Rational approximation with varying weights

in the complex plane

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Abstract. Given an open bounded set G in the complex plane and a weight function $W(z)$ which is analytic and different from zero in G, we consider the problem of locally uniform rational approximation of any function $f(z)$, which is analytic in G, by particular ray sequences of weighted rational functions of the form $W^{m+n}(z)R_{m,n}(z)$, where $R_{m,n}(z) = P_m(z)/Q_n(z)$, with $\deg P_m \le m$ and $\deg Q_n \le n$. The main result of this paper is a necessary and sufficient condition for such an approximation to be valid. We also consider a number of applications of this result to various classical weights, and find explicit criteria for the possibility of weighted approximation in these cases.

§1 Introduction and General Results.

In this paper, we shall develop the ideas of [11] and apply them to the study of the approximation of analytic functions in an open set G by weighted rationals $W^{m+n}(z)R_{m,n}(z)$. Specifically, we examine triples of the form

$$(G, W, \gamma) \tag{1.1}$$

where

$$\left.\begin{array}{ll}
i) & G \text{ is an open bounded set in the complex plane } \mathbb{C}, \\
 & \text{which can be represented as a finite or countable} \\
 & \text{union of disjoint simply connected domains, i.e.,} \\
 & G = \bigcup_{\ell=1}^{\sigma} G_\ell \text{ (where } 1 \le \sigma \le \infty \text{),} \\
ii) & W(z) \text{, the weight function, is analytic in } G \text{ with} \\
 & W(z) \ne 0 \text{ for any } z \in G \text{, and} \\
iii) & \gamma \text{ satisfies } 0 \le \gamma \le 1.
\end{array}\right\} \tag{1.2}$$

We say that the triple (G, W, γ) has the **rational approximation property** if,

$$\left.\begin{array}{ll} & \text{for any } f(z) \text{ which is analytic in } G \text{ and for any compact subset } E \text{ of } G, \text{ there exists a sequence of rational functions } \{R_{m_i,n_i}(z)\}_{i=0}^{\infty}, \text{ where} \\ & R_{m_i,n_i}(z) = P_{m_i}(z)/Q_{n_i}(z), \text{ with } \deg P_{m_i} \le m_i \text{ and } \deg Q_{n_i} \le n_i \text{ for all } i \ge 0, \text{ and where} \\ & (m_i + n_i) \to \infty \text{ as } i \to \infty, \text{ such that} \\ i) & \lim\limits_{i\to\infty} \dfrac{m_i}{m_i + n_i} = \gamma, \\ & \text{and} \\ ii) & \lim\limits_{i\to\infty} \|f - W^{m_i+n_i} R_{m_i,n_i}\|_E = 0, \end{array}\right\} \tag{1.3}$$

where all norms throughout this paper are the uniform (Chebyshev) norms on the indicated sets.

Given a triple (G, W, γ), as in (1.1) which satisfies the conditions of (1.2), we state below our main result, Theorem 1, which gives a characterization, in terms of potential theory, for the triple (G, W, γ) to have the rational approximation property. Let $\mathcal{M}(E)$ be the set of all positive unit Borel measures on $\mathbb{C}$ which are supported on a compact set E, i.e., for any $\mu \in \mathcal{M}(E)$, we have $\mu(\mathbb{C}) = 1$ and $\operatorname{supp}\mu \subset E$. Also, ∂G denotes the boundary of the set G, and the logarithmic potential of an arbitrary compactly supported signed measure μ is defined (see Tsuji [18, p. 53]) by

$$U^{\mu}(z) := \int \log \frac{1}{|z-t|}\, d\mu(t). \tag{1.4}$$

Theorem 1. *A triple (G, W, γ), satisfying (1.2), has the rational approximation property (1.3) if and only if there exist a signed measure*

$$\mu(G, W, \gamma) = \gamma\mu^{+}(G, W, \gamma) - (1-\gamma)\mu^{-}(G, W, \gamma), \tag{1.5}$$

with $\mu^{+}(G, W, \gamma), \mu^{-}(G, W, \gamma) \in \mathcal{M}(\partial G)$, and a constant $F(G, W, \gamma)$ such that

$$U^{\mu(G,W,\gamma)}(z) - \log|W(z)| = F(G, W, \gamma), \qquad \text{for any } z \in G. \tag{1.6}$$

Below, we state some consequences of Theorem 1, while in Section 2, we state applications of Theorem 1 in a number of specific cases.

