_n'(z) = \sum_{k=0}^{n-1} a_k z^k$, write

$$\frac{1}{z}\left(P_n(z) - P_n(0)\right) = \sum_{k=0}^{n-1} \frac{a_k}{k+1} z^k = \Lambda P_n'(z),$$

where

$$\Lambda_{n-1}(z) = \sum_{k=0}^{n-1} \frac{1}{k+1} \binom{n-1}{k} z^k = \frac{(1+z)^n - 1}{nz}.$$

The result of de Bruijn and Springer gives

$$M(P_n(z) - P_n(0)) \le M(\Lambda_{n-1})\, M(P_n'),$$

with

$$M(\Lambda_{n-1}) = \frac{1}{n} M\left((1+z)^n - 1\right) = \frac{1}{n} \prod_{n/6<k<5n/6} 2 \sin \frac{k\pi}{n}.$$

There are many other interesting consequences of (3.2), which we leave for the reader.

We obtain the following generalization of (3.2) for $\|P_n\|_0$.

Theorem 3.1. *For any $\Lambda_n \in \mathbb{C}_n[z]$ and any $P_n \in \mathbb{C}_n[z]$, we have*

$$\|\Lambda P_n\|_0 \le M(\Lambda_n) \|P_n\|_0. \tag{3.3}$$

Note that equality holds in (3.2) and (3.3) for any polynomial $P_n \in \mathbb{C}_n[z]$ when $\Lambda_n(z) = (1+z)^n = \sum_{k=0}^{n} \binom{n}{k} z^k$, because $\Lambda P_n \equiv P_n$ and $M((1+z)^n) = 1$. This inequality allows to treat many problems in a unified way, and it has interesting corollaries stated below.

First, we mention an analog of the de Bruijn-Springer-Mahler inequality.

Corollary 3.2. *For any $P_n \in \mathbb{C}_n[z]$, we have that*

$$\|zP_n'\|_0 \le n \|P_n\|_0 \tag{3.4}$$

and

$$\|P_n'\|_0 \le \sqrt{e}\, n\, \|P_n\|_0, \tag{3.5}$$

where equality holds for $P_n(z) = z^n$.

Another consequence relates $\|P_n\|_0$ to the coefficients of P_n.

Corollary 3.3. *If $P_n(z) = \sum_{k=0}^{n} a_k z^k \in \mathbb{C}_n[z]$ then*

$$|a_k| \le e^{k/2} \binom{n}{k} \|P_n\|_0, \quad k = 0, \ldots, n. \tag{3.6}$$

Recall that we have $|a_k| \le \binom{n}{k} M(P_n)$ for Mahler's measure (see, e.g., [13]), which follows from (3.2) by letting $\Lambda_n(z) = \binom{n}{k} z^k$. One can certainly continue with a list of corollaries by choosing proper polynomials Λ_n.

4. Approximation by polynomials with integer coefficients

We consider a related question of approximation by polynomials with integer coefficients on the unit disk. There is a well known condition necessary for approximation by integer polynomials in essentially any norm on $\mathbb{D}$.

Proposition 4.1. *Suppose that $P_n \in \mathbb{Z}_n[z],\ n \in \mathbb{N}$, converge to f uniformly on compact subsets of $\mathbb{D}$. Then f is analytic in $\mathbb{D}$ and $f^{(k)}(0)/k! \in \mathbb{Z}\ \forall\, k \geq 0,\ k \in \mathbb{Z}$.*

This necessary condition for the convergence is clearly equivalent to the fact that the power series expansion of f at the origin has integer coefficients.

