CHEBYSHEV POLYNOMIALS WITH INTEGER COEFFICIENTS *

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Dedicated to Professor R. S. Varga on the occasion of his seventieth birthday.

Abstract. We study the asymptotic structure of polynomials with integer coefficients and smallest uniform norms on an interval of the real line. Introducing methods of the weighted potential theory into this problem, we improve the bounds for the multiplicities of some factors of the integer Chebyshev polynomials.

1. Introduction

Let $\mathcal{P}_n(\mathbf{C})$ and $\mathcal{P}_n(\mathbf{Z})$ be the sets of algebraic polynomials of degree at most n, respectively with complex and with integer coefficients. Define the uniform norm on the interval $[a,b] \subset \mathbf{R}$ by

$$\|f\|_{[a,b]} := \max_{x\in[a,b]} |f(x)|.$$

It is very well known that the *Chebyshev polynomial*

$$T_n(x) := 2^{1-n}\cos(n \arccos x)$$

is a monic polynomial of degree n, which minimizes the uniform norm on $[-1,1]$ in the class of all *monic* polynomials from $\mathcal{P}_n(\mathbf{C})$ (see [2], [12] and

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[15]). The case of an arbitrary interval $[a, b] \subset \mathbf{R}$ can be reduced to that of $[-1, 1]$ by a change of variable. Thus we immediately obtain that

$$t_n(x) := \left(\frac{b-a}{2}\right)^n T_n\left(\frac{2x-a-b}{b-a}\right)$$

is a monic polynomial with the smallest uniform norm on $[a, b]$ among all *monic* polynomials from $\mathcal{P}_n(\mathbf{C})$. Clearly,

$$||t_n||_{[a,b]} = 2\left(\frac{b-a}{4}\right)^n, \qquad n \in \mathbf{N}, \tag{1.1}$$

and the *Chebyshev constant* for $[a, b]$ is given by

$$cheb([a, b]) := \lim_{n\to\infty} ||t_n||_{[a,b]}^{1/n} = \frac{b-a}{4}. \tag{1.2}$$

Chebyshev polynomials and Chebyshev constant represent very classical topics in analysis. These ideas have applications in many areas of mathematics, see [2], [12] and [15]. We remark that the Chebyshev constant of a compact set in $\mathbf{C}$ is equal to its transfinite diameter and to its logarithmic capacity (cf. [18, pp. 71-75] for the general definitions and a discussion).

A corresponding minimization problem in the class of polynomials with integer coefficients $\mathcal{P}_n(\mathbf{Z})$ also has a long and interesting history, surveyed in [14, Ch. 10] and [3]. An *integer Chebyshev polynomial* $q_n \in \mathcal{P}_n(\mathbf{Z})$ is defined in this case as follows:

$$||q_n||_{[a,b]} = \inf_{0 \not\equiv p_n \in \mathcal{P}_n(\mathbf{Z})} ||p_n||_{[a,b]}, \tag{1.3}$$

where the inf in (1.3) is taken over all polynomials from $\mathcal{P}_n(\mathbf{Z})$, which are not identically zero. Further, one can define the *integer Chebyshev constant* (integer transfinite diameter) for $[a, b]$ similarly to (1.2)

$$inch([a, b]) := \lim_{n\to\infty} ||q_n||_{[a,b]}^{1/n}. \tag{1.4}$$

It is not difficult to see that the above limit exists (cf. [14, Ch. 10] or [3]). Observe from (1.1) and (1.3) that if $b-a \geq 4$ then $q_n(x) \equiv 1$ for any $n \in \mathbf{N}$ and $inch([a,b]) = 1$. However, if $b-a < 4$ then