Remark 1: Results on weighted rational approximation of analytic functions in open sets with *multiply* connected components (as opposed, in (1.2i), to unions of simply connected domains) will be considered elsewhere.

Remark 2: The condition in (1.2ii) that $W(z) \neq 0$ for all $z \in G$ cannot be dropped, for if $W(z_0) = 0$ for some $z_0 \in G_k$, where $G = \bigcup_{\ell=1}^{\sigma} G_\ell$, then the necessarily null sequence

$$\left\{W^{m_i+n_i}(z_0) R_{m_i,n_i}(z_0)\right\}_{i=0}^{\infty}$$

trivially fails to converge to any $f(z)$, analytic in G, with $f(z_0) \neq 0$; whence, the rational approximation property fails.

Corollary 1. *A triple (G, W, γ), satisfying (1.2), has the rational approximation property (1.3) if and only if (1.3) holds for $f(z) \equiv 1$, i.e., if and only if this single function is locally uniformly approximable on compact subsets of G by a corresponding sequence of the weighted rational functions.*

Remark 3: The function $f(z) \equiv 1$ in Corollary 1 can be replaced by any function which is analytic in G and not equal identically to 0 in G.

Corollary 2. *Given a triple (G, W, γ), which satisfies (1.2) with σ finite, assume that there exist a constant F and a signed measure μ with*

$$\operatorname{supp} \mu \subset \partial G \quad \textit{and} \quad \mu(\mathbb{C}) = 2\gamma - 1, \tag{1.7}$$

such that

$$U^{\mu}(z) - \log|W(z)| = F, \qquad \textit{for any } z \in G. \tag{1.8}$$

Then, the triple (G, W, γ) has the rational approximation property (1.3) if and only if the signed measure μ can be decomposed as

$$\mu = \gamma\mu^{+} - (1-\gamma)\mu^{-}, \tag{1.9}$$

with $\mu^{+}, \mu^{-} \in \mathcal{M}(\partial G)$.

Furthermore, let a Jordan decomposition of the signed measure μ, satisfying (1.7) and (1.8), be given by

$$\mu = \tau^{+} - \tau^{-}, \tag{1.10}$$

where τ^{+} and τ^{-} are positive measures with

$$\operatorname{supp} \tau^{+},\ \operatorname{supp} \tau^{-} \subset \partial G \quad \textit{and} \quad \mu(\operatorname{supp} \tau^{+} \cap \operatorname{supp} \tau^{-}) = 0. \tag{1.11}$$

Then, the triple (G, W, γ) *has the rational approximation property* (1.3) *if and only if*

$$\tau^+(\mathbb{C}) \le \gamma. \tag{1.12}$$

If (1.9) *or* (1.12) *holds true for a signed measure* μ *satisfying* (1.7) *and* (1.8), *then*

$$\mu(G, W, \gamma) = \mu \quad \text{and} \quad F(G, W, \gamma) = F. \tag{1.13}$$

The study of weighted *rational* approximation has recently been introduced in papers by Borwein and Chen [1], Borwein, Rakhmanov and Saff [2], and Rakhmanov, Saff, and Simeonov [12]. The last two papers deal with weighted rational approximation only on the real line. Certain special cases of the triples (G, W, γ), in the notation of (1.1), were considered in the complex plane in [1], but that research did not attack the general question of necessary and sufficient conditions for (G, W, γ) to have the rational approximation property of (1.3), as in Theorem 1.

§2 Applications.

Finding the signed measure $\mu(G, W, \gamma)$ of Theorem 1, or verifying its existence, is a nontrivial problem in general. Since $U^{\mu(G,W,\gamma)}(z)$ is harmonic in $\mathbb{C}\setminus$ supp $\mu(G, W, \gamma)$ and, since it can be shown from (1.6), if $\log |W(z)|$ is continuous on $\overline{G}$ and if G is a *finite* union of $G_\ell, \ell = 1, 2, \ldots, \ \ell_0$, that $U^{\mu(G,W,\gamma)}(z)$ is equal to $\log |W(z)| + F(G, W, \gamma)$ on $\operatorname{supp} \mu(G, W, \gamma) \subset \partial G$, then $U^{\mu(G,W,\gamma)}(z)$ can be found as the solution of the corresponding Dirichlet problems. The signed measure $\mu(G, W, \gamma)$ can be recovered from its potential, using the Fourier method described in Section IV.2 of Saff and Totik [13].