Define the Hardy space norm on $\mathbb{D}$ by

$$\|P_n\|_{H^p} := \left(\frac{1}{2\pi}\int_0^{2\pi} |P_n(e^{i\theta})|^p\, d\theta\right)^{1/p}, \quad 0 < p < \infty.$$

It is well known that approximation by polynomials with integer coefficients is possible in H^p only in the trivial case, see [15] and [39]. More precisely, we have

Proposition 4.2. *Suppose that $f \in H^p,\ 0 < p \leq \infty$. If $P_n \in \mathbb{Z}_n[z],\ n \in \mathbb{N}$, satisfy*

$$\lim_{n\to\infty} \|f - P_n\|_{H^p} = 0, \tag{4.1}$$

then f is a polynomial with integer coefficients.

It appears an open question whether this proposition is true for $p = 0$, i.e. for approximation of functions in Mahler's measure. Generally, nontrivial approximation by integer polynomials in the supremum norm is valid on sets with transfinite diameter (capacity) less than 1 [15, 39, 16], and it is not possible if the transfinite diameter is greater than or equal to 1. But the transfinite diameter of $\mathbb{D}$ is exactly equal to 1, so that we deal with a borderline case. However, we show that the Bergman space A^p is different from the Hardy space H^p in this regard, as it does allow approximation by polynomials with integer coefficients.

Theorem 4.3. *Suppose that $f \in A^p,\ 1 < p < \infty$. We have*

$$\lim_{n\to\infty} \|f - P_n\|_p = 0, \tag{4.2}$$

for a sequence of polynomials $P_n \in \mathbb{Z}_n[z],\ n \in \mathbb{N}$, if and only if f has a power series expansion about $z = 0$ with integer coefficients. Clearly, this is equivalent to $f^{(k)}(0)/k! \in \mathbb{Z}\ \forall\, k \geq 0,\ k \in \mathbb{Z}$.

Thus there are many functions in A^p that can be approximated by polynomials with integer coefficients. In fact, one can use partial sums of the power series for this purpose, see the proof of Theorem 4.3. However, we do not know whether Theorem 4.3 is valid in the case $0 \leq p \leq 1$. Note that if $f \in A^p,\ p > 1$, has a Taylor expansion with integer coefficients, then $f \in A^q$ for any $q \in [0, p)$ and the partial sums P_n of this expansion satisfy $\|f - P_n\|_q \le \|f - P_n\|_p \to 0$ as $n \to \infty$.

5. Multivariate polynomials

The definition of $\|P_n\|_0$ is easily generalized to the case of multivariate polynomials $P_n(z_1, \dots, z_d)$ as follows:

$$\|P_n\|_0 := \exp\left(\frac{1}{\pi^d}\int_{\mathbb{D}} \dots \int_{\mathbb{D}} \log|P_n(z_1, \dots, z_d)|\, dA(z_1) \dots dA(z_d)\right).$$

It is also parallel to multivariate Mahler's measure

$$M(P_n) := \exp\left(\frac{1}{(2\pi)^d}\int_{\mathbb{T}} \dots \int_{\mathbb{T}} \log|P_n(z_1, \dots, z_d)|\, |dz_1| \dots |dz_d|\right).$$

We note that many of the properties of $\|P_n\|_0$ are preserved in the multivariate case. Thus it still defines a multiplicative height on the space of polynomials. If P_n is a polynomial with complex coefficients and the constant term a_0, then we can apply the area mean value inequality to the (pluri)subharmonic function $\log|P_n(z_1, \dots, z_d)|$ in each variable, which gives together with Fubini's theorem that

$$\|P_n\|_0 \ge |a_0|.$$

Furthermore, the above inequality turns into equality if $P_n(z_1, \dots, z_d) \neq 0$ on $\mathbb{D}^d$, by the area mean value theorem for the (pluri)harmonic function $\log|P_n(z_1, \dots, z_d)|$. However, it is rather unlikely that some kind of explicit relation such as (1.4) exists for general multivariate polynomials.

We now state an estimate generalizing Corollary 1.2.