$$\frac{b-a}{4} = cheb([a,b]) \leq inch([a,b]). \tag{1.5}$$

On the other hand, the results of Hilbert [10] and Fekete [5] imply that

$$inch([a,b]) \leq \sqrt{\frac{b-a}{4}} \tag{1.6}$$

(see [3]). The exact value of the integer Chebyshev constant and an explicit (or even asymptotic) form of the integer Chebyshev polynomials is not known for any $[a,b]$ with $b-a<4$. Perhaps the most studied case, due to the interest in the distribution of prime numbers, is the case of $[0,1]$ (cf. [14, Ch. 10], [3] and [4]). The best known bounds for $inch([0,1])$ are as follows:

$$0.42072638 < inch([0,1]) \leq 0.42347945. \tag{1.7}$$

The lower bound in (1.7) is obtained with the help of the Gorshkov-Wirsing polynomials (see [14, Ch. 10]). It was believed to be the precise value of $inch([0,1])$, but Borwein and Erdélyi [3] recently showed that there must be a *strict* inequality on the left of (1.7). The upper bound can be found from the very definition of the integer Chebyshev constant in (1.3)-(1.4), using various optimization techniques. Although this is by no means straightforward, both theoretically and practically, this nevertheless becomes more accessible for computations with growing power of modern computers. Thus, the upper bound in (1.7) has recently been improved several times (cf. [3], [6], [7] and [9]). The value in (1.7) is taken from [9], and to our knowledge is the best computed upper bound.

2. Asymptotic structure of the integer Chebyshev polynomials

We are interested in the asymptotic structure of the polynomials $Q_n \in \mathcal{P}_n(\mathbf{Z})$ satisfying

$$\|Q_n\|_{[0,1]} = \inf_{0 \not\equiv p_n \in \mathcal{P}_n(\mathbf{Z})} \|p_n\|_{[0,1]}, \quad n \in \mathbf{N}. \tag{2.1}$$

This problem was originally proposed by A. O. Gelfond (cf. [1]). It is known that the polynomials Q_n satisfying (2.1) have factors that tend to repeat and to increase in power as $n \to \infty$ (see [14, Ch. 10] and [3] for a discussion). In particular, Aparicio (cf. Theorem 3 in [1]) showed that if $\{Q_n\}_{n=1}^{\infty} \subset \mathcal{P}_n(\mathbf{Z})$ satisfy (2.1), then

$$Q_n(x) = (x(1-x))^{[\alpha_1 n]}(2x-1)^{[\alpha_2 n]}(5x^2-5x+1)^{[\alpha_3 n]} R_n(x), \quad \text{as } n \to \infty, \tag{2.2}$$

where

$$\alpha_1 \geq 0.1456, \quad \alpha_2 \geq 0.0166 \quad \text{and} \quad \alpha_3 \geq 0.0037, \tag{2.3}$$

and $R_n \in \mathcal{P}_n(\mathbf{Z})$, $n \in \mathbf{N}$. Borwein and Erdélyi proved that

$$\alpha_1 \geq 0.26 \tag{2.4}$$

in (2.2) (see Theorem 3.1 of [3]). Flammang, Rhin and Smyth [8] recently generalized the ideas of [3] and obtained the following lower bounds

$$\alpha_1 \geq 0.264151, \quad \alpha_2 \geq 0.021963 \quad \text{and} \quad \alpha_3 \geq 0.005285. \tag{2.5}$$

They also considered six additional factors of $Q_n(x)$ and studied other intervals (cf. [8] for the details).

We use the methods of the weighted potential theory, developed during the last two decades, to study the integer Chebyshev problem and to improve the bounds for α_1 and α_2. A complete account on the weighted potential theory is contained in [16].

