Inequalities for integral norms of polynomials via multipliers

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Dedicated to the memory of Professor Q. I. Rahman

Abstract We consider a wide range of polynomial inequalities for norms defined by the contour or the area integrals over the unit disk. Special attention is devoted to the inequalities obtained by using the Schur-Szegő composition.

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1 The Schur-Szegő composition and polynomial inequalities

We survey and develop a large variety of polynomial inequalities for the integral norms on the unit disk. An especially important tool in this study is the Schur-Szegő composition (or convolution) of polynomials, which is defined via certain coefficient multipliers. In particular, it played prominent role in the development of polynomial inequalities in Hardy spaces. Let $\mathbb{C}_n[z]$ be the set of all polynomials of degree at most n with complex coefficients. Define the standard Hardy space H^p norm for $P_n \in \mathbb{C}_n[z]$ by

$$\|P_n\|_{H^p} = \left(\frac{1}{2\pi} \int_0^{2\pi} |P_n(e^{i\theta})|^p \, d\theta \right)^{1/p}, \quad 0 < p < \infty.$$

It is well known that the supremum norm of the space H^∞ satisfies

$$\|P_n\|_{H^\infty} = \max_{|z|=1} |P_n(z)| = \lim_{p\to\infty} \|P_n\|_{H^p}.$$

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We note the other limiting case [12, p. 139] of the so-called H^0 norm:

$$\|P_n\|_{H^0} = \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \log|P_n(e^{i\theta})|\,d\theta\right) = \lim_{p\to 0+} \|P_n\|_{H^p}.$$

It is also known as the *contour geometric mean* or the *Mahler measure* of a polynomial $P_n \in \mathbb{C}_n[z]$. An application of Jensen's inequality for $P_n(z) = a_n \prod_{j=1}^n (z-z_j) \in \mathbb{C}_n[z]$ immediately gives that

$$\|P_n\|_{H^0} = |a_n| \prod_{j=1}^n \max(|z_j|, 1).$$

The above explicit expression is very convenient, and it is frequently used in our paper and other literature. This direct connection with the roots of P_n explains why the Mahler measure and its close counterpart the Weil height play an important role in number theory, see a survey of Smyth [20].

For a polynomial $\Lambda_n(z) = \sum_{k=0}^n \lambda_k \binom{n}{k} z^k \in \mathbb{C}_n[z]$, we define the *Schur-Szegő composition* with another polynomial $P_n(z) = \sum_{k=0}^n a_k z^k \in \mathbb{C}_n[z]$ by

$$\Lambda P_n(z) := \sum_{k=0}^n \lambda_k a_k z^k. \tag{1}$$

If Λ_n is a fixed polynomial, then ΛP_n is a multiplier (or convolution) operator acting on a space of polynomials P_n. More information on the history and applications of this composition may be found in [6], [1], [2] and [18]. De Bruijn and Springer [6] proved a remarkable inequality stated below.

Theorem 1. *Suppose that $\Lambda_n \in \mathbb{C}_n[z]$ and $P_n \in \mathbb{C}_n[z]$. If $\Lambda P_n \in \mathbb{C}_n[z]$ is defined by* (1)*, then*

$$\|\Lambda P_n\|_{H^0} \le \|\Lambda_n\|_{H^0} \|P_n\|_{H^0}. \tag{2}$$

If $\Lambda_n(z) = (1+z)^n$ then $\Lambda P_n(z) \equiv P_n(z)$ and $\|\Lambda_n\|_{H^0} = 1$, so that (2) turns into equality, showing sharpness of Theorem 1. This result was not sufficiently recognized for a long time. In fact, Mahler [14] proved the following special case of (2) nearly 15 years later by using a more complicated argument.

Corollary 1. $\|P_n'\|_{H^0} \le n\|P_n\|_{H^0}$

We add that equality holds in Corollary 1 if and only if the polynomial P_n has all zeros in the closed unit disk, and present a proof of this fact in Section 3. To see how Theorem 1 implies the above estimate for the derivative, just note that if $\Lambda_n(z) = nz(1+z)^{n-1} = \sum_{k=0}^n k\binom{n}{k} z^k$, then $\Lambda P_n(z) = zP_n'(z)$ and $\|\Lambda_n\|_{H^0} = n$. Furthermore, (2) immediately answers the question about a lower bound for the Mahler measure of derivative raised in [9, pp. 12 and 194]. Following Storozhenko [21], we consider $P_n'(z) = \sum_{k=0}^{n-1} a_k z^k$ and write

$$\frac{1}{z}\left(P_n(z)-P_n(0)\right)=\sum_{k=0}^{n-1}\frac{a_k}{k+1}z^k=\Lambda P_n'(z),$$

where

$$\Lambda_{n-1}(z)=\sum_{k=0}^{n-1}\frac{1}{k+1}\binom{n-1}{k}z^k=\frac{(1+z)^n-1}{nz}.$$

The result of de Bruijn and Springer (2) gives

Corollary 2. *[21] For any $P_n\in\mathbb{C}_n[z]$, we have*

$$\|P_n(z)-P_n(0)\|_{H^0}\le c_n\,\|P_n'\|_{H^0},$$

where

$$c_n:=\frac{1}{n}\|(z+1)^n-1\|_{H^0}=\frac{1}{n}\prod_{n/6<k<5n/6}2\sin\frac{k\pi}{n}.$$

It is easy to see that $c_n\approx(1.4)^n$ as $n\to\infty$. Moreover, equality holds in Corollary 2 for $P_n(z)=(z+1)^n-1$.

Another interesting consequence of (2) is the well known estimate for coefficients (usually attributed to Mahler).

Corollary 3. *If $P_n(z)=\sum_{k=0}^{n}a_kz^k$ then*

$$|a_k|\le\binom{n}{k}\|P_n\|_{H^0},\quad k=0,\ldots,n.$$

The above inequality follows at once from (2) by letting $\Lambda_n(z)=\binom{n}{k}z^k$, $k=1,\ldots,n$, and taking into account that $\|\Lambda P_n\|_{H^0}=\|a_kz^k\|_{H^0}=|a_k|$ and $\|\Lambda_n\|_{H^0}=\binom{n}{k}$.

Many other inequalities may be obtained from Theorem 1, including the one below, found in [16].

Corollary 4. *Let $P_n(z)=\sum_{k=0}^{n}a_kz^k$ and $m=0,\ldots,n$. We have*

$$\left\|\sum_{k\ne m}a_kz^k\right\|_{H^0}\le\left\|(1+z)^n-\binom{n}{m}z^m\right\|_{H^0}\|P_n\|_{H^0}.$$

In particular, if $m=0$ then $\|(1+z)^n-1\|_{H^0}=\prod_{n/6<k<5n/6}2\sin\frac{k\pi}{n}\approx(1.4)^n$ as $n\to\infty$.

