Weighted Approximation on Compact Sets

Igor E. Pritsker

Abstract. For a compact set E with connected complement in the complex plane, we consider a problem of the uniform approximation on E by the weighted polynomials $W^n(z)P_n(z)$, where $W(z)$ is a continuous non-vanishing weight function on E, analytic in the interior of E. Let $A(E,W)$ be the set of functions uniformly approximable on E by such weighted polynomials. If E has empty interior, then $A(E,W)$ is completely characterized by a zero set $Z_W \subset E$, where all functions from $A(E,W)$ must vanish. This generalizes recent results of Totik and Kuijlaars for the real line case. However, if E is a closure of Jordan domain, the description of $A(E,W)$ also involves an inner function. In both cases, we exhibit the role of the support of a certain extremal measure, which is the solution of a weighted logarithmic energy problem, played in the descriptions of $A(E,W)$.

§1. Introduction

Let E be a compact set in the complex plane $\mathbf{C}$ with the connected complement $\overline{\mathbf{C}}\backslash E$. We denote the uniform algebra of functions which are continuous on E and analytic in the interior of E by $A(E)$ (see, e.g., [4, p. 25]). Clearly, the corresponding uniform norm for any $f \in A(E)$ is defined by

$$\|f\|_E := \max_{z\in E} |f(z)|. \tag{1.1}$$

Consider a weight function $W \in A(E)$ such that $W(z) \neq 0$ for any $z \in E$, and define the *weighted polynomials* $W^n(z)P_n(z)$, where $P_n(z)$ is an algebraic polynomial in z with complex coefficients, $\deg P_n \le n$. We are interested in a description of the function set $A(E,W)$, consisting of the *uniform limits on* E of sequences of the weighted polynomials $\{W^n(z)P_n(z)\}_{n=0}^{\infty}$, as $n \to \infty$. It is well known that if $W(z) \equiv 1$ on E then $A(E,1) = A(E)$ by Mergelyan's theorem [4, p. 48]. In general, we have that $A(E,W) \subset A(E)$.

Our problem originated in the work of Lorentz [10] on incomplete polynomials on the real line. Surveys of results in this area, dealing with weighted

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approximation on the real line, can be found in [16] and [14, Ch. VI]. The most recent developments are in [6]-[8].

The questions of density of the weighted polynomials in the set of analytic functions in a domain have been considered in [3], [11] and [12]. In particular, [12] contains a necessary and sufficient condition such that any function analytic in a bounded open set is uniformly approximable by the weighted polynomials $W^n(z)P_n(z)$ on *compact subsets.* However, the description of $A(E,W)$ seems to be much more complicated, in that no general necessary and sufficient condition is known (in terms of the weight $W(z)$), even for the real interval case, i.e., for $E=[a,b]\subset \mathbf{R}$.

We shall approach the above mentioned problems on $A(E,W)$, using ideas of the theories of uniform algebras and of weighted potentials.

§2. $A(E,W)$ as a Closed Ideal and Weighted Potentials

Proposition 2.1. *$A(E,W)$, endowed with norm (1.1), is a closed function algebra (not necessarily containing constants and separating points).*

We have already remarked that $A(E,W)\subset A(E)$. To make this inclusion more precise, let us introduce the algebra $[W(z),zW(z)]$ generated by the two functions $W(z)$ and $zW(z)$, which is the uniform closure of all polynomials in $W(z)$ and $zW(z)$ (with constant terms included) on E. Clearly, $[W(z),zW(z)]\subset A(E)$. Since any weighted polynomial $W^n(z)P_n(z)$ is an element of $[W(z),zW(z)]$, then $A(E,W)\subset[W(z),zW(z)]$. Thus, we arrive at the following

Proposition 2.2. $A(E,W)\subset[W(z),zW(z)]\subset A(E)$.

Proposition 2.3. *$A(E,W)$ is a closed ideal of $[W(z),zW(z)]$.*

It turns out that in many cases $[W(z),zW(z)]=A(E)$, so that $A(E,W)$ becomes a closed ideal of $A(E)$ by Proposition 2.3.

Proposition 2.4. $[W(z),zW(z)]=A(E)$ *iff* $1/W(z)\in[W(z),zW(z)]$.

