Means of algebraic numbers in the unit disk

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Abstract

Schur studied limits of the arithmetic means s_n of zeros for polynomials of degree n with integer coefficients and simple zeros in the closed unit disk. If the leading coefficients are bounded, Schur proved that $\limsup_{n\to\infty} |s_n| \le 1 - \sqrt{e}/2$. We show that $s_n \to 0$, and estimate the rate of convergence by generalizing the Erdős-Turán theorem on the distribution of zeros. *To cite this article: I. E. Pritsker, C. R. Acad. Sci. Paris, Ser. I 336 (2003).*

Résumé

Moyennes de nombres algébriques dans le disque unité. Schur a étudié les limites des moyennes arithmétiques s_n des zéros pour les polynômes à coefficients entiers de degré n ayant des zéros simples dans le disque unité fermé. Lorsque les coefficients dominants restent bornés, Schur a démontré que $\limsup_{n\to\infty} |s_n| \le 1 - \sqrt{e}/2$. Nous prouvons que $s_n \to 0$. Nous donnons une estimation du taux de convergence, grâce à une généralisation d'un théorème de Erdős-Turán sur la distribution des zéros. *Pour citer cet article : I. E. Pritsker, C. R. Acad. Sci. Paris, Ser. I 336 (2003).*

1. Schur's problem and equidistribution of zeros

Let $\mathbb{Z}_n(D)$ be the set of polynomials of degree n with integer coefficients and all zeros in the closed unit disk D. We denote the subset of $\mathbb{Z}_n(D)$ with simple zeros by $\mathbb{Z}_n^1(D)$. Given $M > 0$, we write $P_n = a_n z^n + \ldots \in \mathbb{Z}_n^1(D, M)$ if $|a_n| \le M$ and $P_n \in \mathbb{Z}_n^1(D)$ (respectively $P_n \in \mathbb{Z}_n(D, M)$ if $|a_n| \le M$ and $P_n \in \mathbb{Z}_n(D)$). Schur [9, §8] studied the limiting behavior of the arithmetic means s_n of zeros for polynomials from $\mathbb{Z}_n^1(D, M)$ as $n \to \infty$, where $M > 0$ is an arbitrary fixed number. He showed that $\limsup_{n\to\infty} |s_n| \le 1 - \sqrt{e}/2$, and remarked that this lim sup is equal to 0 for *monic* polynomials from $\mathbb{Z}_n(D)$ by Kronecker's theorem [6]. We prove that $\lim_{n\to\infty} s_n = 0$ for any sequence of polynomials from Schur's class $\mathbb{Z}_n^1(D, M)$, $n \in \mathbb{N}$. This result is obtained as a consequence of the asymptotic equidistribution of zeros near the unit circle. Namely, if $\{\alpha_k\}_{k=1}^n$ are the zeros of P_n, we define the counting measure $\tau_n :=$

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$\frac{1}{n}\sum_{k=1}^n \delta_{\alpha_k}$, where δ_{α_k} is the unit point mass at α_k. Consider the normalized arclength measure μ on the unit circumference $\mathbb{T}$, with $d\mu(e^{it}) := \frac{1}{2\pi}dt$. If the τ_n converge weakly to μ as $n \to \infty$ ($\tau_n \xrightarrow{*} \mu$) then $\lim_{n\to\infty} s_n = \lim_{n\to\infty} \int z\, d\tau_n(z) = \int z\, d\mu(z) = 0$. Thus Schur's problem is solved by the following result.

Theorem 1.1 *If $P_n(z) = a_n z^n + \ldots \in \mathbb{Z}_n^1(D),\ n \in \mathbb{N}$, satisfy $\lim\limits_{n\to\infty} |a_n|^{1/n} = 1$, then $\tau_n \xrightarrow{*} \mu$ as $n \to \infty$.*

Ideas on the equidistribution of zeros date back to Jentzsch and Szegő, cf. [1, Ch. 2]. They were developed further by Erdős and Turán [4], and many others; see [1] for history and additional references. More recently, this topic received renewed attention in number theory, e.g. in the work of Bilu [2]. If the leading coefficients of polynomials are bounded, then we can allow even certain multiple zeros. Define the multiplicity of an irreducible factor Q of P_n as the integer $m_n \geq 0$ such that Q^{m_n} divides P_n, but Q^{m_n+1} does not divide P_n. If a factor Q occurs infinitely often in a sequence $P_n,\ n \in \mathbb{N}$, then $m_n = o(n)$ means $\lim_{n\to\infty} m_n/n = 0$. If Q is present only in finitely many P_n, then $m_n = o(n)$ by definition.