Again, the proof is a simple application of (2) with $\Lambda_n(z)=(1+z)^n-\binom{n}{m}z^m$, so that $\lambda_m=0$ and $\lambda_k=1,\ k\ne m$.

An important generalization of Theorem 1 for the H^p norms was obtained by Arestov [1].

Theorem 2. *Suppose that $\Lambda_n \in \mathbb{C}_n[z]$ and $P_n \in \mathbb{C}_n[z]$. If $\Lambda P_n \in \mathbb{C}_n[z]$ is defined by* (1)*, then*

$$\|\Lambda P_n\|_{H^p} \le \|\Lambda_n\|_{H^0} \|P_n\|_{H^p}, \quad 0 \le p \le \infty. \tag{3}$$

In fact, Arestov obtained an even more general inequality, and also described the set of extremal polynomials for it, see [1] for details. One of the main motivations for such a result was the Bernstein inequality for derivative of a polynomial in $H^p,\ p \in (0,1)$.

Corollary 5. *For any $P_n \in \mathbb{C}_n[z]$ we have*

$$\|P_n'\|_{H^p} \le n\|P_n\|_{H^p},\ 0 \le p \le \infty. \tag{4}$$

If $p > 0$ then equality holds in (4) *only for polynomials of the form $P_n(z) = cz^n,\ c \in \mathbb{C}$.*

This inequality was originally proved by Bernstein for $p = \infty$ [5, 15, 18], and generalized to $p \ge 1$ by Zygmund, see [24]. For $p = 0$, (4) reduces to the result of de Bruijn-Springer-Mahler stated in Corollary 1. The case $p \in (0,1)$ remained open for a long time, and was finally settled by Arestov [1]. A more complete history of this result can be found in the book [18] and the recent survey [3].

Lower bounds for the derivative are also of interest. While Theorem 2 immediately gives the analogue of Corollary 2 for H^p (in the same manner as before), the resulting constant c_n of Corollary 2 is no longer sharp. In fact, one can prove much better estimates.

Theorem 3. *If $P_n \in \mathbb{C}_n[z]$ then*

$$\|P_n - P_n(0)\|_{H^\infty} \le \pi \|P_n'\|_{H^1} \tag{5}$$

and

$$\|P_n - P_n(0)\|_{H^\infty} \le \pi n^{1/p-1} \|P_n'\|_{H^p}, \quad 0 < p < 1. \tag{6}$$

The constant π in (5) *cannot be replaced by a smaller number.*

A different application of Theorem 2 gives the solution of the Chebyshev minimization problem in H^p.

Corollary 6. *Any monic polynomial $P_n(z) = z^n + \ldots \in \mathbb{C}_n[z]$ satisfies*

$$\|P_n\|_{H^p} \ge 1, \quad 0 \le p \le \infty. \tag{7}$$

If $p > 0$ then equality holds in (7) *only for the monomial $P_n(z) = z^n$.*

The case of $p = \infty$ in (7) reduces to the classical Chebyshev problem for the unit disk. It is readily seen that for $p = 0$ equality holds in Corollary 6 if and only if P_n has all zeros in the closed unit disk.

Yet another useful application of (3) is the following sharp estimate of the growth for the circular means of polynomials.

Corollary 7. *For any $P_n \in \mathbb{C}_n[z]$ and any $R > 1$, we have*

$$\|P_n(Rz)\|_{H^p} \le R^n \|P_n\|_{H^p}, \quad 0 \le p \le \infty. \tag{8}$$

If $p > 0$ then equality holds in (8) *only for polynomials of the form $P_n(z) = cz^n$, $c \in \mathbb{C}$.*

The above estimate is a special case of the classical Bernstein-Walsh Lemma on the growth of polynomials outside the set, when $p = \infty$.

If we use Theorem 2 to estimate the coefficients of a polynomial as in Corollary 3, then the result is certainly valid, but is not best possible. Given any polynomial $P_n(z) = \sum_{k=0}^n a_k z^k$, we obtain that

$$|a_k| \le \binom{n}{k} \|P_n\|_{H^p}, \quad k = 0, \dots, n, \quad 0 \le p \le \infty.$$

Apart from the cases $k = 0$ and $k = n$, this is far from being precise. In particular, recall the well known elementary (and sharp) estimate:

$$|a_k| \le \|P_n\|_{H^1}, \quad k = 0, \dots, n.$$

Many more interesting estimates for the coefficients of a polynomial may be found in Chapter 16 of [18].

It is useful to have a bound for the regular convolution (or the Hadamard product) of two polynomials, in addition to the Schur-Szegő convolution we mainly consider here. In fact, one version of such an estimate follows directly from Theorem 2, as observed by Tovstolis [22].

Theorem 4. *If $P_n(z) = \sum_{k=0}^n a_k z^k \in \mathbb{C}_n[z]$ and $Q_n(z) = \sum_{k=0}^n b_k z^k \in \mathbb{C}_n[z]$ then we have for $P_n * Q_n(z) = \sum_{k=0}^n a_k b_k z^k$ that*

$$\|P_n * Q_n\|_{H^p} \le \|\Theta_n\|_{H^0} \|P_n\|_{H^0} \|Q_n\|_{H^p}, \quad 0 \le p \le \infty,$$

where

$$\Theta_n(z) = \sum_{k=0}^n \binom{n}{k}^2 z^k \quad \text{and} \quad \lim_{n\to\infty} \|\Theta_n\|_{H^0}^{1/n} \approx 3.20991230072\dots.$$

We conclude this section with a bound for the derivative of a polynomial without zeros in the unit disk that was originally proved by Lax for $p = \infty$, then by de Bruijn for $p \ge 1$, and finally by Rahman and Schmeisser for all $p \ge 0$. See [18, p. 553] for a detailed account.

Theorem 5. *If $P_n \in \mathbb{C}_n[z]$ has no zeros in the unit disk, then*

$$\|P_n'\|_{H^p} \le \frac{n}{\|z+1\|_{H^p}} \|P_n\|_{H^p},\ 0 \le p \le \infty,$$

where

$$\|z+1\|_{H^p} = 2\left(\frac{\Gamma(p/2+1/2)}{\sqrt{\pi}\,\Gamma(p/2+1)}\right)^{1/p}, \quad 0 < p < \infty,$$
$$\|z+1\|_{H^0} = 1 \quad \textit{and} \quad \|z+1\|_{H^\infty} = 2.$$

Note that Theorem 5 is sharp as equality holds for polynomials of the form $P_n(z) = az^n + b$ with $|a| = |b| \neq 0$. Since $\|z+1\|_{H^p} > 1$ for $p > 0$, this result is an improvement over the standard Bernstein inequality stated in Corollary 5. Arestov [2] considered generalizations of Theorem 5 in the spirit of Theorem 2.