Theorem 2.1. *Let* $\{Q_n\}_{n=1}^{\infty}$ *be a sequence of polynomials with integer coefficients satisfying* (2.1)*. Then* (2.2) *holds with*

$$0.2961 \le \alpha_1 \le 0.3634 \quad \textit{and} \quad 0.0952 \le \alpha_2 \le 0.1767. \tag{2.6}$$

Furthermore, the pair (α_1, α_2) *must belong to the region* G *pictured below in* Figure 1*, which is determined by* (3.25)*,* (3.26) *and* (3.28)*.*

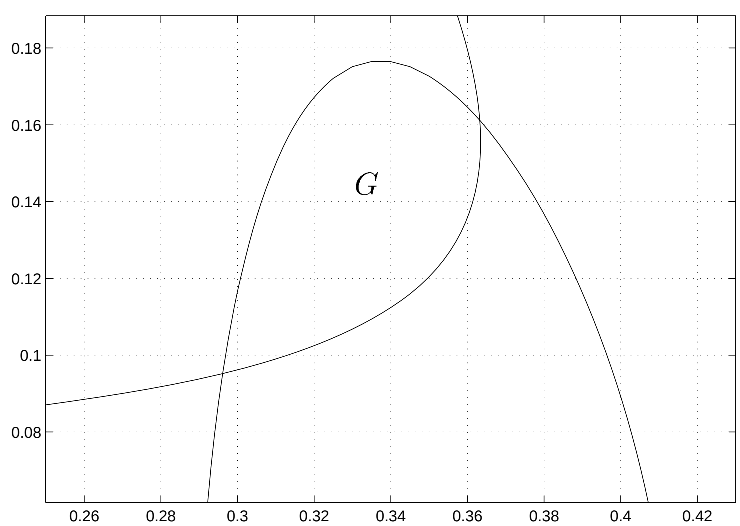

Figure 1: Region G for α_1 and α_2.

Note that (2.6) also gives the upper bounds for α_1 and α_2. Moreover, we believe that the methods introduced here can produce bounds for α_3 and for the multiplicities of other factors of the integer Chebyshev polynomials (see [3], [8] and [9] for lists of such factors). The proof of Theorem 2.1 is presented in Section 3, after the development of necessary techniques.

3. Weighted polynomials and weighted potentials

Using the idea of symmetry and the change of variable $x(1-x) \to x$, one can see that

$$(inch([0,1]))^2 = inch([0,1/4]). \tag{3.1}$$

Furthermore, we have by Lemmas 1-2 of [9] that

$$Q_{2k}(x) = q_k(x(1-x)) \tag{3.2}$$

and

$$Q_{2k+1}(x) = (1-2x)q_k(x(1-x)), \tag{3.3}$$

where

$$\lim_{k\to\infty} \|q_k\|_{[0,1/4]}^{1/k} = inch([0,1/4]). \tag{3.4}$$

Hence we can study the sequences $\{q_n\}_{n=0}^{\infty} \subset \mathcal{P}_n(\mathbf{Z})$ satisfying (3.4) instead of considering the original sequence $\{Q_n\}_{n=0}^{\infty}$ satisfying (2.1), which is more convenient for technical reasons. Note that the factors $x(1-x)$ and $(1-2x)^2$ for $Q_{2k}(x)$ are transformed into the factors x and $(1-4x)$ for $q_k(x)$, under the change of variable $x(1-x) \to x$. On writing

$$q_n(x) = x^{k_1(n)}(1-4x)^{k_2(n)} r_{n-k_1(n)-k_2(n)}(x), \quad n \in \mathbf{N}, \tag{3.5}$$

where $r_{n-k_1(n)-k_2(n)}(0) \neq 0$ and $r_{n-k_1(n)-k_2(n)}(1/4) \neq 0$, we can assume that the following limits exist:

$$\lim_{n\to\infty} \frac{k_1(n)}{n} =: u \quad \text{and} \quad \lim_{n\to\infty} \frac{k_2(n)}{n} =: v \tag{3.6}$$

(passing to subsequences, if necessary). It follows that in the study of the n-th root behavior of (3.5) one may equivalently consider

$$\left(x^{\frac{u}{1-u-v}}(1-4x)^{\frac{v}{1-u-v}}\right)^{n-k_1(n)-k_2(n)} r_{n-k_1(n)-k_2(n)}(x), \quad x \in [0,1/4], \tag{3.7}$$

as $n \to \infty$, where (3.7) is the so-called weighted polynomial with varying weight $(w(x))^{n-k_1(n)-k_2(n)}$ and where

$$w(x) := x^{\frac{u}{1-u-v}}(1-4x)^{\frac{v}{1-u-v}}, \quad x \in [0,1/4] \tag{3.8}$$