Proposition 5.1. *For a polynomial*

$$P_n(z_1, \dots, z_d) = \sum_{k_1 + \dots + k_d \le n} a_{k_1 \dots k_d} z_1^{k_1} \dots z_d^{k_d} \tag{5.1}$$

of degree at most n with complex coefficients, we have

$$e^{-n/2} M(P_n) \le \|P_n\|_0 \le M(P_n). \tag{5.2}$$

Equality holds in the lower estimate for any $P_n(z_1, \dots, z_d) = a_{k_1 \dots k_d} z_1^{k_1} \dots z_d^{k_d}$ with $k_1 + \dots + k_d = n$. The upper estimate turns into equality for any polynomial not vanishing in $\mathbb{D}^d$.

It is of interest to find explicit values of the multivariate $\|P_n\|_0$. This problem has received a considerable attention in Mahler's measure setting (see [8], [34, 35], [13], [18]), and it remains a very active area of research. In particular, it is of importance to characterize multivariate polynomials with integer coefficients satisfying $\|P_n\|_0 = 1$. Smyth [35] proved a complete Kronecker-type characterization for the multivariate Mahler's measure $M(P_n) = 1$. Thus we expect that one should be able to produce an analog for $\|P_n\|_0$, generalizing Theorem 1.3. We postpone a detailed study of the multivariate $\|P_n\|_0$ for another occasion, and conclude with simple examples.

Example 5.2. *The following identities hold for the multivariate* $\|P_n\|_0$*:*
(a) $\|z_1 + z_2\|_0 = e^{-1/4}$
(b) $\|1 + z_1^{k_1} \dots z_d^{k_d}\|_0 = 1, \quad k_1, \dots, k_d \ge 0$
(c) *If the polynomial* P_n *of the form* (5.1) *satisfies*

$$|a_{0\dots 0}| \ge \sum_{0<k_1+\dots+k_d\le n} |a_{k_1\dots k_d}|,$$

then $\|P_n\|_0 = M(P_n) = |a_{0\dots 0}|$.

6. Proofs

6.1. **Proofs for Section 1.**

Proof of Theorem 1.1. If P_n does not vanish in $\mathbb{D}$, then $\log|P_n(z)|$ is harmonic in $\mathbb{D}$. Hence $M(P_n) = |a_0|$ and $\|P_n\|_0 = |a_0|$ follow from the contour and area mean value theorems. Assume now that P_n has zeros in $\mathbb{D}$. Applying Jensen's formula, we obtain that

$$\log M(P_n) = \frac{1}{2\pi}\int_0^{2\pi} \log|P_n(e^{i\theta})|\,d\theta = \log|a_n| + \sum_{|z_j|\ge 1} \log|z_j|.$$

Furthermore,

$$\begin{aligned}
\log\|P_n\|_0 &= \frac{1}{\pi}\int_0^1\int_0^{2\pi} \log|P_n(re^{i\theta})|\,rdrd\theta \\
&= 2\int_0^1 \left(\frac{1}{2\pi}\int_0^{2\pi} \log|P_n(re^{i\theta})|\,d\theta\right) rdr \\
&= 2\int_0^1 \left(\log|a_n| + \sum_{|z_j|\ge r}\log|z_j| + \sum_{|z_j|<r}\log r\right) rdr \\
&= \log|a_n| + \sum_{|z_j|\ge 1}\log|z_j| + \frac{1}{2}\sum_{|z_j|<1}(|z_j|^2 - 1).
\end{aligned}$$

Hence

$$\|P_n\|_0 = M(P_n)\,\exp\left(\frac{1}{2}\sum_{|z_j|<1}(|z_j|^2-1)\right).$$

$\square$

Proof of Corollary 1.2. Inequality (1.5) follows from (1.4) after observing that the smallest value of the exponential is achieved when all $z_j = 0$, while the largest value is 1 when all $|z_j| \ge 1$.