2 Polynomial inequalities in Bergman spaces

Polynomial inequalities for Bergman spaces (with norms defined by the area measure) are not developed as well as those for Hardy spaces considered in the previous section. Given a non-negative radial function $w(z) = w(|z|),\ z \in \mathbb{D}$, with $b_w = \iint_{\mathbb{D}} w\,dA > 0$, we define the weighted Bergman space A_w^p norm by setting

$$\|P_n\|_{A_w^p} := \left(\frac{1}{b_w}\iint_{\mathbb{D}} |P_n(z)|^p\, w(z)dA(z)\right)^{1/p}, \quad 0 < p < \infty,$$

where dA is the Lebesgue area measure. If $w \equiv 1$ then we use the standard notation A^p for the regular Bergman space, with $b_w = \pi$. Detailed information on Bergman spaces can be found in the books [8] and [13]. We also define the A_w^0 norm by

$$\|P_n\|_{A_w^0} := \exp\left(\frac{1}{b_w}\iint_{\mathbb{D}} \log|P_n(z)|\, w(z)dA(z)\right).$$

This norm was studied in [16] and [17], and it has the same relation to Bergman spaces as H^0 norm to Hardy spaces:

$$\|P_n\|_{A_w^0} = \lim_{p\to 0+} \|P_n\|_{A_w^p},$$

see [12, p. 139]. If $w \equiv 1$ then the following explicit form for $\|P_n\|_{A^0}$ is found in [16, 17].

Theorem 6. *Let* $P_n(z) = a_n\prod_{j=1}^n (z - z_j) = \sum_{k=0}^n a_k z^k \in \mathbb{C}_n[z]$. *If* P_n *has no roots in* $\mathbb{D}$, *then* $\|P_n\|_{A^0} = |a_0|$. *Otherwise,*

$$\|P_n\|_{A^0} = \|P_n\|_{H^0} \exp\left(\frac{1}{2}\sum_{|z_j|<1}(|z_j|^2-1)\right). \tag{9}$$

We immediately obtain the following comparison result from Theorem 6.

Corollary 8. *For any $P_n \in \mathbb{C}_n[z]$, we have*

$$e^{-n/2}\|P_n\|_{H^0} \le \|P_n\|_{A^0} \le \|P_n\|_{H^0}.$$

Equality holds in the lower estimate if and only if $P_n(z) = cz^n,\ c \in \mathbb{C}$. The upper estimate turns into equality if and only if P_n has no zeros in $\mathbb{D}$.

We state the following generalization of Theorem 2 for the weighted Bergman space.

Theorem 7. *Suppose that $\Lambda_n \in \mathbb{C}_n[z]$ and $P_n \in \mathbb{C}_n[z]$. If $\Lambda P_n \in \mathbb{C}_n[z]$ is defined by* (1)*, then*

$$\|\Lambda P_n\|_{A^p_w} \le \|\Lambda_n\|_{H^0}\|P_n\|_{A^p_w}, \quad 0 \le p \le \infty. \tag{10}$$

Note that equality holds in (10) for any polynomial $P_n \in \mathbb{C}_n[z]$ when $\Lambda_n(z) = (1+z)^n = \sum_{k=0}^n \binom{n}{k} z^k$, because $\Lambda P_n \equiv P_n$ and $\|(1+z)^n\|_{H^0} = 1$. This result allows to treat many problems in a unified way, and it has numerous interesting consequences.

We start with the following version of the Bernstein inequality for derivative of a polynomial in Bergman spaces.

Theorem 8. *For any $P_n \in \mathbb{C}_n[z]$, we have that*

$$\|zP_n'\|_{A^p_w} \le n\|P_n\|_{A^p_w}, \qquad 0 \le p < \infty.$$

If $p > 0$ then equality holds here only for polynomials on the form $P_n(z) = cz^n,\ c \in \mathbb{C}$. The same is true for $p = 0$ provided $0 \in \operatorname{supp} w$.

By writing $0 \in \operatorname{supp} w$ we mean that $\iint_{|z|<\varepsilon} w\,dA > 0$ for all $\varepsilon > 0$, which is the same as $\int_0^\varepsilon w(r)\,dr > 0\ \forall\,\varepsilon > 0$. While the set of extremal polynomials remains the same, note the difference in the left hand side comparing to the classical H^p case. It is clear that the norms of H^∞ and A^∞ coincide, and that Theorem 8 reduces to Corollary 5 in this case.

Continuing in the parallel pattern to the results for H^p spaces, we turn to the lower bounds of the derivative for polynomials in Bergman norms. The approach used in Corollary 2 can be applied to produce a similar inequality for A^p_w (with the same constant c_n). But that inequality is not sharp even for $p = 0$ now, in contrast with Corollary 2. Instead, we follow different approaches to obtain the following estimates for A^p.

Theorem 9. *Any $P_n \in \mathbb{C}_n[z]$ satisfies*

$$\|P_n - P_n(0)\|_{A^\infty} \le \frac{n^{2/p-1}}{(2-p)^{1/p}} \|P_n'\|_{A^p}, \quad 1 \le p < 2, \tag{11}$$

$$\|P_n - P_n(0)\|_{A^\infty} \le \left(\sum_{k=1}^n \frac{1}{k}\right)^{1/2} \|P_n'\|_{A^2} \le \sqrt{1+\log n}\, \|P_n'\|_{A^2}, \quad n \in \mathbb{N}, \tag{12}$$

and

$$\|P_n - P_n(0)\|_{A^\infty} \le \frac{p}{p-2} \|P_n'\|_{A^p}, \quad p > 2. \tag{13}$$

Note that the first inequality in (12) turns into equality for $Q_n(z) = \sum_{k=1}^n z^k/k$, as

$$\|Q_n - Q_n(0)\|_{A^\infty} = \sum_{k=1}^n \frac{1}{k} \quad \text{and} \quad \|Q_n'\|_{A^2} = \left(\sum_{k=1}^n \frac{1}{k}\right)^{1/2}.$$

We also show in the proof of Theorem 9 that the exponent of n in (11) is optimal.

Theorem 7 implies, among many other results, that z^n has the smallest Bergman space norm among all monic polynomials.

Corollary 9. *If $P_n \in \mathbb{C}_n[z]$ is a monic polynomial, then*

$$\|P_n\|_{A^p_w} \ge \|z^n\|_{A^p_w}, \quad 0 \le p < \infty. \tag{14}$$

If $p > 0$ then equality holds above only for $P_n(z) = z^n$. This is also true for $p = 0$ provided $0 \in \operatorname{supp} w$.

For $w \equiv 1$ we have

$$\|z^n\|_{A^p} = \begin{cases} e^{-n/2}, & p = 0, \\ \left(\dfrac{2}{pn+2}\right)^{1/p}, & 0 < p < \infty. \end{cases}$$

Since $\|P_n\|_{A^\infty} = \|P_n\|_{H^\infty}$, the inequality $\|P_n\|_{A^\infty} \ge \|z^n\|_\infty = 1$ is well known, see Corollary 6 and [5, 18].