Unfortunately, we do not know any effectively verifiable necessary and sufficient condition on the weight $W(z)$, so that the equality $[W(z),zW(z)]=A(E)$ is valid. Nevertheless, a number of sufficient conditions can be given, guaranteeing that the two algebras $[W(z),zW(z)]$ and $A(E)$ coincide.

Proposition 2.5. *Each of the following conditions implies that $[W(z),\ zW(z)]=A(E)$:*

(a) The point $\zeta=0$ belongs to the unbounded component of $\overline{\mathbf{C}}\backslash W(E)$;

(b) E is the closure of a Jordan domain or a Jordan arc, and $W(z)$ is one-to-one on E;

(c) E is a Jordan arc and $W(z)$ is of bounded variation on E;

(d) E is a Jordan arc and $W(z)$ is locally one-to-one on E;

(e) $E=\overline{G}$, where G is a Jordan domain bounded by an analytic curve, and $W'(z)\in A(\overline{G})$.

Assuming that E has positive logarithmic capacity, then

$$w(z) := \begin{cases} |W(z)|, & z \in E, \\ 0, & z \notin E, \end{cases} \tag{2.1}$$

is an *admissible weight* for the *weighted logarithmic energy problem* on E considered in Section I.1 of [14]. This enables us to use certain results of [14], which we summarize below for the convenience of the reader. Recall that the logarithmic potential of a compactly supported Borel measure μ is given by

$$U^{\mu}(z) := \int \log \frac{1}{|z-t|} d\mu(t). \tag{2.2}$$

Proposition 2.6. *There exists a positive unit Borel measure μ_w, with support $S_w :=$ supp $\mu_w \subset \partial E$, such that for any polynomial $P_n(z), \deg P_n \le n$, we have*

$$|W^n(z)P_n(z)| \le \|W^nP_n\|_{S_w} \exp(n(F_w - U^{\mu_w}(z) + \log|W(z)|)), \tag{2.3}$$

where $z \in E$ and where F_w is a constant. Furthermore, the inequality

$$U^{\mu_w}(z) - \log|W(z)| \ge F_w \tag{2.4}$$

holds quasi-everywhere on E, and

$$U^{\mu_w}(z) - \log|W(z)| \le F_w, \text{ for any } z \in S_w. \tag{2.5}$$

By saying quasi-everywhere (q.e.), we mean that a property holds everywhere, with the exception of a set of zero logarithmic capacity. The measure μ_w is the solution of a weighted energy problem, corresponding to the weight $w(z)$ of (2.1) (see Section I.1 of [14]). It follows from (2.3) and (2.4) that the norm of a weighted polynomial W^nP_n essentially "lives" on S_w. In particular, the following is valid (see Corollary III.2.6 of [14]).

Proposition 2.7. *Suppose that for every point $z_0 \in E$, the set $\{z : |z-z_0| < \delta,\ z \in E\}$ has positive capacity for any $\delta > 0$. Then*

$$\|W^nP_n\|_E = \|W^nP_n\|_{S_w} \tag{2.6}$$

for any polynomial $P_n, \deg P_n \le n$.

§3. Sets with Empty Interior

Let E be a compact set with connected complement and empty interior. Obviously, $A(E) = C(E)$ in this case. We characterize $A(E,W)$ in terms of a certain zero set.

Theorem 3.1. *Suppose that E has a connected complement and an empty interior, and that $W \in C(E)$ is a nonvanishing weight on E. Assume that $[W(z), zW(z)] = C(E)$. Then, there exists a closed set $Z_W \subset E$ such that $f \in A(E,W)$ if and only if $f \in C(E)$ and $f|_{Z_W} \equiv 0$.*

It is clear that $A(E,W) = C(E)$ if and only if the set Z_W is empty. This is true, for example, for $W(z) \equiv 1$ on E.

Theorem 3.1 generalizes a recent result of Kuijlaars (see Theorem 3 of [8]), related to polynomial approximation with varying weights on the real line. However, it has a new part even in the latter case, allowing us to consider the *complex valued weights* $W(z)$ on subsets of the real line.

A description of the set Z_W in terms of the weight $W(z)$ is unknown in general. We can only show that Z_W must contain the complement of S_w (see Proposition 2.6) in E.