However, we next consider a different method, dealing with specific weight functions, which allows us to deduce "explicit" expressions for the signed measure $\mu(G, W, \gamma)$ of Theorem 1. For simplicity, we assume throughout this section that G is given as in (1.2i), but with σ finite. We denote the unbounded component of $\overline{\mathbb{C}}\setminus\overline{G}$ by Ω. Let ν^+ and ν^- be two *positive* Borel measures on $\mathbb{C}$, with compact supports satisfying

$$\operatorname{supp} \nu^+ \subset \overline{\mathbb{C}}\setminus G \text{ and } \operatorname{supp} \nu^- \subset \overline{\mathbb{C}}\setminus G, \tag{2.1}$$

such that

$$\nu^+(\mathbb{C}) = \nu^-(\mathbb{C}) = 1. \tag{2.2}$$

For real numbers $\alpha \ge 0$ and $\beta \ge 0$, assume that the weight function $W(z)$, satisfying

$$\log |W(z)| = -\left(\alpha U^{\nu^+}(z) - \beta U^{\nu^-}(z)\right) = -U^{\nu}(z),\ z \in G, \tag{2.3}$$

with $\nu := \alpha\nu^+ - \beta\nu^-$ being a signed measure, is analytic in G. Then, we state, as an application of Theorem 1, our next result as

Theorem 2. *Given any pair of real numbers $\alpha \geq 0$ and $\beta \geq 0$, given an open bounded set $G = \bigcup_{\ell=1}^{\sigma} G_\ell$, as in* (1.2i) *with σ finite, and given the weight function $W(z)$ of* (2.3)*, then the triple (G, W, γ) has the rational approximation property* (1.3) *if and only if the signed measure*

$$\mu := (2\gamma - 1 + \alpha - \beta)\omega(\infty, \cdot, \Omega) - \alpha\hat{\nu}^+ + \beta\hat{\nu}^- \tag{2.4}$$

can be decomposed as

$$\mu = \gamma\mu^+ - (1-\gamma)\mu^-, \tag{2.5}$$

where $\mu^+, \mu^- \in \mathcal{M}(\partial G)$. Here, $\omega(\infty, \cdot, \Omega)$ is the harmonic measure at ∞ with respect to Ω, and $\hat{\nu}^+$ and $\hat{\nu}^-$ are, respectively, the balayages of ν^+ and ν^- from $\overline{\mathbb{C}}\backslash\overline{G}$ to $\overline{G}$.

Furthermore, if μ of (2.4) *satisfies* (2.5)*, then* (see Theorem 1)

$$\mu(G, W, \gamma) = \mu. \tag{2.6}$$

We point out that the harmonic measure $\omega(\infty, \cdot, \Omega)$ (defined in Nevanlinna [8] or Tsuji [18]) is the same as the equilibrium distribution measure for $\overline{G}$, in the sense of classical logarithmic potential theory [18]. For the notion of balayage of a measure, we refer the reader to Chapter IV of Landkof [6] or Section II.4 of Saff and Totik [13].

In the following series of subsections, we consider various classical weight functions and we find their corresponding signed measures, associated with the weighted rational approximation problem in G, as given in Theorem 1.

Incomplete Rationals. With $\mathbb{N}_0$ and $\mathbb{N}$ denoting respectively the sets of nonnegative and positive integers, the *incomplete polynomials* of Lorentz [7] are a sequence of polynomials of the form

$$\left\{z^{m(i)}P_{n(i)}(z)\right\}_{i=0}^{\infty}, \quad \deg P_{n(i)} \leq n(i),\ (m(i), n(i) \in \mathbb{N}_0), \tag{2.7}$$

where it is assumed that $\lim\limits_{i\to\infty} \dfrac{m(i)}{n(i)} =: \alpha$, where $\alpha > 0$ is a real number. The question of the possibility of the approximation of functions by incomplete polynomials is closely connected to that of the approximation of functions by the weighted polynomials

$$\{z^{\alpha n} P_n(z)\}_{n=0}^{\infty}, \quad \deg P_n \le n. \tag{2.8}$$

The approximation question for the incomplete polynomials of (2.7) was completely settled, by Saff and Varga [14] and by v. Golitschek [4], for the real interval $[0,1]$ (see Totik [17] and Saff and Totik [13] for the associated history and later developments), and by the authors [11] in the complex plane. We consider now the analogous problem for *incomplete rational functions* in the complex plane. A special case of incomplete rational approximation in the complex plane was studied by Borwein and Chen in [1]. The latest such developments, on the real line, are in Borwein, Rakhmanov and Saff [2] and Rakhmanov, Saff and Simeonov [12].