Another useful estimate compares norms on the concentric disks $D_R := \{z : |z| < R\}$ to that on the unit disk.

Corollary 10. *If $P_n \in \mathbb{C}_n[z]$ and $R \ge 1$, then*

$$\left(\frac{1}{\pi R^2} \iint_{D_R} |P_n(z)|^p \, dA(z)\right)^{1/p} \le R^n \|P_n\|_{A^p}, \qquad p \in (0, \infty),$$

and

$$\exp\left(\frac{1}{\pi R^2}\iint_{D_R}\log|P_n(z)|\,dA(z)\right) \le R^n\,\|P_n\|_{A^0},$$

where equality holds for $P_n(z) = z^n$.

Again, in the case $p = \infty$, it is already known that $\max_{z\in D_R}|P_n(z)| \le R^n\|P_n\|_\infty$ (cf. Corollary 7 and [18]).

Another consequence relates $\|P_n\|_p$ to the coefficients of P_n.

Corollary 11. *If* $P_n(z) = \sum_{k=0}^n a_k z^k \in \mathbb{C}_n[z]$ *then*

$$|a_k| \le \left(\|z^k\|_{A^p_w}\right)^{-1}\binom{n}{k}\|P_n\|_{A^p_w}, \quad k = 0,\ldots,n, \quad 0 \le p < \infty.$$

If $w \equiv 1$ *then we have*

$$|a_k| \le \left(\frac{pk+2}{2}\right)^{1/p}\binom{n}{k}\|P_n\|_{A^p}, \quad k = 0,\ldots,n, \quad 0 < p < \infty,$$

and

$$|a_k| \le e^{k/2}\binom{n}{k}\|P_n\|_{A^0}, \quad k = 0,\ldots,n.$$

If $k = 0$ or $k = n$, then the estimates of Corollary 11 are sharp for the corresponding monomials, but this is not generally so because binomial coefficients grow very fast with n. One can often improve the estimates of Corollary 11 by using the coefficient estimates for general functions from Bergman spaces. For example, the result of Horowitz (cf. [8, p. 81]) states that for any $f(z) = \sum_{k=0}^\infty a_k z^k \in A^p$ we have

$$\left(\sum_{k=0}^{\infty}\frac{|a_k|^q}{(k+1)^{q-1}}\right)^{1/q} \le \|f\|_{A^p}, \quad 1 < p \le 2,\ q = p/(p-1). \tag{15}$$

It is certainly possible to extend the list of corollaries by choosing appropriate coefficient multipliers through the polynomials Λ_n.

A somewhat different kind of inequalities are related to comparing the norms of polynomials in Hardy and Bergman spaces. It is well known [8, 13] that for any function $f \in H^p$ we have

$$\|f\|_{A^p} \le \|f\|_{H^p}, \qquad 0 \le p \le \infty.$$

Clearly, we have equality for $p = \infty$. One can prove inequalities for polynomials in the opposite direction, of the form

$$\|P_n\|_{H^p} \le C(n,p)\,\|P_n\|_{A^p}.$$

For example, we have by Corollary 8 that

$$\|P_n\|_{H^0} \le e^{n/2}\,\|P_n\|_{A^0},$$

where equality holds for $P_n(z) = z^n$.

The case $p = 2$ is easy to handle, because

$$\|P_n\|_{H^2}^2 = \sum_{k=0}^{n} |a_k|^2 \le (n+1)\sum_{k=0}^{n} \frac{|a_k|^2}{k+1} = (n+1)\,\|P_n\|_{A^2}^2,$$

where $P_n(z) = \sum_{k=0}^{n} a_k z^k$. Hence we obtain that

$$\|P_n\|_{H^2} \le \sqrt{n+1}\,\|P_n\|_{A^2}, \quad P_n \in \mathbb{C}_n[z],$$

with equality for $P_n(z) = z^n$. It is plausible that more generally

$$\|P_n\|_{H^p} \le (pn/2+1)^{1/p}\,\|P_n\|_{A^p}, \quad 0 < p < \infty,$$

with equality for $P_n(z) = z^n$.

Estimates for the Bergman space norms of zero-free polynomials in the unit disk are not available to the best of our knowledge. We give a bound for the derivative of a polynomial without zeros in the unit disk that generalizes Theorem 5.

Theorem 10. *If $P_n \in \mathbb{C}_n[z]$ has no zeros in the unit disk, then*

$$\left(\frac{1}{b_w}\int_0^{2\pi}\int_0^1 |P_n'(re^{i\theta})|^p\,\|rz+1\|_{H^p}^p\,w(r)\,r\,dr\,d\theta\right)^{1/p} \le n\,\|P_n\|_{A_w^p},\ 0 < p < \infty,$$

where $b_w = \iint_{\mathbb{D}} w\,dA$. In particular, we have

$$\|P_n'\|_{A_w^p} \le n\,\|P_n\|_{A_w^p}, \quad 0 \le p \le \infty.$$

It is a peculiar fact that the original form of the Bernstein inequality holds for the zero-free polynomials in this case. However, the above estimates are not sharp, see the proof of Theorem 10.

3 Proofs

We prove all new results, and also selected known results where reasonably concise proofs can be given. In particular, the proofs of Theorems 1 and 2 are not included, and may be respectively found in the original papers of de Bruijn and Springer [6], and of Arestov [1]. An alternative exposition of methods that include a proof of Theorem 2 is contained in Section 13.2 of [18]. Proofs of Corollaries 1-4 are already outlined in Section 1. We start with characterization of all extremal polynomials in Corollary 1 by the location of their zeros in the closed unit disk. We are not aware of this observation made previously in the literature.

Proof of Corollary 1. We present an alternative proof of this corollary, independent of Theorem 1, that gives a description of all polynomials achieving equality. Consider any $P_n(z) = a_n \prod_{k=1}^n (z - z_k) \in \mathbb{C}_n[z],\ a_n \neq 0$, and note that the inequality of Corollary 1 is equivalent to the following

$$\log \frac{\|P_n'\|_{H^0}}{\|P_n\|_{H^0}} = \frac{1}{2\pi} \int_0^{2\pi} \log \frac{|P_n'(e^{i\theta})|}{|P_n(e^{i\theta})|}\, d\theta \le \log n.$$

On the other hand, we have that

$$\frac{1}{2\pi} \int_0^{2\pi} \log \left| \frac{P_n'(e^{i\theta})}{P_n(e^{i\theta})} \right| d\theta = \frac{1}{2\pi} \int_0^{2\pi} \log \left| \sum_{k=1}^n \frac{1}{e^{i\theta} - z_k} \right| d\theta.$$