(see [16]). Clearly, the uniform norms of both (3.5) and (3.7) on $[0,1/4]$ cannot be attained at the endpoints. Furthermore, if $u>0$ and $v>0$ then these uniform norms "live" on an interval $[a,b] \subset (0,1/4)$. This type of problem for the Jacobi weights (see Example IV.1.17 in [16, p. 206]) has first been considered in [13], [11] and [17], where the sharp values for a and b were found. We follow the modern and general approach to the problem via the weighted potential theory, described in [16].

For $[a,b] \subset \mathbf{R}$, let $\Omega := \overline{\mathbf{C}} \setminus [a,b]$ and let $g_\Omega(z,p)$ be the Green function of Ω with pole at $p \in \Omega$ (cf. [18, p. 14]). Consider the following natural extension for $w(x)$ of (3.8):

$$w(z) := |z|^{\frac{u}{1-u-v}}|1-4z|^{\frac{v}{1-u-v}}, \quad z \in \mathbf{C}, \tag{3.9}$$

where $u \geq 0$, $v \geq 0$ and $u+v \leq 1$.

Lemma 3.1. *Suppose that $P_n \in \mathcal{P}_n(\mathbf{C})$ and w is defined by (3.9). Then there exists an interval $[a,b] \subset [0,1/4]$ with*

$$a := (u^2 - v^2 - \sqrt{\Delta} + 1)/8 \text{ and } b := (u^2 - v^2 + \sqrt{\Delta} + 1)/8, \tag{3.10}$$

where $\Delta := (1-(u+v)^2)(1-(u-v)^2)$, and a continuous in $\mathbf{C}$ and harmonic in $\mathbf{C} \setminus [a,b]$ function

$$\begin{aligned} h(z) \quad := \quad & (g_\Omega(z,\infty) - u(\log|z| + g_\Omega(z,0)) \\ - \quad & v(\log|4z-1| + g_\Omega(z,1/4)))/(1-u-v), \end{aligned} \tag{3.11}$$

such that

$$|P_n(z)| \le \|w^n P_n\|_{[a,b]} e^{nh(z)}, \qquad z \in \mathbf{C}. \tag{3.12}$$

Moreover,

$$\|w^n P_n(z)\|_{[0,1/4]} = \|w^n P_n\|_{[a,b]}. \tag{3.13}$$

Proof. It follows from Theorem III.2.1 of [16] that

$$|P_n(z)| \le \|w^n P_n\|_{S_w} \exp(n(F_w - U^{\mu_w}(z))), \quad z \in \mathbf{C}, \tag{3.14}$$

where μ_w is a positive unit Borel measure with the support $S_w \subset [0,1/4]$, which is the solution of the weighted energy problem for the weight w of (3.9) on $[0,1/4]$, considered in Section I.1 of [16]. Here, $U^{\mu_w}(z)$ is the logarithmic potential of μ_w

$$U^{\mu_w}(z) := \int \log \frac{1}{|z-t|} d\mu_w(t), \tag{3.15}$$

and F_w is the *modified Robin constant* for w. Note that the weight w of (3.9) is just a special case of the Jacobi weights of Example IV.1.17 in [16]. Thus our problem on the interval $[0,1/4]$ is easily reduced to that on the interval $[-1,1]$ considered there, with the help of the change of variable $x \to (x+1)/8$. We obtain from Example IV.1.17 of [16] that $S_w = [a,b]$, with a and b given by (3.10) (see (1.27) and (1.28) in [16, p. 207]). Equation (3.13) now follows from Corollary III.2.6 of [16]. Also, Theorem I.1.3 of [16] yields that