□

Proof of Theorem 1.3. If P_n is cyclotomic, then $\|P_n\|_0 = 1$ by Theorem 1.1, because $|z_j| = 1,\ j = 1, \ldots, n$, and $M(P_n) = 1$. Assume now that $\|P_n\|_0 = 1$. Let $z_j,\ j = 1, \ldots, m,\ m \le n$, be the zeros of P_n in $\mathbb{D}$. We recall that $M(P_n) = |a_n| \prod_{|z_j|>1} |z_j| = |a_0| \prod_{|z_j|<1} |z_j|^{-1}$, where $a_0 \ne 0$ is the constant term of P_n. Thus we have from (1.4) that

$$\|P_n\|_0 = |a_0| \prod_{j=1}^{m} \frac{e^{(|z_j|^2-1)/2}}{|z_j|} \ge \prod_{j=1}^{m} \frac{e^{(|z_j|^2-1)/2}}{|z_j|}. \tag{6.1}$$

Define $g(x) := e^{(x^2-1)/2}/x,\ x > 0$, and observe that $g'(x) < 0$ when $x \in (0, 1)$, while $g'(x) > 0$ when $x \in (1, \infty)$. Hence

$$g(1) = 1 \text{ is the strict global minimum for } g(x) \text{ on } (0, \infty). \tag{6.2}$$

It follows from (6.1)-(6.2) that

$$1 < \prod_{j=1}^{m} g(|z_j|) = \prod_{j=1}^{m} \frac{e^{(|z_j|^2-1)/2}}{|z_j|} \le \|P_n\|_0 = 1,$$

which is a contradiction. Hence P_n has no zeros in $\mathbb{D}$, and $M(P_n) = \|P_n\|_0 = 1$ by Theorem 1.1. This implies that P_n is cyclotomic by Kronecker's theorem.

We could also proceed in a different way, by assuming that $\|P_n\|_0 = 1$ and observing from (6.1) that

$$\exp\left(\sum_{j=1}^{m} \frac{|z_j|^2 - 1}{2}\right) = \frac{1}{|a_0|} \prod_{j=1}^{m} |z_j|$$

Since the expression on the right is an algebraic number, as well as the sum in the exponent on the left, we obtain that equality is only possible when the latter sum is zero, by the well known result of Lindemann that the exponential of a nonzero algebraic number is transcendental [3]. Hence $|z_j| \ge 1,\ j = 1, \ldots, n$, and $M(P_n) = \|P_n\|_0 = 1$ as before.

□

Proof of Proposition 1.5. Assume that the zeros of P_n in $\mathbb{D}$ are given by $z_j,\ j = 1, \ldots, m$, and observe from (1.4) that

$$\exp\left(\sum_{j=1}^{m} \frac{|z_j|^2 - 1}{2}\right) = \frac{\|P_n\|_0}{M(P_n)}$$

Since the sum in the exponent on the left is algebraic, we obtain that the left hand side is transcendental by the well known result of Lindemann [3]. Note that $M(P_n)$ is always algebraic. If $\|P_n\|_0$ were algebraic, then the right hand side would be algebraic too, a contradiction. When P_n has no zeros in $\mathbb{D}$, we have $\|P_n\|_0 = M(P_n) = |a_0|$ by Theorem 1.1.

□

6.2. **Proofs for Section 2.**

Proof of Theorem 2.1. We first show that P_n has $o(n)$ zeros in $D_r := \{z : |z| < r\}$ as $n \to \infty$, for any $r < 1$. Assume to the contrary that there is a subsequence of n such that P_n has at least αn zeros, with $\alpha > 0$, in some $D_r,\ r < 1$. Suppose that those zeros are $z_j \neq 0,\ j = 1, \dots, m,\ m \le n$, and proceed as in the proof of Theorem 1.3 to obtain

$$\prod_{j=1}^{m} g(|z_j|) = \prod_{j=1}^{m} \frac{e^{(|z_j|^2-1)/2}}{|z_j|} \le \|P_n\|_0 \tag{6.3}$$