Denote the above expression by $u(z_1, \dots, z_n)$, and observe that it is a continuous function of the roots $z_k \in \mathbb{C}$. Moreover, u is subharmonic in each $z_k \in \mathbb{D}, k = 1, \dots, n$, by Theorem 2.4.8 of [19, p. 38]. It is also subharmonic in each variable in $\Omega := \{z \in \mathbb{C} : |z| > 1\}$. Applying the maximum principle for u with respect to every variable z_k in the domains $\mathbb{D}$ and Ω, we obtain that the largest value of u is attained for a polynomial $Q_n(z) = b_n z^n + \dots$ with all roots $w_k, k = 1, \dots, n$, located on the unit circumference. But we can explicitly find that $\|Q_n\|_{H^0} = |b_n|$ for such an extremal polynomial. Since all zeros of Q_n' are contained in the closed unit disk by the Gauss-Lukas Theorem, we also find that $\|Q_n'\|_{H^0} = n|b_n|$. Thus the largest value of u is $\log n$ for all n-tuples of points $\{z_k\}_{k=1}^n$, i.e., for all polynomials P_n. Furthermore, the same argument gives that $\|P_n\|_{H^0} = |a_n|$ and $\|P_n'\|_{H^0} = n|a_n|$ for any polynomial P_n with all zeros in the closed unit disk, so that equality holds in Corollary 1 as claimed. If P_n has a zero in Ω, then we have a strict inequality. Indeed, assume to the contrary that $z_n \in \Omega$ and $u(z_1 \dots, z_n) = \log n$. Since u is subharmonic and achieves maximum in Ω, it must be constant with respect to $z_n \in \Omega$. Letting $z_n \to \infty$ (and keeping other roots fixed), we now have that

$$\begin{aligned}
\log n = \lim_{z_n \to \infty} u(z_1, \dots, z_n) &= \frac{1}{2\pi} \int_0^{2\pi} \log \left| \sum_{k=1}^{n-1} \frac{1}{e^{i\theta} - z_k} \right| d\theta \\
&= \frac{1}{2\pi} \int_0^{2\pi} \log \left| \frac{R_{n-1}'(e^{i\theta})}{R_{n-1}(e^{i\theta})} \right| d\theta,
\end{aligned}$$

where $R_{n-1}(z) = \prod_{k=1}^{n-1} (z - z_k)$ is of degree $n-1$. This is in contradiction with the already proved inequality

$$\log \frac{\|R_{n-1}'\|_{H^0}}{\|R_{n-1}\|_{H^0}} = \frac{1}{2\pi} \int_0^{2\pi} \log \left| \frac{R_{n-1}'(e^{i\theta})}{R_{n-1}(e^{i\theta})} \right| d\theta \le \log(n-1).$$

□

Proof of Corollary 5. Inequality (4) is obtained from (3) by using the polynomial

$$\Lambda_n(z) = nz(1+z)^{n-1} = \sum_{k=0}^{n} k \binom{n}{k} z^k.$$

Indeed, given any polynomial $P_n(z) = \sum_{k=0}^n a_k z^k \in \mathbb{C}_n[z]$, we obtain from the definition of the Schur-Szegő composition in (1) that

$$\Lambda P_n(z) = \sum_{k=0}^{n} k a_k z^k = z P_n'(z).$$

Furthermore, it is immediate that $\|\Lambda_n\|_{H^0} = n$, so that (4) follows.

The uniqueness part for $0 < p < \infty$ is a consequence of Theorem 5 from [1], because the coefficients of $\Lambda_n(z) = \sum_{k=0}^n \lambda_k z^k$ satisfy $\lambda_n = n > \lambda_0 = 0$, and the function $\phi(u) = u^p$ clearly satisfies that $u\phi'(u)$ is strictly increasing. The case of $p = \infty$ is classical. Uniqueness is also explicitly discussed in Theorem 6 of [10]. $\square$

Proof of Theorem 3. Let f be analytic in $\mathbb{D}$, with the derivative f' in the Hardy space H^1, and apply the Fejér-Riesz inequality, see [7, p. 46]. For any $r \in [0,1]$ and $\theta \in [0, 2\pi)$, we obtain that

$$\begin{aligned} |f(re^{i\theta}) - f(0)| &= \left| \int_0^r f'(te^{i\theta}) e^{i\theta}\, dt \right| \le \int_0^r |f'(te^{i\theta})|\, dt \\ &\le \int_{-1}^1 |f'(te^{i\theta})|\, dt \le \frac{1}{2} \int_0^{2\pi} |f'(e^{i(\theta+\phi)})|\, d\phi = \pi \|f'\|_{H^1}. \end{aligned}$$

It follows that

$$\|f - f(0)\|_{H^\infty} \le \pi \|f'\|_{H^1},$$

which contains (5) as $P_n' \in H^1$ for any polynomial $P_n \in \mathbb{C}_n[z]$.

We now prove that the constant π in the above inequality and in (5) is sharp. Consider the conformal mapping ψ of the unit disk $\mathbb{D}$ onto the rectangle $R := (-\varepsilon, 1) \times (-\varepsilon, \varepsilon),\ \varepsilon > 0$, that satisfies $\psi(0) = 0$ and $\psi'(0) > 0$. It is easy to see that $\|\psi - \psi(0)\|_{H^\infty} = \|\psi\|_{H^\infty} = \sqrt{1+\varepsilon^2}$. Moreover, the perimeter of R is expressed as

$$2 + 6\varepsilon = \int_0^{2\pi} |\psi'(e^{i\theta})|\, d\theta = 2\pi \|\psi'\|_{H^1}.$$

Hence we have that

$$\lim_{\varepsilon \to 0} \frac{\|\psi - \psi(0)\|_{H^\infty}}{\pi \|\psi'\|_{H^1}} = \lim_{\varepsilon \to 0} \frac{\sqrt{1+\varepsilon^2}}{1 + 3\varepsilon} = 1,$$

which shows asymptotic sharpness for $f = \psi$ as $\varepsilon \to 0$. On the other hand, polynomials are dense in H^1, and there is a sequence of polynomials $Q_n \in \mathbb{C}_n[z]$ such that $\|\psi' - Q_n'\|_{H^1} \to 0$ as $n \to \infty$. The Fejér-Riesz inequality again gives

$$\|\psi - Q_n - (\psi(0) - Q_n(0))\|_{H^\infty} \le \pi \|\psi' - Q_n'\|_{H^1} \to 0 \quad \text{as } n \to \infty.$$

Thus $\lim_{n\to\infty}\|Q_n'\|_{H^1} = \|\psi'\|_{H^1}$ and $\lim_{n\to\infty}\|Q_n - Q_n(0)\|_{H^\infty} = \|\psi - \psi(0)\|_{H^\infty}$, so that (5) is asymptotically sharp for Q_n as $n \to \infty$.