$$F_w - U^{\mu_w}(z) = -\log w(z), \tag{3.16}$$

for quasi every $z \in [a,b]$ (i.e., with the exception of a set of zero capacity). Hence $F_w - U^{\mu_w}(z) - h(z)$ is a harmonic function in Ω such that

$$F_w - U^{\mu_w}(z) - h(z) = 0$$

for quasi every $z \in [a, b] = \partial\Omega$ by (3.11), (3.9), (3.16) and the basic properties of Green functions (see [18, p. 14]). Using the uniqueness theorem for the solution of the Dirichlet problem in Ω (cf. Theorem III.28 and its Corollary in [18]), we conclude that

$$F_w - U^{\mu_w}(z) \equiv h(z), \quad z \in \mathbf{C}.$$

Thus (3.12) follows from (3.14). ∎

Proof of Theorem 2.1. Suppose that $\{q_n\}_{n=0}^{\infty}$ is a sequence of polynomials with integer coefficients, satisfying (3.2)-(3.4). We also assume that (3.5)-(3.6) hold for this sequence, as before. It follows from (3.6) that

$$\lim_{n\to\infty} |z|^{\frac{k_1(n)}{n-k_1(n)-k_2(n)}} |1-4z|^{\frac{k_2(n)}{n-k_1(n)-k_2(n)}} = w(z), \tag{3.17}$$

where $w(z)$ is given by (3.9) and where the above convergence is uniform on compact subsets of $\mathbf{C}$. Since $r_{n-k_1(n)-k_2(n)}(0) \neq 0$ by (3.5), we obtain from (3.12) that

$$\begin{aligned} 1 &\leq |r_{n-k_1(n)-k_2(n)}(0)| \\ &\leq \|w^{n-k_1(n)-k_2(n)} r_{n-k_2(n)-k_2(n)}\|_{[a,b]} e^{(n-k_1(n)-k_2(n))h(0)} \\ &\leq \|w^{n-k_1(n)-k_2(n)} r_{n-k_1(n)-k_2(n)}\|_{[0,1/4]} e^{(n-k_1(n)-k_2(n))h(0)}. \end{aligned}$$

Extracting the n-th root in the above inequality, passing to the limit as $n \to \infty$ and, using (3.4)-(3.6) and (3.17), we arrive at

$$1 \leq inch([0,1/4]) e^{(1-u-v)h(0)}. \tag{3.18}$$

A similar argument applied in the case $z = 1/4$ gives that

$$\left(\frac{1}{4}\right)^{n-k_1(n)-k_2(n)} \leq \left| r_{n-k_1(n)-k_2(n)}\left(\frac{1}{4}\right)\right|$$

and that

$$4^{u+v-1} \leq inch([0,1/4])e^{(1-u-v)h(1/4)}. \tag{3.19}$$

The Green functions in the definition of $h(z)$ in (3.11) can be found explicitly, by using the conformal mappings of Ω onto the exterior of the unit disk $D' := \{w : |w| > 1\}$. Indeed, introducing these conformal mappings by

$$\Phi_\infty(z) := \frac{2z - a - b + 2\sqrt{(z-a)(z-b)}}{b-a}, \quad z \in \Omega, \tag{3.20}$$

$$\Phi_0(z) := \frac{2z^{-1} - b^{-1} - a^{-1} + 2\sqrt{(z^{-1}-b^{-1})(z^{-1}-a^{-1})}}{b^{-1}-a^{-1}}, \quad z \in \Omega, \tag{3.21}$$

and

$$\Phi_{1/4}(z) := \frac{2(z-1/4)^{-1} - (b-1/4)^{-1} - (a-1/4)^{-1}}{(b-1/4)^{-1} - (a-1/4)^{-1}} +$$
$$\frac{2\sqrt{((z-1/4)^{-1} - (a-1/4)^{-1})((z-1/4)^{-1} - (b-1/4)^{-1})}}{(b-1/4)^{-1} - (a-1/4)^{-1}}, \quad z \in \Omega, \tag{3.22}$$

we observe that

$$\Phi_\infty(\infty) = \infty, \quad \Phi_0(0) = \infty \quad \text{and} \quad \Phi_{1/4}(1/4) = \infty. \tag{3.23}$$