by (6.1). Since $g(x) = e^{(x^2-1)/2}/x$ is strictly decreasing on $(0, 1)$, we have that

$$\prod_{j=1}^{m} g(|z_j|) \ge (g(r))^{\alpha n}.$$

It immediately follows from (6.2) and (6.3) that

$$\limsup_{n\to\infty} \|P_n\|_0^{1/n} \ge (g(r))^{\alpha} > 1,$$

which is in direct conflict with assumptions of this theorem. If P_n has a simple zero at $z = 0$, then $P_n(z) = zQ_{n-1}(z)$ and $\|P_n\|_0 = \|Q_{n-1}\|_0/\sqrt{e}$. Hence we can apply the above argument to Q_{n-1} and come to the same conclusion that P_n has $o(n)$ zeros in $D_r := \{z : |z| < r\},\ r < 1$, as $n \to \infty$.

The second step is to show that $\lim_{n\to\infty}(M(P_n))^{1/n} = 1$. Note that

$$1 \le M(P_n) = \|P_n\|_0 \exp\left(\frac{1}{2} \sum_{|z_j|<1} (1 - |z_j|^2)\right). \tag{6.4}$$

If P_n has $m = o(n)$ zeros in $D_r,\ r < 1$, then

$$\exp\left(\frac{1}{2} \sum_{|z_j|<1} (1 - |z_j|^2)\right) \le e^{m/2+n(1-r^2)/2}.$$

Using this in (6.4), we obtain that

$$\begin{aligned} 1 &\le \liminf_{n\to\infty}(M(P_n))^{1/n} \le \limsup_{n\to\infty}(M(P_n))^{1/n} \\ &\le e^{(1-r^2)/2} \lim_{n\to\infty} \|P_n\|_0^{1/n} = e^{(1-r^2)/2}. \end{aligned}$$

Hence $\lim_{n\to\infty}(M(P_n))^{1/n} = 1$ follows by letting $r \to 1-$. The proof may now be completed by applying Bilu's result [4] (at least when P_n is irreducible for all $n \in \mathbb{N}$), but we prefer to continue with an independent proof via a standard potential theoretic argument.

Observe that $P_n(z) = a_n \prod_{j=1}^n (z - z_j)$ has $o(n)$ zeros in $\mathbb{C} \setminus D_r,\ r > 1$, for otherwise we would have $\liminf_{n\to\infty}(M(P_n))^{1/n} > 1$ as

$$M(P_n) = |a_n| \prod_{|z_j|>1} |z_j| \ge \prod_{|z_j|>1} |z_j|.$$

This also implies that

$$\lim_{n\to\infty} |a_n|^{1/n} = 1. \tag{6.5}$$

Hence any weak* limit ν of the sequence ν_n must satisfy $\operatorname{supp}\nu \subset \mathbb{T}$. Define the logarithmic energy of ν by

$$I(\nu) := \iint \log \frac{1}{|z-t|} d\nu(z)\, d\nu(t).$$

Our goal is to show that $I(\nu) = 0$, which implies that ν has the smallest possible energy among all positive Borel measures of mass 1 supported on $\mathbb{T}$. On the other hand, it is well known in potential theory that the equilibrium measure minimizing the energy integral is unique, and it is equal to the normalized arclength on $\mathbb{T}$ [30, 40]. Thus $\nu = d\theta/(2\pi)$ and the proof would be completed.