We obtain (6) from (5) with the help of the Nikolskii-type inequality [18, p. 463]:

$$\|P_n'\|_{H^1} \le \left((n-1)\lceil p/2\rceil + 1\right)^{1/p-1}\|P_n'\|_{H^p} = n^{1/p-1}\|P_n'\|_{H^p}, \quad 0 < p < 1,$$

where $\lceil\cdot\rceil$ is the standard ceiling function. □

Proof of Corollary 6. Let $\Lambda_n(z) = z^n$, so that for any monic polynomial $P_n \in \mathbb{C}_n[z]$ we have the Schur-Szegő composition $\Lambda P_n(z) = z^n$. Since $\|\Lambda_n\|_{H^0} = 1$, (7) follows from (3).

The uniqueness part for $0 < p < \infty$ follows from Theorem 5 of [1], because the coefficients of Λ_n satisfy $\lambda_n = 1 > \lambda_0 = 0$, and the function $\phi(u) = u^p$ satisfies that $u\phi'(u)$ is strictly increasing. If $p = \infty$ then uniqueness of the extremal polynomial in (7) is the content of Tonelli's theorem [23, p. 72]. □

Proof of Corollary 7. For $\Lambda_n(z) = (Rz+1)^n = \sum_{k=0}^n \binom{k}{n} R^k z^k$ and $P_n(z) = \sum_{k=0}^n a_k z^k \in \mathbb{C}_n[z]$, we have that

$$\Lambda P_n(z) = \sum_{k=0}^n a_k R^k z^k = P_n(Rz).$$

Note that $\|\Lambda_n\|_{H^0} = R^n$, because the only root of Λ_n is in $\mathbb{D}$. Thus (8) follows from (3). The case of equality is again a consequence of Theorem 5 of [1], as $\lambda_n = R^n > \lambda_0 = 1$. □

Proof of Theorem 4. We apply Theorem 2 and the definition of the Schur-Szegő composition to obtain that

$$\|P_n * Q_n\|_{H^p} = \left\|\sum_{k=0}^n a_k b_k z^k\right\|_{H^p} \le \left\|\sum_{k=0}^n \binom{n}{k} a_k z^k\right\|_{H^0} \|Q_n\|_{H^p}, \quad 0 \le p \le \infty.$$

Using Theorem 1 for the first factor on the right (or Theorem 2 again), we have

$$\left\|\sum_{k=0}^n \binom{n}{k} a_k z^k\right\|_{H^0} \le \left\|\sum_{k=0}^n \binom{n}{k}^2 z^k\right\|_{H^0} \left\|\sum_{k=0}^n a_k z^k\right\|_{H^0} = \|\Theta_n\|_{H^0}\|P_n\|_{H^0}.$$

The asymptotic value

$$\lim_{n\to\infty} \|\Theta_n\|_{H^0}^{1/n} \approx 3.20991230072\ldots$$

is found from the product of zeros of Θ_n outside the unit disk, see [22] for details and more precise asymptotic results. □

A proof of Theorem 5 may be found in [18, pp. 554-555].

Proof of Theorem 6. If P_n does not vanish in $\mathbb{D}$, then $\log|P_n(z)|$ is harmonic in $\mathbb{D}$. Hence $\|P_n\|_{H^0} = |a_0|$ and $\|P_n\|_{A^0} = |a_0|$ follow from the contour and the area mean value theorems respectively. Assume now that P_n has zeros in $\mathbb{D}$. Applying Jensen's formula, we obtain that

$$\log \|P_n\|_{H^0} = \frac{1}{2\pi}\int_0^{2\pi} \log|P_n(e^{i\theta})|\,d\theta = \log|a_n| + \sum_{|z_j|\geq 1} \log|z_j|.$$

Furthermore,

$$\begin{aligned}
\log \|P_n\|_{A^0} &= \frac{1}{\pi}\int_0^1 \int_0^{2\pi} \log|P_n(re^{i\theta})|\,r dr d\theta \\
&= 2\int_0^1 \left(\frac{1}{2\pi}\int_0^{2\pi} \log|P_n(re^{i\theta})|\,d\theta\right) r dr \\
&= 2\int_0^1 \left(\log|a_n| + \sum_{|z_j|\geq r} \log|z_j| + \sum_{|z_j|<r} \log r\right) r dr \\
&= \log|a_n| + \sum_{|z_j|\geq 1} \log|z_j| + \frac{1}{2}\sum_{|z_j|<1} (|z_j|^2 - 1).
\end{aligned}$$

Hence

$$\|P_n\|_{A^0} = \|P_n\|_{H^0} \exp\left(\frac{1}{2}\sum_{|z_j|<1} (|z_j|^2-1)\right).$$

□

Proof of Corollary 8. The lower bound for $\|P_n\|_{A^0}$ follows from (9) because the smallest value of the sum

$$\sum_{|z_j|<1} (|z_j|^2-1)$$

is equal to $-n$, which is achieved if and only if all $z_j = 0$. The largest value of this sum is clearly 0 iff all $|z_j| \geq 1$, giving us the upper bound. □

Proof of Theorem 7. Appling (2) to the polynomial $P_n(rz),\ r \in [0,1]$, instead of $P_n(z)$, we obtain that

$$\int_0^{2\pi} \log|\Lambda P_n(re^{i\theta})|\,d\theta \leq 2\pi \log\|\Lambda_n\|_{H^0} + \int_0^{2\pi} \log|P_n(re^{i\theta})|\,d\theta.$$

Next we integrate the above inequality with respect to $w(r)r\,dr$ from 0 to 1:

$$\begin{aligned}
\int_0^{2\pi}\int_0^1 \log|\Lambda P_n(re^{i\theta})|\,w(r)r\,dr d\theta &\leq \log\|\Lambda_n\|_{H^0}\, 2\pi\int_0^1 w(r)r\,dr \\
&\quad + \int_0^{2\pi}\int_0^1 \log|P_n(re^{i\theta})|\; w(r)r\,dr d\theta.
\end{aligned}$$

Dividing by $b_w = 2\pi\int_0^1 w(r)r\,dr$ and taking exponential, we prove (10) for $p = 0$. Similarly, we obtain from (3) that

$$\int_0^{2\pi} |\Lambda P_n(re^{i\theta})|^p \, d\theta \le \|\Lambda_n\|_{H^0}^p \int_0^{2\pi} |P_n(re^{i\theta})|^p \, d\theta, \quad 0 < p < \infty,$$

which implies that

$$\int_0^{2\pi}\int_0^1 |\Lambda P_n(re^{i\theta})|^p \, w(r) r \, dr d\theta \le \|\Lambda_n\|_{H^0}^p \int_0^{2\pi}\int_0^1 |P_n(re^{i\theta})|^p \, w(r) r \, dr d\theta.$$

Dividing by b_w and taking the power $1/p$, we now have (10) for $p > 0$. If $p = \infty$ then (10) follows from (3) again:

$$\begin{aligned}\|\Lambda P_n\|_{A_w^\infty} &= \sup_{\substack{0\le\theta<2\pi \\ 0\le r<1}} |\Lambda P_n(re^{i\theta})| w(r) \le \sup_{\substack{0\le\theta<2\pi \\ 0\le r<1}} \|\Lambda_n\|_{H^0} \max_{0\le\theta<2\pi} |P_n(re^{i\theta})| \, w(r) \\ &= \|\Lambda_n\|_{H^0} \|P_n\|_{A_w^\infty}.\end{aligned}$$

□

Proof of Theorem 8. Observe that the derivative of P_n can be expressed in the form of the Schur-Szegő convolution as in the proof of Corollary 5:

$$zP_n'(z) = \Lambda P_n(z) \quad \text{with} \quad \Lambda_n(z) = nz(1+z)^{n-1} = \sum_{k=0}^n k \binom{n}{k} z^k.$$

Since $\|\Lambda_n\|_{H^0} = n$, the inequality of Theorem 8 follows from (10).