Since the weight $W(z) := z^{\alpha}$ in (2.8) is multiple-valued in $\mathbb{C}$ if $\alpha \notin \mathbb{N}$, we then restrict ourselves to the slit domain $S_1 := \mathbb{C} \setminus (-\infty, 0]$ and the single-valued branch of $W(z)$ in S_1 satisfying $W(1) = 1$. Thus,

$$W(z) := z^{\alpha}, \quad z \in S_1 := \mathbb{C} \setminus (-\infty, 0], \tag{2.9}$$

where $\alpha > 0$ is a real number.

Theorem 3. *Given an open set G, as in* (1.2i) *with σ finite, such that $\overline{G} \subset S_1$, and given the weight function $W(z)$ of* (2.9)*, then the triple (G, z^{α}, γ) has the rational approximation property* (1.3) *if and only if the signed measure*

$$\mu = (2\gamma - 1 + \alpha)\omega(\infty, \cdot, \Omega) - \alpha\omega(0, \cdot, \Omega) \tag{2.10}$$

can be decomposed as

$$\mu = \gamma\mu^{+} - (1-\gamma)\mu^{-},$$

where $\mu^{+}, \mu^{-} \in \mathcal{M}(\partial G)$. Here, $\omega(\infty, \cdot, \Omega)$ and $\omega(0, \cdot, \Omega)$ are the harmonic measures with respect to the unbounded component Ω of $\overline{\mathbb{C}} \setminus \overline{G}$, respectively, at $z = \infty$ and at $z = 0$.

In special cases where the geometric shape of G is given explicitly, it is possible to determine the explicit form of the signed measure in (2.10). As a simple example, we consider below the special case of a disk and $\gamma = 1/2$.

Corollary 3. *Given the disk $D_r(a) := \{z \in \mathbb{C} : |z - a| < r\}$, where $a \in (0, +\infty)$ and where $\overline{D}_r(a) \subset S_1 = \mathbb{C} \setminus (-\infty, 0]$, i.e., $r < a$, and given the weight function of* (2.9)*, then the triple $(D_r(a), z^{\alpha}, 1/2)$ has the rational approximation property* (1.3) *if and only if*

$$r \le r_{\max}(a,\ \alpha) := \begin{cases} a,\ \alpha \in (0, 1/2], \\ a \sin \dfrac{\pi}{4\alpha},\ \alpha \in (1/2, +\infty). \end{cases} \tag{2.11}$$

Furthermore, if (2.11) *is satisfied, then the associated signed measure* $\mu(D_r(a), z^\alpha, 1/2)$ (see Theorem 1) *is given by*

$$d\mu(D_r(a), z^\alpha, 1/2) = \frac{\alpha}{2\pi r}\left(1 - \frac{a^2 - r^2}{|z|^2}\right)\ ds, \tag{2.12}$$

where ds *is the arclength measure on the circle* $|z-a| = r$.

Remark 4: Generally, it is possible to show that the triple $(D_r(a),\ z^\alpha,\ \gamma)$, as in Corollary 3 but with *any* $\gamma \in [0,1]$, has the approximation property (1.3) if and only if

$$r \le \begin{cases} a & ,\quad \alpha + \gamma \le 1, \\ au_0 & ,\quad \alpha + \gamma > 1, \end{cases}$$

where $u_0 \in (0,1]$ is the largest solution of the equation

$$(2\gamma - 1)\ \ \arccos\left[\frac{1 - 2\gamma - (2\gamma - 1 + 2\alpha)u^2}{2u(2\gamma - 1 + \alpha)}\right]$$

$$+2\alpha \arccos\left[\frac{(2\gamma - 1 + 2\alpha)\sqrt{1-u^2}}{2\sqrt{\alpha(2\gamma - 1 + \alpha)}}\right] = \gamma\pi,$$

in the interval $(0,1]$.