Define the discriminant of P_n as $\Delta_n := a_n^{2n-2} \prod_{1\le j<k\le n}(z_j - z_k)^2$. Observe that it is an integer, being a symmetric form with integer coefficients in the roots of $P_n \in \mathbb{Z}_n[z]$. Since P_n has no multiple roots, we have $\Delta_n \ne 0$ and $|\Delta_n| \ge 1$. Therefore,

$$\log \frac{1}{|\Delta_n|} = -(2n-2)\log|a_n| + \sum_{j\ne k} \log \frac{1}{|z_j - z_k|} \le 0. \tag{6.6}$$

Let

$$K_M(z,t) := \min\left(\log \frac{1}{|z-t|}, M\right), \qquad M > 0.$$

It is clear that $K_M(z,t)$ is a continuous function in z and t on $\mathbb{C}\times\mathbb{C}$, and that $K_M(z,t)$ increases to $\log \frac{1}{|z-t|}$ as $M \to \infty$. Using the Monotone Convergence

Theorem and the weak* convergence of $\nu_n \times \nu_n$ to $\nu \times \nu$, we obtain that

$$\begin{aligned} I(\nu) &= \lim_{M\to\infty} \iint K_M(z,t)\,d\nu(z)\,d\nu(t) \\ &= \lim_{M\to\infty} \left(\lim_{n\to\infty} \iint K_M(z,t)\,d\nu_n(z)\,d\nu_n(t) \right) \\ &= \lim_{M\to\infty} \left(\lim_{n\to\infty} \left(\frac{1}{n^2} \sum_{j\neq k} K_M(z_j,z_k) + \frac{M}{n} \right) \right) \\ &\le \lim_{M\to\infty} \left(\liminf_{n\to\infty} \frac{1}{n^2} \sum_{j\neq k} \log \frac{1}{|z_j - z_k|} \right) \\ &= \liminf_{n\to\infty} \frac{1}{n^2} \log \frac{|a_n|^{2n-2}}{\Delta_n}. \end{aligned}$$

Hence $I(\nu) \le 0$ follows from (6.5)-(6.6). But $I(\mu) > 0$ for any positive unit Borel measure supported on $\mathbb{T}$, with the only exception for the equilibrium measure $d\mu_{\mathbb{T}} := d\theta/(2\pi)$, $I(\mu_{\mathbb{T}}) = 0$, see [40, pp. 53-89].

□

Proof of Theorem 2.3. (a) We use the same notation and approach as in the proof of Theorem 1.3. If P_n has no zeros in $\mathbb{D}$, then $\min_{1\le j\le n} |z_j| \ge 1$. Otherwise, let $z_j,\ j = 1,\dots,m,\ m \le n$, be the zeros of P_n in $\mathbb{D}$. It follows from (6.1)-(6.2) that

$$\|P_n\|_0 = |a_0| \prod_{j=1}^{m} \frac{e^{(|z_j|^2-1)/2}}{|z_j|} \ge g\left(\min_{1\le j\le n} |z_j| \right) > 1.$$

Thus we obtain the result by the continuity of $g(x) = e^{(x^2-1)/2}/x,\ x > 0$, and (6.2).

(b) Note that $\lim_{n\to\infty} \|P_n\|_0 = 1$ in this case too, by (1.1) and (1.2). Hence (2.1) holds true. Furthermore, we have for any zero $z_k \in \mathbb{C} \setminus \mathbb{D}$ that

$$1 \le |z_k| \le |a_n| \prod_{|z_j|>1} |z_j| = M(P_n).$$

Thus

$$\lim_{n\to\infty} \max_{1\le j\le n} |z_j| = 1,$$

and (2.2) follows.

□

6.3. **Proofs for Section 3.**

Proof of Theorem 3.1. Using (3.2) for the polynomial $P_n(rz),\ r \in [0,1]$, we obtain that

$$\int_0^{2\pi} \log|\Lambda P_n(re^{i\theta})|\,d\theta \le 2\pi \log M(\Lambda_n) + \int_0^{2\pi} \log|P_n(re^{i\theta})|\,d\theta.$$

Hence (3.3) follows immediately, if we multiply this inequality by $r\,dr/\pi$ and integrate from 0 to 1.