Theorem 1.2 *Assume that $P_n \in \mathbb{Z}_n(D, M),\ n \in \mathbb{N}$. If every irreducible factor in the sequence of polynomials P_n has multiplicity $o(n)$, then $\tau_n \xrightarrow{*} \mu$ as $n \to \infty$.*

Corollary 1.3 *If $P_n(z) = a_n \prod_{k=1}^n (z - \alpha_k),\ n \in \mathbb{N}$, satisfy the assumptions of Theorem 1.1 or 1.2, then*

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^n \alpha_k^m = 0, \quad m \in \mathbb{N}.$$

We also show that the norms $\|P_n\|_\infty := \max_{|z|=1} |P_n(z)|$ have at most subexponential growth.

Corollary 1.4 *If $P_n,\ n \in \mathbb{N}$, satisfy the assumptions of Theorem 1.1 or Theorem 1.2, then*

$$\lim_{n\to\infty} \|P_n\|_\infty^{1/n} = 1.$$

This result is somewhat unexpected, as we have no direct control of the norm or coefficients (except for the leading one). For example, $P_n(z) = (z-1)^n$ has norm $\|P_n\|_\infty = 2^n$.

We now consider quantitative aspects of the convergence $\tau_n \xrightarrow{*} \mu$. As an application, we obtain estimates of the convergence rate of s_n to 0 in Schur's problem. A classical result on the distribution of zeros is due to Erdős and Turán [4]. For $P_n(z) = \sum_{k=0}^n a_k z^k$ with $a_k \in \mathbb{C}$, let $N(\phi_1, \phi_2)$ be the number of zeros in the sector $\{z \in \mathbb{C} : 0 \leq \phi_1 \leq \arg(z) \leq \phi_2 < 2\pi\}$, where $\phi_1 < \phi_2$. Erdős and Turán [4] proved that

$$\left| \frac{N(\phi_1, \phi_2)}{n} - \frac{\phi_2 - \phi_1}{2\pi} \right| \leq 16 \sqrt{\frac{1}{n} \log \frac{\|P_n\|_\infty}{\sqrt{|a_0 a_n|}}}. \tag{1}$$

The constant 16 was improved by Ganelius, and $\|P_n\|_\infty$ was replaced by weaker integral norms by Amoroso and Mignotte; see [1] for more history and references. Our main difficulty in applying (1) to Schur's problem is the absence of an effective estimate for $\|P_n\|_\infty,\ P_n \in \mathbb{Z}_n^1(D, M)$. We prove a new "discrepancy" estimate via energy considerations from potential theory. These ideas originated in part in the work of Kleiner, and were developed by Sjögren and Hüsing, see [1, Ch. 5]. We also use the Mahler measure of a polynomial $P_n(z) = a_n \prod_{k=1}^n (z - \alpha_k)$, defined by $M(P_n) := \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \log|P_n(e^{it})|\, dt\right)$. Note that $M(P_n) = \lim_{p\to 0} \|P_n\|_p$, where $\|P_n\|_p := \left(\frac{1}{2\pi}\int_0^{2\pi} |P_n(e^{it})|^p\, dt\right)^{1/p},\ p > 0$. Jensen's formula readily gives $M(P_n) = |a_n| \prod_{k=1}^n \max(1, |\alpha_k|)$ [3, p. 3]. Hence $M(P_n) = |a_n| \leq M$ for any $P_n \in \mathbb{Z}_n(D, M)$.

Theorem 1.5 *Let $\phi : \mathbb{C} \to \mathbb{R}$ satisfy $|\phi(z) - \phi(t)| \leq A|z - t|,\ z, t \in \mathbb{C}$, and $\operatorname{supp}(\phi) \subset \{z : |z| \leq R\}$. If $P_n(z) = a_n \prod_{k=1}^n (z - \alpha_k)$ is a polynomial with integer coefficients and simple zeros, then*

$$\left| \frac{1}{n}\sum_{k=1}^n \phi(\alpha_k) - \int \phi\, d\mu \right| \leq A(2R+1)\sqrt{\frac{\log\max(n, M(P_n))}{n}}, \quad n \geq 55. \tag{2}$$

This theorem is related to recent results of Favre and Rivera-Letelier [5], obtained in a different setting. Choosing ϕ appropriately, we obtain an estimate of the means s_n in Schur's problem.