Theorem 3.2. *Let E be an arbitrary compact set with the connected complement $\overline{\mathbf{C}}\setminus E$ and let $W \in A(E)$ be a nonvanishing weight on E. Suppose that for every point $z_0 \in E$, the set $\{z : |z - z_0| < \delta,\ z \in E\}$ has positive logarithmic capacity for any $\delta > 0$. Assume further that $\overline{\mathbf{C}}\setminus S_w$ is connected and $[W(z), zW(z)] = C(S_w)$ on S_w. If $f \in A(E,W)$, then $f(z) = 0$ for any $z \in \overline{E\setminus S_w}$. In particular, if E has empty interior, then $\overline{E\setminus S_w} \subset Z_W$.*

The proof of Theorem 3.2 is based on an idea of Kuijlaars (see Theorem 2 and its proof in [8]).

If E is a compact subset of the real line and the weight $W(z)$ is real valued, then condition (a) of Proposition 2.5 is clearly satisfied, so that $[W(z), zW(z)] = C(E)$. Therefore, the conclusion of Theorem 3.1 is valid, and coincides with that of Theorem 3 of [8]. Furthermore, if for any point in E, the intersection of its arbitrary neighborhood with E has positive logarithmic capacity, then $\overline{E\setminus S_w} \subset Z_W$. Since $[W(z), zW(z)] = C(S_w)$ on S_w by Proposition 2.5(a), Theorem 3.2 essentially reduces to Theorem 2 of [8] in this case, which in turn contains an earlier result of Theorem 4.1 of [16].

§4. Unit Disk and Jordan Domains

The first result of this section is a consequence of the well-known description of closed ideals of $A(\overline{D})$, where D is the unit disk, due to Beurling (unpublished) and Rudin [13] (see also [5, pp. 82-87] for a discussion). Recall that g is an *inner function* if it is analytic in D, with $\|g\|_{\overline{D}} \le 1$, and $|g(e^{i\theta})| = 1$ almost everywhere on the unit circle (cf. [5, p. 62]). By the factorization theorem, every inner function can be uniquely expressed in the form

$$g(z) = B(z)S(z), \quad z \in D, \tag{4.1}$$

where $B(z)$ is a Blaschke product and $S(z)$ is a *singular function*, i.e.,

$$S(z) := \exp\left(-\int \frac{e^{i\theta}+z}{e^{i\theta}-z}\, d\nu_s(\theta)\right), \quad z \in D, \tag{4.2}$$

with ν_s being a positive measure on the unit circle, singular with respect to $d\theta$ (see [5, pp. 63-67]).

Theorem 4.1. *Let a nonvanishing weight $W \in A(\overline{D})$ be such that $[W(z),\ zW(z)] = A(\overline{D})$. Assume that $A(\overline{D}, W)$ contains a function not identically zero.*

Then there exist a closed set $H_W \subset \partial D$ of Lebesgue measure zero and an inner function g_W satisfying

(i) *every accumulation point of the zeros of its Blaschke product is in H_W,*

(ii) *the measure ν_s of its singular function is supported on H_W;*

such that

$$f \in A(\overline{D}, W) \text{ if and only if } f = g_W h, \text{ where } h \in A(\overline{D}) \text{ and } h|_{H_W \equiv 0}.$$

The case of a Jordan domain G can be reduced to that of the unit disk, using a canonical conformal mapping $\phi : G \to D$ and its inverse $\psi := \phi^{-1}$.

Our next goal is to exhibit the role of the set S_w (see Proposition 2.6) in the case of weighted approximation on Jordan domains. Since $W \in A(\overline{G})$ is analytic in G, then $S_w \subset \partial G$ by Theorem IV.1.10(a) of [14] and (2.1). The following result shows that $S_w = \partial G$ is necessary for nontrivial weighted approximation on $\overline{G}$.

Theorem 4.2. *Let G be a Jordan domain and let $W \in A(\overline{G})$ be a nonvanishing weight. Assume that S_w is a proper subset of ∂G and that $[W(z),\ zW(z)] = C(S_w)$ on S_w. Then $A(\overline{G}, W)$ contains the identically zero function only.*