□

Proof of Corollary 3.2. We follow [9] by setting $\Lambda_n(z) = nz(1+z)^{n-1} = \sum_{k=0}^n k\binom{n}{k}z^k$. This gives $\Lambda P_n(z) = zP_n'(z)$ and $M(\Lambda_n) = n$. Hence (3.4) is a consequence of (3.3). In order to deduce (3.5) from (3.4), we only need to observe that $\|zP_n'\|_0 = \|z\|_0\|P_n'\|_0 = \|P_n'\|_0/\sqrt{e}$.

□

Proof of Corollary 3.3. Let $\Lambda_n(z) = \binom{n}{k}z^k, 0 \le k \le n$, where k is fixed. Then $\Lambda P_n(z) = a_k z^k$ and $M(\Lambda_n) = \binom{n}{k}$. It follows from (3.3) that

$$\|a_k z^k\|_0 = |a_k|e^{-k/2} \le \binom{n}{k}\|P_n\|_0,$$

because $\|z^k\|_0 = e^{-k/2}$.

□

6.4. **Proofs for Section 4.**

Proof of Proposition 4.1. Recall that the uniform convergence of P_n to f on compact subsets of $\mathbb{D}$ implies that f is analytic in $\mathbb{D}$, and that $P_n^{(k)}$ converge to $f^{(k)}$ on compact subsets of $\mathbb{D}$ for any $k \in \mathbb{N}$. In particular,

$$\lim_{n\to\infty} P_n^{(k)}(0) = f^{(k)}(0) \qquad \forall\ k \ge 0,\ k \in \mathbb{Z}.$$

But $P_n^{(k)}(0) = k!a_k$, where $a_k \in \mathbb{Z}$ is a corresponding coefficient of P_n. Hence the result follows.

□

Proof of Proposition 4.2. We have that

$$\|P_n - P_{n-1}\|_{H^p} \le \|f - P_n\|_{H^p} + \|f - P_{n-1}\|_{H^p}$$

by the triangle inequality for $p \ge 1$, and

$$\|P_n - P_{n-1}\|_{H^p}^p \le \|f - P_n\|_{H^p}^p + \|f - P_{n-1}\|_{H^p}^p$$

for $0 < p < 1$. In both cases, (4.1) implies that

$$\lim_{n\to\infty} \|P_n - P_{n-1}\|_{H^p} = 0, \qquad 0 < p \le \infty.$$

If $P_n \not\equiv P_{n-1}$ then we let $a_k z^k$ be the lowest nonzero term of $P_n - P_{n-1}$, where $|a_k| \in \mathbb{N}$. Using the mean value inequality [11], we obtain

$$\|P_n - P_{n-1}\|_{H^p} \geq |a_k| \geq 1, \qquad 0 < p \leq \infty.$$

This is obviously impossible as $n \to \infty$, so that we have $P_n \equiv P_{n-1}$ for all sufficiently large $n \in \mathbb{N}$. Hence the limit function f is also a polynomial with integer coefficients.

□

Proof of Theorem 4.3. If (4.2) holds then P_n converge to f on compact subsets of $\mathbb{D}$ by the area mean value inequality:

$$\begin{aligned}|f(z) - P_n(z)|^p &\leq \frac{1}{\pi(1-|z|)^2} \iint_{|t-z|<1-|z|} |f(t) - P_n(t)|^p \, dA \\ &\leq \frac{\|f(t) - P_n(t)\|_p^p}{(1-|z|)^2} \to 0, \quad n \to \infty, \ z \in \mathbb{D}.\end{aligned}$$

Hence f has a power series expansion at $z = 0$ with integer coefficients by Proposition 4.1.

Conversely, suppose that $f \in A^p$ is represented by a power series with integer coefficients. Since the partial sums of this series converge to f in A^p norm for $1 < p < \infty$ by Theorem 4 [11, p. 31], we can select the sequence P_n be the sequence of the partial sums.