Corollary 1.6 *If $P_n \in \mathbb{Z}_n^1(D,M)$ then*

$$\left|\frac{1}{n}\sum_{k=1}^{n}\alpha_k\right| \le 8\sqrt{\frac{\log n}{n}}, \quad n \ge \max(M,55).$$

We also have an improvement of Corollary 1.4 for Schur's class $\mathbb{Z}_n^1(D,M)$.
Corollary 1.7 *If $\{P_n\}_{n=1}^\infty \in \mathbb{Z}_n^1(D,M)$ then there is some $c>0$ such that $\|P_n\|_\infty \le e^{c\sqrt{n}\log n}$ as $n\to\infty$.*

The proof of Theorem 1.5 gives a result for arbitrary polynomials with simple zeros, and for any continuous ϕ with finite Dirichlet integral $D[\phi] = \iint(\phi_x^2+\phi_y^2)\,dA$. Moreover, all arguments may be extended to general sets of logarithmic capacity 1, e.g. to $[-2,2]$. Using the characteristic function $\phi=\chi_E$, we can prove general discrepancy estimates on arbitrary sets, and obtain an Erdős-Turán-type theorem. Our results have a number of applications to the problems on integer polynomials considered in [3].

2. Proofs

Proof of Theorem 1.1. Observe that the discriminant $\Delta(P_n) := a_n^{2n-2}\prod_{1\le j<k\le n}(\alpha_j-\alpha_k)^2$ is an integer, as a symmetric form in the zeros of P_n. Since P_n has simple roots, we have $\Delta(P_n)\neq 0$ and $|\Delta(P_n)|\ge 1$. Using weak compactness, we assume that $\tau_n \xrightarrow{*} \tau$, where τ is a probability measure on D. Let $K_M(x,t) := \min(-\log|x-t|, M)$. Since $\tau_n\times\tau_n \xrightarrow{*} \tau\times\tau$, we obtain for the energy of τ that

$$\begin{aligned} I[\tau] &:= -\iint \log|x-t|\,d\tau(x)\,d\tau(t) = \lim_{M\to\infty}\left(\lim_{n\to\infty}\iint K_M(x,t)\,d\tau_n(x)\,d\tau_n(t)\right) \\ &= \lim_{M\to\infty}\left(\lim_{n\to\infty}\left(\frac{1}{n^2}\sum_{j\neq k}K_M(\alpha_j,\alpha_k)+\frac{M}{n}\right)\right) \le \lim_{M\to\infty}\left(\liminf_{n\to\infty}\frac{1}{n^2}\sum_{j\neq k}\log\frac{1}{|\alpha_j-\alpha_k|}\right) \\ &= \liminf_{n\to\infty}\frac{1}{n^2}\log\frac{|a_n|^{2n-2}}{\Delta(P_n)} \le \liminf_{n\to\infty}\frac{1}{n^2}\log|a_n|^{2n-2} = 0. \end{aligned}$$

Thus $I[\tau]\le 0$. But $I[\nu]>0$ for any probability measure ν on D, except for μ [7]. Hence $\tau=\mu$.
Proof of Theorem 1.2. Let $\phi\in C(\mathbb{C})$. Note that for any $\epsilon>0$ there are finitely many irreducible factors Q in the sequence P_n such that $|\int\phi\,d\tau(Q)-\int\phi\,d\mu|\ge\epsilon$, where $\tau(Q)$ is the zero counting measure for Q. Indeed, if we have an infinite sequence of such Q_m, then $\deg(Q_m)\to\infty$, as there are only finitely many $Q_m\in\mathbb{Z}_n(D,M)$ of bounded degree. Hence $\int\phi\,d\tau(Q_m)\to\int\phi\,d\mu$ by Theorem 1.1. Let the number of such exceptional factors Q_m be N. Then we have $|n\int\phi\,d\tau_n - n\int\phi\,d\mu| \le No(n)\max_D|\phi-\int\phi\,d\mu| + (n-N)\epsilon,\ n\in\mathbb{N}$. Hence $\limsup_{n\to\infty}|\int\phi\,d\tau_n-\int\phi\,d\mu|\le\epsilon$, and $\lim_{n\to\infty}\int\phi\,d\tau_n=\int\phi\,d\mu$ after letting $\epsilon\to 0$.
Proof of Corollary 1.3. Let $\phi(z)=z^m$ and write $\lim_{n\to\infty}\int z^m\,d\tau_n(z)=\int z^m\,d\mu(z)=0$.
Proof of Corollary 1.4. Let $\|P_n\|_\infty=|P_n(z_n)|,\ z_n\in D$, and assume $\lim_{n\to\infty}z_n=z_0\in D$ by compactness. Then $\|P_n\|_\infty=\exp(\log|P_n(z_n)|)=|a_n|\exp\left(n\int\log|z_n-t|\,d\tau_n(t)\right)$. Since $\tau_n\xrightarrow{*}\mu$, Theorem I.6.8 of [8] gives $\limsup_{n\to\infty}\|P_n\|_\infty^{1/n}\le\exp\left(\int\log|z_0-t|\,d\mu(t)\right)=1$ [8, p. 22]. But $\|P_n\|_\infty\ge|a_n|\ge 1$, see [1, p. 16].
Proof of Theorem 1.5. Given $r>0$, define the measures ν_k^r with $d\nu_k^r(\alpha_k+re^{it})=dt/(2\pi),\ t\in[0,2\pi)$. Let $\tau_n^r := \frac{1}{n}\sum_{k=1}^n\nu_k^r$, and estimate $\left|\int\phi\,d\tau_n-\int\phi\,d\tau_n^r\right| \le \frac{1}{n}\sum_{k=1}^n\frac{1}{2\pi}\int_0^{2\pi}\left|\phi(\alpha_k)-\phi(\alpha_k+re^{it})\right|\,dt\le\omega_\phi(r)$, where $\omega_\phi(r):=\sup_{|z-\zeta|\le r}|\phi(z)-\phi(\zeta)|$ is the modulus of continuity of ϕ.