§5. Proofs

Proof of Proposition 2.1: We have to show that $A(E, W)$ is closed under addition, multiplication by constants and by functions of $A(E, W)$, and under uniform limits. Suppose that $W^n P_n \to f \in A(E, W)$ and $W^n Q_n \to g \in A(E, W)$ uniformly on E, as $n \to \infty$. Then $W^n(P_n + Q_n) \to (f + g)$, as $n \to \infty$, so that $(f + g) \in A(E, W)$. If $\alpha \in \mathbf{C}$ then $W^n \alpha P_n \to \alpha f$, as $n \to \infty$, and $\alpha f \in A(E, W)$. Observe that

$$\|fg - W^{2n} P_n Q_n\|_E \le \|fg - fW^n Q_n\|_E \ + \ \|fW^n Q_n - W^{2n} P_n Q_n\|_E \le$$

$$\|f\|_E \ \|g - W^n Q_n\|_E + \|W^n Q_n\|_E \ \|f - W^n P_n\|_E \to 0,$$

as $n \to \infty$, i.e., $fg \in A(E, W)$. Applying the standard diagonalization argument, we see that $A(E, W)$ is closed in norm (1.1). $\square$

Proof of Proposition 2.3: Assume that $f \in A(E,W)$ and $W^n P_n \to f$ uniformly on E, as $n \to \infty$. Then, for any pair of nonnegative integers k and ℓ such that $k \geq \ell$, we have

$$\|f(z)W^k(z)z^\ell - W^{n+k}(z)z^\ell P_n(z)\|_E \leq$$

$$\|W^k(z)z^\ell\|_E \, \|f - W^n P_n\|_E \to 0, \qquad \text{as } n \to \infty,$$

which gives that $f(z)W^k(z)z^\ell \in A(E,W)$. Since $A(E,W)$ is closed under addition and multiplication by constants (by Proposition 2.1), the product of f and any polynomial in $W(z)$ and $zW(z)$ belongs to $A(E,W)$. Thus, if $g \in [W(z),\ zW(z)]$ then $fg \in A(E,W)$ follows immediately, because $A(E,W)$ is closed in the uniform norm on E (cf. Proposition 2.1). The proof is now complete in view of Propositions 2.1 and 2.2. $\square$

Proof of Proposition 2.4: Obviously, if $[W(z), zW(z)] = A(E)$ then $1/W(z) \in A(E) = [W(z), zW(z)]$.

Assume that $1/W(z) \in [W(z), zW(z)]$. It follows that $z \in [W(z), zW(z)]$ and, consequently, every polynomial in z is in $[W(z),\ zW(z)]$. Since $[W(z), zW(z)]$ is uniformly closed on E by definition, then $A(E) \subset [W(z), zW(z)]$ by Mergelyan's theorem [4, p. 48]. Thus, Proposition 2.2 implies that $A(E) = [W(z), zW(z)]$. $\square$

Proof of Proposition 2.5: First, we remark that $W(z)$ and $zW(z)$ together separate points of any set E.

(a) Observe that $W(E)$, the image of E in ζ-plane under the mapping $\zeta = W(z)$, is compact. By assumption, function $1/\zeta$ is analytic on the polynomially convex hull of $W(E)$ and can be uniformly approximated there by polynomials in ζ (by Mergelyan's theorem). Returning to z-plane, we obtain that $1/W(z)$ is uniformly approximable on E by polynomials in $W(z)$. It follows that $[W(z), zW(z)] = A(E)$ by Proposition 2.4.

(b) The mapping $\zeta = W(z)$ can be extended to a homeomorphism between z-plane and ζ-plane (cf. [9, p. 535]). Since $W(z)$ doesn't vanish on E, $\zeta = 0$ belongs to the domain $\overline{\mathbf{C}} \setminus W(E) = W(\overline{\mathbf{C}} \setminus E)$, which contains $\zeta = \infty$. Hence, (b) follows from (a).

(c) If $E = [0,1]$ then (c) is a direct consequence of Theorem 2 of [2]. For E being a Jordan arc, we consider a homeomorphic parametrization of E by $\tau : [0,1] \to E$. Since $W \circ \tau(x)$ is of bounded variation on $[0,1]$, we have, as before, that $[W \circ \tau(x), \tau(x)(W \circ \tau)(x)] = C([0,1])$. Clearly, τ induces an isometric isomorphism between $C([0,1])$ and $C(E)$. Thus, the result follows after returning to E with the help of τ^{-1}.

(d) is implied by Theorem 1 of [1] for $E = [0,1]$. The case of a Jordan arc can be reduced to that of the interval as in the proof of (c).