Turning to the case of equality in Theorem 8, we first let $p > 0$. We assume that

$$\int_0^{2\pi}\int_0^1 |rP_n'(re^{i\theta})|^p \, w(r) r \, dr d\theta = n^p \int_0^{2\pi}\int_0^1 |P_n(re^{i\theta})|^p \, w(r) r \, dr d\theta$$

holds for a polynomial P_n. Note that Corollary 5 applied to the polynomial $P_n(rz), r > 0$, gives that

$$\int_0^{2\pi} |rP_n'(re^{i\theta})|^p \, d\theta \le n^p \int_0^{2\pi} |P_n(re^{i\theta})|^p \, d\theta.$$

Since we have equality for the area integrals over $\mathbb{D}$, we must also have equality in the latter inequality for almost every $r \in \operatorname{supp} w$. But this is only possible when $P_n(z) = cz^n,\ c \in \mathbb{C}$, by Corollary 5.

For $p = 0$, we argue in a similar fashion to show that

$$\int_0^{2\pi} \log |rP_n'(re^{i\theta})| \, d\theta = 2\pi \log n + \int_0^{2\pi} \log |P_n(re^{i\theta})| \, d\theta$$

holds for almost every $r \in \operatorname{supp} w$, provided we have equality in Theorem 8. It follows that the family of polynomials $Q_n(z) = P_n(rz)$ is extremal in Corollary 1 for all such r. Hence $P_n(rz)$ has all zeros in the closure of $\mathbb{D}$, while $P_n(z)$ has all zeros in $\{z \in \mathbb{C} : |z| \le r\}$. Since this holds for a sequence of radii $r \to 0$ such that $r \in \operatorname{supp} w$, we conclude that all zeros of P_n are at the origin. □

Proof of Theorem 9. We start with the case $1 < p \le 2$. Let $P_n(z) = \sum_{k=0}^n a_k z^k$, so that $P_n'(z) = \sum_{k=1}^n k a_k z^{k-1}$. Applying Theorem 2 of [8, p. 81] (also see (15)), we obtain that

$$\|P_n'\|_{A^p} \ge \left(\sum_{k=1}^n \frac{k^q |a_k|^q}{k^{q-1}} \right)^{1/q} = \left(\sum_{k=1}^n k|a_k|^q \right)^{1/q}, \quad 1 < p \le 2,\ q = p/(p-1).$$

Using this inequality together with Hölder's inequality, we estimate

$$\begin{aligned} \|P_n - P_n(0)\|_{A^\infty} &\le \sum_{k=1}^n k^{1/q}|a_k| k^{-1/q} \le \left(\sum_{k=1}^n k|a_k|^q \right)^{1/q} \left(\sum_{k=1}^n k^{-p/q} \right)^{1/p} \\ &\le \|P_n'\|_{A^p} \left(\sum_{k=1}^n k^{1-p} \right)^{1/p} \le \|P_n'\|_{A^p} \left(1 + \int_1^n x^{1-p} \right)^{1/p}. \end{aligned}$$

Evaluating the latter integral, we arrive at (11) and (12). The case $p = 1$ in (11) is obtained by letting $p \to 1+$.

We now show that the exponent of n in (11) is sharp. Consider the polynomial

$$Q_{2n-1}(z) = \int_0^z \left(\sum_{k=1}^n k t^{k-1} \right)^2 dt, \quad \deg(Q_{2n-1}) = 2n-1.$$

The second part of Theorem 2 in [8, p. 81] states a reverse inequality to (15) for $p \ge 2$. Although $p \in (1,2]$ in our case, we use this fact for $2p \in (2,4]$ and $r = 2p/(2p-1)$ to estimate that

$$\begin{aligned} \|Q_{2n-1}'\|_{A^p}^p &= \left\| \left(\sum_{k=0}^n k z^{k-1} \right)^2 \right\|_{A^p}^p = \left\| \sum_{k=0}^n k z^{k-1} \right\|_{A^{2p}}^{2p} \le \left(\sum_{k=1}^n \frac{k^r}{k^{r-1}} \right)^{2p/r} \\ &= \left(\sum_{k=1}^n k \right)^{2p-1} = \left(\frac{n(n+1)}{2} \right)^{2p-1}. \end{aligned}$$

Hence the right hand side of (11) for Q_{2n-1} is of the order $O(n^3)$. Note that both Q_{2n-1} and its derivative have positive coefficients. This immediately implies that

$$\|Q_{2n-1}\|_{A^\infty} = Q_{2n-1}(1) = \int_0^1 \left(\sum_{k=1}^n k t^{k-1} \right)^2 dt.$$

Given any polynomial $P_m(z) = \sum_{k=0}^m a_k z^k$ of degree m with positive coefficients, we have that

$$\int_0^1 P_m(x)\, dx = \sum_{k=0}^m \frac{a_k}{k+1} \ge \frac{P_m(1)}{m+1}.$$

The latter inequality applied to Q'_{2n-1} gives that

$$\|Q_{2n-1}\|_{A^\infty} = \int_0^1 \left(\sum_{k=1}^n kt^{k-1}\right)^2 dt \geq \frac{1}{2n-1}\left(\sum_{k=1}^n k\right)^2 = \frac{1}{2n-1}\left(\sum_{k=1}^n \frac{n(n+1)}{2}\right)^2.$$

Hence the left hand side of (11) for Q_{2n-1} grows like n^3 as $n \to \infty$, matching the right hand side.