Let $p_\nu(z) := -\int\log|z-t|d\nu(t)$ be the potential of a measure ν. A direct evaluation gives that $p_{\nu_k^r}(z) = -\log\max(r,|z-\alpha_k|)$ and $p_\mu(z)=-\log\max(1,|z|)$ [8, p. 22]. Consider $\sigma:=\tau_n^r-\mu,\ \sigma(\mathbb{C})=0$. One computes (or see [8, p. 92]) that $d\sigma=-\frac{1}{2\pi}(\partial p_\sigma/\partial n_+ + \partial p_\sigma/\partial n_-)\,ds$, where ds is the arclength on $\mathrm{supp}(\sigma)=\{|z|=1\}\cup(\cup_{k=1}^n\{|z-\alpha_k|=r\})$, and $n_\pm$ are the inner and the outer normals. We now use

Green's identity $\iint_G u\Delta v\, dA = \int_{\partial G} u \frac{\partial v}{\partial n}\, ds - \iint_G \nabla u \cdot \nabla v\, dA$ with $u = \phi$ and $v = p_\sigma$ in each component G of $\{|z| < R\} \setminus \mathrm{supp}(\sigma)$. Since $\Delta p_\sigma = 0$ in G, adding the identities for all G, we obtain that

$$\left|\int \phi\, d\sigma\right| = \frac{1}{2\pi}\left|\iint_{|z|\le R} \nabla\phi\cdot\nabla p_\sigma\, dA\right| \le \frac{1}{2\pi}\sqrt{D[\phi]}\,\sqrt{D[p_\sigma]},$$

where $D[\phi] = \iint(\phi_x^2 + \phi_y^2)\, dA$ is the Dirichlet integral of ϕ. It is known that $D[p_\sigma] = 2\pi I[\sigma]$ [7, Thm 1.20], where $I[\sigma] = -\iint \log|z-t|\, d\sigma(z)\, d\sigma(t) = \int p_\sigma\, d\sigma$. Since $p_\mu(z) = -\log\max(1,|z|)$, we observe that $\int p_\mu\, d\mu = 0$, so that $I[\sigma] = \int p_{\tau_n^r}\, d\tau_n^r - 2\int p_\mu\, d\tau_n^r$. Further, $-\int p_\mu\, d\tau_n^r = \int \log\max(1,|z|)\, d\tau_n^r(z) \le \left(\sum_{|\alpha_k|\le 1+r}\log(1+2r) + \sum_{|\alpha_k|>1+r}\log|\alpha_k|\right)/n \le \log(1+2r) + \frac{1}{n}\log M(P_n) - \frac{1}{n}\log|a_n|$. We also have that $\int p_{\tau_n^r}\, d\tau_n^r \le \left(-\sum_{j\ne k}\log|\alpha_j - \alpha_k| - n\log r\right)/n^2$. We next combine the energy estimates to obtain

$$I[\sigma] \le \frac{2}{n}\log M(P_n) - \frac{1}{n^2}\log\left|a_n^2\Delta(P_n)\right| - \frac{1}{n}\log r + 4r.$$