Hence

$$\begin{aligned} g_\Omega(z,\infty) = \log|\Phi_\infty(z)|, \quad g_\Omega(z,0) = \log|\Phi_0(z)| \\ \text{and} \quad g_\Omega(z,1/4) = \log|\Phi_{1/4}(z)|, \quad z \in \Omega, \end{aligned} \tag{3.24}$$

by Theorem I.17 of [18, p. 18]. Taking (3.24) into account, we rewrite (3.18) as

$$1 \leq inch([0,1/4])|\Phi_\infty(0)| \left(\lim_{z\to 0} |z\Phi_0(z)|\right)^{-u} |\Phi_{1/4}(0)|^{-v} \tag{3.25}$$

and (3.19) as

$$4^{2v-1} \leq inch([0,1/4])|\Phi_\infty(1/4)||\Phi_0(1/4)|^{-u} \left(\lim_{z \to 1/4} |z-1/4||\Phi_{1/4}(z)| \right)^{-v}. \tag{3.26}$$

Thus $u \in [0,1]$ and $v \in [0,1]$ must satisfy the inequalities (3.25) and (3.26). Applying (3.1) and the upper bound of (1.7) in (3.25)-(3.26), we obtain the region H of Figure 2 below and the following bounds for u and v:

$$0.5923 \leq u \leq 0.7268 \quad \text{and} \quad 0.0952 \leq v \leq 0.1767. \tag{3.27}$$

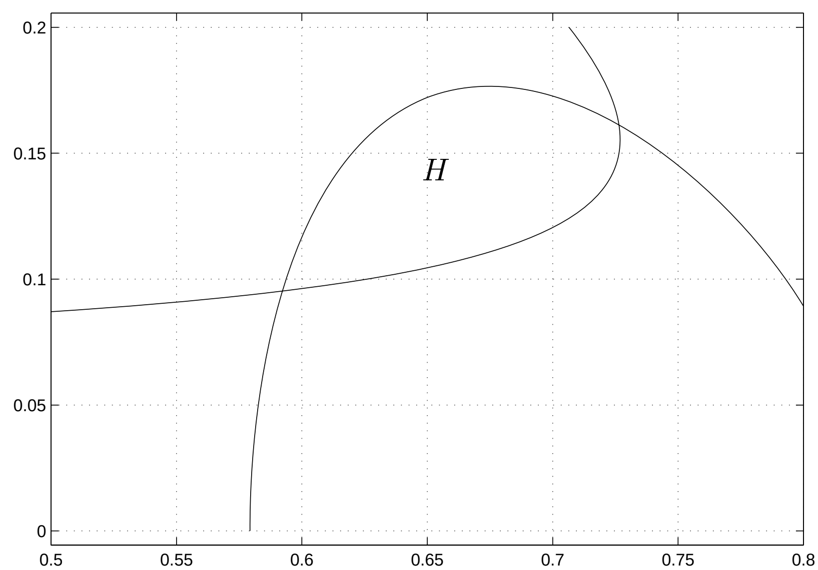

Figure 2: Region H for u and v.

Using (3.2) - (3.4), we conclude that (2.2) holds with

$$\alpha_1 = u/2 \quad \text{and} \quad \alpha_2 = v, \tag{3.28}$$

so that Theorem 2.1 follows from the results for u and v. ∎

Remark. As a consequence of the above proof and (3.13), we have that

$$inch([0,1/4]) = inch([A,B]),$$

where

$$A := \inf_{(u,v)\in H} a(u,v) \approx 0.089 \quad \text{and} \quad B := \sup_{(u,v)\in H} b(u,v) \approx 0.247,$$

with $a(u,v)$ and $b(u,v)$ as in (3.10).

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