Turning to the case $p > 2$, we apply the area submean inequality for a subharmonic function $|P'_n(z)|^p$ on the disk $\{t \in \mathbb{C} : |t-z| < 1-|z|\}$ contained in $\mathbb{D}$ for any $z \in \mathbb{D}$:

$$|P'_n(z)|^p \leq \frac{1}{\pi(1-|z|)^2}\int_{\{|t-z|<1-|z|\}} |P'_n(t)|^p\, dA(t) \leq \frac{\|P'_n\|_{A^p}^p}{(1-|z|)^2}, \quad z \in \mathbb{D}.$$

Hence (13) follows from

$$|P_n(e^{i\theta}) - P_n(0)| \leq \int_0^1 |P'_n(re^{i\theta})|\, dr \leq \|P'_n\|_{A^p} \int_0^1 (1-r)^{-2/p}\, dr = \frac{p}{p-2}\|P'_n\|_{A^p}.$$

□

Proof of Corollary 9. Consider any monic polynomial $P_n(z) = z^n + \ldots$ and the multiplier polynomial $\Lambda_n(z) = z^n$. Then the Schur-Szegő convolution is given by $\Lambda P_n(z) = z^n$. Hence (14) follows from (10). Equality holds trivially for $P_n(z) = z^n$, and we now show that this is the only extremal polynomial. Assume first that $p > 0$. Equality in (14) is equivalent to

$$\int_0^{2\pi}\int_0^1 |P_n(re^{i\theta})|^p\, w(r) r\, dr d\theta = \int_0^{2\pi}\int_0^1 r^{np}\, w(r) r\, dr d\theta$$

holding for a monic polynomial P_n. Corollary 6 gives by a scaling change of variable that

$$\frac{1}{2\pi}\int_0^{2\pi} |P_n(re^{i\theta})|^p\, d\theta \geq r^{np}, \quad 0 \leq r \leq 1,$$

for any monic polynomial $P_n \in \mathbb{C}_n[z]$, with equality only for $P_n(z) = z^n$. Hence equality in (14) implies that equality must hold in the above inequality for a.e. $r \in \operatorname{supp} w$, which means that $P_n(z) = z^n$ by Corollary 6. The case $p = 0$ is handled similarly. It is immediate to see that

$$\frac{1}{2\pi}\int_0^{2\pi} \log|P_n(re^{i\theta})|\, d\theta \geq \log r^n, \quad 0 \leq r \leq 1,$$

for any monic polynomial $P_n \in \mathbb{C}_n[z]$, with equality only if all zeros of $P_n(rz)$ are in the closed unit disk. Equality in (14) for $p = 0$ can be written as

$$\int_0^{2\pi}\int_0^1 \log|P_n(re^{i\theta})|\, w(r) r\, dr d\theta = \int_0^{2\pi}\int_0^1 \log r^n\, w(r) r\, dr d\theta$$

for a monic polynomial P_n, which implies that

$$\frac{1}{2\pi}\int_0^{2\pi}\log|P_n(re^{i\theta})|\,d\theta=\log r^n$$

for almost every $r\in\operatorname{supp}w$. Thus $P_n(rz)$ has all zeros in the closure of $\mathbb{D}$, and $P_n(z)$ has all zeros in $\{z\in\mathbb{C}:|z|\le r\}$ for a sequence of radii $r\to 0$ such that $r\in\operatorname{supp}w$. It follows that $P_n(z)=z^n$.

The values of $\|z^n\|_{A^p}$ given in this corollary are found by a routine computation. □

Proof of Corollary 10. Let $\Lambda_n(z)=(1+Rz)^n=\sum_{k=0}^n\binom{n}{k}R^kz^k$. Then $\Lambda P_n(z)=P_n(Rz)$ and $\|\Lambda_n\|_{H^0}=R^n$. Hence (10) gives that

$$\|P_n(Rz)\|_{A^p}\le R^n\|P_n\|_{A^p},\quad 0\le p<\infty,$$

for any $R\ge 1$. Changing variable and passing to the integral over D_R, we obtain that

$$\|P_n(Rz)\|_{A^p}=\left(\frac{1}{\pi R^2}\iint_{D_R}|P_n(z)|^p\,dA(z)\right)^{1/p},\quad 0\le p<\infty,$$

and

$$\|P_n(Rz)\|_{A^0}=\exp\left(\frac{1}{\pi R^2}\iint_{D_R}\log|P_n(z)|\,dA(z)\right),\quad p=0.$$

The case of equality for $P_n(z)=z^n$ is verified by the same substitution. □

Proof of Corollary 11. Let $\Lambda_n(z)=\binom{n}{k}z^k,\ 0\le k\le n$. Then $\Lambda P_n(z)=a_kz^k$ and $\|\Lambda_n\|_{H^0}=\binom{n}{k}$. It follows from (10) that

$$|a_k|\|z^k\|_{A^p_w}=\|a_kz^k\|_{A^p_w}\le\binom{n}{k}\|P_n\|_{A^p_w},\quad 0\le p\le\infty.$$

If $w\equiv 1$ then we can use explicit values of $\|z^k\|_{A^p_w}$ as given in Corollary 9 to obtain the last two inequalities of Corollary 11. □

Proof of Theorem 10. We recall the following estimate for a polynomial P_n without zeros in the disk $\{z\in\mathbb{C}:|z|<R\},\ R\ge 1$:

$$\|P_n'\|_{H^p}\le\frac{n}{\|z+R\|_{H^p}}\,\|P_n\|_{H^p},\ 0\le p\le\infty.$$

This extension of Theorem 5 was originally proved by Govil and Rahman [11] for $p\ge 1$, and later by Aziz and Shah [4] for any $p>0$. While equality may hold in Theorem 5 as explained after its statement, the above inequality cannot turn into equality for any P_n without zeros in the disk $\{z\in\mathbb{C}:|z|<R\}$, $R>1$. The cases $p=0$ and $p=\infty$ follow immediately by taking limits as $p\to 0$ and $p\to\infty$. We apply the stated result to the family of polynomials $P_n(rz),\ r\in(0,1]$. It is clear that if P_n is zero-free in $\mathbb{D}$, then $P_n(rz)$ has no zeros in the disk $\{z\in\mathbb{C}:|z|<1/r\},\ r\in(0,1]$. Hence

$$\int_0^{2\pi} |rP_n'(re^{i\theta})|^p \, d\theta \le \frac{n^p}{\|z+1/r\|_{H^p}^p} \int_0^{2\pi} |P_n(re^{i\theta})|^p \, d\theta, \ 0 < p < \infty.$$

Simplifying, we obtain that

$$\|rz+1\|_{H^p}^p \int_0^{2\pi} |P_n'(re^{i\theta})|^p \, d\theta \le n^p \int_0^{2\pi} |P_n(re^{i\theta})|^p \, d\theta, \ 0 < p < \infty.$$

We now integrate the above inequality with respect to $w(r)r\,dr$ from 0 to 1:

$$\int_0^{2\pi}\int_0^1 |P_n'(re^{i\theta})|^p \, \|rz+1\|_{H^p}^p \, w(r)r\,dr d\theta \le n^p \int_0^{2\pi}\int_0^1 |P_n(re^{i\theta})|^p \, w(r)r\,dr d\theta.$$

Thus the first inequality follows for $p \in (0,\infty)$. It remains to observe that $\|rz+1\|_{H^p}^p \ge 1$ by the submean inequality for the subharmonic function $|rz+1|^p$, so that the second inequality is a consequence of the first one for $p \in (0,\infty)$. The endpoints are handled by the standard limits as $p \to 0$ and $p \to \infty$. $\square$

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