Collecting all estimates, we proceed with $\left|\int\phi\, d\tau_n - \int\phi\, d\mu\right| \le \left|\int\phi\, d\tau_n - \int\phi\, d\tau_n^r\right| + \left|\int\phi\, d\tau_n^r - \int\phi\, d\mu\right| \le \omega_\phi(r) + \sqrt{D[\phi]}\,\sqrt{D[p_\sigma]}/(2\pi) = \omega_\phi(r) + \sqrt{D[\phi]}\,\sqrt{I[\sigma]/(2\pi)}$. Thus we arrive at the main inequality:

$$\left|\int\phi\, d\tau_n - \int\phi\, d\mu\right| \le \omega_\phi(r) + \sqrt{\frac{D[\phi]}{2\pi}}\left(\frac{2}{n}\log M(P_n) - \frac{1}{n^2}\log\left|a_n^2\Delta(P_n)\right| - \frac{1}{n}\log r + 4r\right)^{1/2}. \qquad (3)$$

Note that $D[\phi] \le 2\pi R^2 A^2$, as $|\phi_x| \le A$ and $|\phi_y| \le A$ a.e. in $\mathbb{C}$. Also, $\omega_\phi(r) \le Ar$. Since $|\Delta(P_n)| \ge 1$ and $|a_n| \ge 1$, we have $|a_n^2\Delta(P_n)| \ge 1$. Hence (2) follows from (3) by letting $r = 1/\max(n, M(P_n))$.

Proof of Corollary 1.6. Since P_n has real coefficients, we have that $s_n = \int z\, d\tau_n(z) = \int \Re(z)\, d\tau_n(z)$. We let $\phi(z) = \Re(z),\ |z| \le 1;\ \phi(z) = \Re(z)(1 - \log|z|),\ 1 \le |z| \le e$; and $\phi(z) = 0,\ |z| \ge e$. An elementary computation shows that $|\phi_x(z)| \le 1$ and $|\phi_y(z)| \le 1/2$ for all $z = x + iy \in \mathbb{C}$. The Mean Value Theorem gives $|\phi(z) - \phi(t)| \le |z-t|\max_{\mathbb{C}}\sqrt{\phi_x^2 + \phi_y^2}$. Hence we can use Theorem 1.5 with $A = \sqrt{5}/2$ and $R = e$.

Proof of Corollary 1.7. Note that $\log|P_n(z)| = n\int\log|z-w|\, d\tau_n(w)$. For $|z| = 1 + 1/n$, we let $\phi(w) = \log|z-w|,\ |w| \le 1;\ \phi(w) = (1 - \log|w|)\log|1 - \bar{z}w|,\ 1 \le |w| \le e$; and $\phi(z) = 0,\ |w| \ge e$. Then $|\phi_x(w)| = O(|z-w|^{-1}),\ |w| \le 1; |\phi_x(w)| = O(|1 - \bar{z}w|^{-1}),\ 1 \le |w| \le e$; and the same estimates hold for $|\phi_y|$. Hence $D[\phi] = O\left(\iint_{|w|\le 1}|z-w|^{-2}dA(w)\right) = O\left(\int_{1/n}^1 r^{-1}dr\right) = O(\log n)$, and $\omega_\phi(r) \le r\max_{\mathbb{C}}\sqrt{\phi_x^2 + \phi_y^2} = rO(n)$. Let $r = 1/n^2$ and use (3) to obtain $|\log|P_n(z)| - n\log|z|| = O(\sqrt{n}\log n)$.

References

[1] V. V. Andrievskii and H.-P. Blatt, Discrepancy of Signed Measures and Polynomial Approximation, Springer-Verlag, New York, 2002.

[2] Y. Bilu, Limit distribution of small points on algebraic tori, Duke Math. J. 89 (1997), 465-476.

[3] P. Borwein, Computational Excursions in Analysis and Number Theory, Springer-Verlag, New York, 2002.

[4] P. Erdős and P. Turán, On the distribution of roots of polynomials, Ann. Math. 51 (1950), 105-119.

[5] C. Favre and J. Rivera-Letelier, Equidistribution quantitative des points de petite hauteur sur la droite projective, Math. Ann. 335 (2006), 311-361; Corrigendum in Math. Ann. 339 (2007), 799-801.

[6] L. Kronecker, Zwei Sätze über Gleichungen mit ganzzahligen Coëfficienten, J. reine ang. Math. 53 (1857), 173-175.

[7] N. S. Landkof, Foundations of Modern Potential Theory, Springer-Verlag, New York, 1972.

[8] E. B. Saff and V. Totik, Logarithmic Potentials with External Fields, Springer-Verlag, Berlin, 1997.

[9] I. Schur, Über die Verteilung der Wurzeln bei gewissen algebraischen Gleichungen mit ganzzahligen Koeffizienten, Math. Zeit. 1 (1918), 377-402.