□

6.5. **Proofs for Section 5.**

Proof of Proposition 5.1. We apply (1.5) in each variable z_j, $j = 1, \ldots, d$, and use Fubini's theorem to prove (5.2). Indeed, (1.5) gives that

$$\begin{aligned}\frac{1}{2\pi} \int_{\mathbb{T}} \log |P_n(z_1, \ldots, z_d)| \, |dz_1| - \frac{k_1}{2} &\leq \frac{1}{\pi} \int_{\mathbb{D}} \log |P_n(z_1, \ldots, z_d)| \, dA(z_1) \\ &\leq \frac{1}{2\pi} \int_{\mathbb{T}} \log |P_n(z_1, \ldots, z_d)| \, |dz_1|\end{aligned}$$

is true for all $z_2, \ldots, z_d \in \mathbb{C}$. Integrating the above inequality with respect to $dA(z_2)/\pi$, interchanging the order of integration in the lower and upper bounds, and applying (1.5) in the variable z_2, we obtain

$$\begin{aligned}&\frac{1}{(2\pi)^2} \int_{\mathbb{T}} \int_{\mathbb{T}} \log |P_n(z_1, \ldots, z_d)| \, |dz_1||dz_2| - \frac{k_1 + k_2}{2} \\ &\leq \frac{1}{\pi^2} \int_{\mathbb{D}} \int_{\mathbb{D}} \log |P_n(z_1, \ldots, z_d)| \, dA(z_1) dA(z_2) \\ &\leq \frac{1}{(2\pi)^2} \int_{\mathbb{T}} \int_{\mathbb{T}} \log |P_n(z_1, \ldots, z_d)| \, |dz_1||dz_2|\end{aligned}$$

is true for all $z_3, \ldots, z_d \in \mathbb{C}$. After carrying out this argument for each variable z_j, we arrive at (5.2) in d steps. When $P_n(z_1, \ldots, z_d) \neq 0$ in $\mathbb{D}^d$, we have that

$\|P_n\|_0 = M(P_n) = |a_{0\ldots 0}|$ by the iterative application of Theorem 1.1. If $P_n(z_1,\ldots,z_d) = a_{k_1\ldots k_d} z_1^{k_1}\ldots z_d^{k_d}$, where $k_1+\ldots+k_d = n$, then we evaluate directly that $M(P_n) = |a_{k_1\ldots k_d}|$ and $\|P_n\|_0 = |a_{k_1\ldots k_d}|e^{-n/2}$, because $\|z_j\|_0 = e^{-1/2},\ j=1,\ldots,n.$

$\square$

Proof of Example 5.2. (a) Applying (1.4), we have that

$$\frac{1}{\pi^2}\int_{\mathbb{D}}\int_{\mathbb{D}} \log|z_1+z_2|\,dA(z_1)dA(z_2) = \frac{1}{\pi}\int_{\mathbb{D}} \frac{|z_2|^2-1}{2}\,dA(z_2) = -\frac{1}{4}.$$

(b) is an immediate consequence of (c).
(c) Let $a_{0\ldots 0} = |a_{0\ldots 0}|e^{i\phi}$. Observe that $P_n(z_1,\ldots,z_d)+\varepsilon e^{i\phi} \neq 0$ in $\mathbb{D}^d$ for any $\varepsilon > 0$, because

$$|P_n(z_1,\ldots,z_d)+\varepsilon e^{i\phi}| \ge |a_{0\ldots 0}| + \varepsilon - \sum_{0<k_1+\ldots+k_d\le n} |a_{k_1\ldots k_d}| > 0$$

by the triangle inequality. We obtain that $\|P_n+\varepsilon e^{i\phi}\|_0 = M(P_n+\varepsilon e^{i\phi}) = |a_{0\ldots 0}|+\varepsilon$ by the area and contour mean value properties of the (pluri)harmonic function $\log|P_n(z_1,\ldots,z_d)+\varepsilon e^{i\phi}|$ in $\mathbb{D}^d$, and the result follows by letting $\varepsilon \to 0$.

$\square$

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