(e) First, assume that $E = \overline{D}$. Then (e) follows at once from [17, p. 135]. It is well known that the conformal mapping $\phi : G \to D$ extends as a diffeomorphism between $\overline{G}$ and $\overline{D}$ (with nonvanishing derivatives of ϕ and $\psi := \phi^{-1}$), because G is bounded by an analytic Jordan curve. Using ϕ, the result for $E = \overline{G}$ is a consequence of [17, p. 135], too. $\square$

Proof of Proposition 2.6: Since $W(z)$ is a continuous nonvanishing function on E and $w(z)$ of (2.1) is so too, the existence of μ_w and inequalities (2.4)-(2.5) follow from Theorem I.1.3 of [14]. Moreover, $W(z)$ is analytic in the interior of E, which implies that $S_w \subset \partial E$ by Theorem IV.1.10(a) of [14] and (2.1). The inequality (2.3) is a direct consequence of Theorem III.2.1 of [14]. $\square$

Proof of Theorem 3.1: We have that $[W(z), zW(z)] = C(E)$ by the assumption of the theorem. Thus, $A(E, W)$ is a closed ideal of $C(E)$ (cf. Proposition 2.3), which is known to be described by its zero set (see [15, p. 32]). $\square$

Proof of Theorem 3.2: We essentially follow the proof of Theorem 2 of [8]. Suppose that there exist $f_0 \in A(E, W)$ and $z_0 \in E \backslash S_w$ such that $f_0(z_0) \neq 0$ and $W^n P_n \to f_0$ uniformly on E, as $n \to \infty$.

It is clear that $f_0|_{S_w} \in A(S_w, W)$. Recall that $S_w \subset \partial E$ by Proposition 2.6, i.e., S_w has empty interior. Applying Theorem 3.1, with E replaced by S_w, we obtain that $A(S_w, W)$ is described by the zero set $Z_W^* \subset S_w$. Observe that multiplying $A(S_w, W)$ by $(z - z_0)W(z)$, we obtain a closed ideal of $[W(z), zW(z)] = C(S_w)$ (cf. Proposition 2.3), which consists of all functions, uniformly approximable on S_w by the weighted polynomials $W^n(z)Q_n(z)$ such that $Q_n(z_0) = 0$, as $n \to \infty$. On the other hand, the zero set of the ideal $(z - z_0)W(z)A(S_w, W)$ coincides with that of $A(S_w, W)$. It follows that $(z - z_0)W(z)A(S_w, W) = A(S_w, W)$ (see [15, p. 32]) and that $f_0|_{S_w} \in (z - z_0)W(z)A(S_w, W)$.

Thus, there exists a sequence of the weighted polynomials $\{W^n Q_n\}_{n=0}^{\infty}$, with $Q_n(z_0) = 0$, uniformly convergent to f_0 on S_w, as $n \to \infty$. Since $W^n(z)(P_n(z) - Q_n(z))$ converges to zero uniformly on S_w and converges to $f_0(z_0) \neq 0$ for $z = z_0 \in E \backslash S_w$, as $n \to \infty$, we obtain a direct contradiction with (2.6) for some sufficiently large n.

Consequently, if $f \in A(E, W)$ then $f(z) = 0$ for any $z \in E \backslash S_w$. Furthermore, the same is true for any $z \in \overline{E \backslash S_w}$ by the continuity of $f(z)$. $\square$

Proof of Theorem 4.1: Since $[W(z),\ zW(z)] = A(\overline{D})$ by the assumption of the theorem, $A(\overline{D}, W)$ is a closed ideal of $A(\overline{D})$ by Proposition 2.3. The result now follows from the description of nontrivial closed ideals of the disk algebra (see [13] and [5, pp. 82-87]). $\square$

Proof of Theorem 4.2: Since G is a Jordan domain, the set $\{z : |z - z_0| < \delta,\ z \in \overline{G}\}$ has positive logarithmic capacity for any $z_0 \in \overline{G}$ and $\delta > 0$. It is clear that S_w is contained in some Jordan arc, as a proper closed subset of ∂G, so that $\overline{\mathbf{C}} \backslash S_w$ is connected. Observe that all conditions of Theorem 3.2 are satisfied in this case, which yields that any function $f \in A(\overline{G}, W)$ must vanish on $\overline{(\overline{G} \backslash S_w)} = \overline{G}$. $\square$

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Igor E. Pritsker
Department of Mathematics
Case Western Reserve University
10900 Euclid Avenue
Cleveland, OH 44106-7058
`iep@po.cwru.edu`