Exponential Weight. Consider the weight function

$$W(z) := e^{-z}, \quad z \in \mathbb{C}. \tag{2.13}$$

This section is devoted to the study of weighted rational approximation, with respect to the exponential weight of (2.13), in disks centered at the origin and in certain domains, arising in connection with Padé approximations of the exponential function. Our next result treats the case of disks.

Theorem 4. *Given* $D_r(0) := \{z \in \mathbb{C} : |z| < r\}$ *and given the weight* $W(z)$ *of* (2.13), *then the triple* $(D_r(0), e^{-z}, \gamma)$ *has the rational approximation property* (1.3) *if and only if*

$$r \le r_{\max}(\gamma), \quad 0 \le \gamma \le 1, \tag{2.14}$$

where $r_{\max}(\gamma)$ *is the unique positive solution, for* r *in the interval* $[|\gamma-\frac{1}{2}|,+\infty)$, *of the following equation:*

$$\begin{aligned}\sqrt{r^2-\left(\gamma-\tfrac{1}{2}\right)^2} \quad &- \left(\gamma-\tfrac{1}{2}\right)\arccos\left(\tfrac{\gamma-\frac{1}{2}}{r}\right) \\ &= \tfrac{\pi}{2}\left(1-\gamma\right).\end{aligned} \tag{2.15}$$

Moreover, if (2.14) *holds, then the associated signed measure* $\mu(D_r(0),\, e^{-z},\gamma)$ *is given by*

$$d\mu(D_r(0), e^{-z}, \gamma) = (2\gamma - 1 - 2r\cos\theta)\frac{d\theta}{2\pi}, \tag{2.16}$$

where $d\theta$ *is the angular measure on* $|z| = r$ *and where* $z = re^{i\theta}$.

In particular, $r_{\max}(1) = \frac{1}{2}$ (see also Theorem 3.8 of [10]), $r_{\max}\left(\frac{1}{2}\right) = \frac{\pi}{4}$ *and* $r_{\max}(0) = \frac{1}{2}$.

We remark that the solution, $r_{\max}(\gamma)$ of (2.15), can be verified to be symmetric about $\gamma = \dfrac{1}{2}$, as a function of γ in the interval $[0,1]$.

Next, we again consider the weight function $W(z) := e^{-z}$ of (2.13), but we now consider the triple $(G_\gamma,\ e^{-z},\ \gamma)$, where G_γ, a *generalized Szegő domain*, is defined below. To begin, first assume that $0 < \gamma < 1$. Then following [15], the two conjugate complex numbers, defined by

$$z_\gamma^{\pm} := \exp\left\{\pm\ i\arccos(2\gamma-1)\right\}, \tag{2.17}$$

have modulus unity, and we consider the complex plane $\mathbb{C}$ slit along the two rays

$$\mathbb{R}_\gamma := \left\{z\in\mathbb{C} : z = z_\gamma^+ + i\tau \text{ or } z = z_\gamma^- - i\tau,\ \text{for } \tau\geq 0\right\}. \tag{2.18}$$

This is shown below in Figure 1. (For readers who are familiar with [15], the quantity $\sigma := \lim\limits_{i\to\infty}\dfrac{n_i}{m_i}$ in that paper and γ of (1.3i) are related through $\gamma = \frac{1}{1+\sigma}$.) Next, the function

$$\hat{g}_\gamma(z) := \sqrt{1+z^2-2z(2\gamma-1)} \tag{2.19}$$

has z_γ^+ and z_γ^- as branch points, which are the finite extremities of $\mathbb{R}_\gamma$. On taking the principal branch for the square root, i.e., on setting $\hat{g}_\gamma(0) = 1$ and extending g_γ analytically on the doubly slit domain $\mathbb{C}\backslash\mathbb{R}_\gamma$, then g_γ is analytic and single-valued on $\mathbb{C}\backslash\mathbb{R}_\gamma$. It can also be verified that $1\pm z+\hat{g}_\gamma(z)$ does not vanish on $\mathbb{C}\backslash\mathbb{R}_\gamma$.

Next, we define the functions $(1+z+\hat{g}_\gamma(z))^{2\gamma}$ and $(1-z+\hat{g}_\gamma(z))^{2(1-\gamma)}$ so that their values at $z=0$ are respectively, $2^{2\gamma}$ and $2^{2(1-\gamma)}$, with remaining

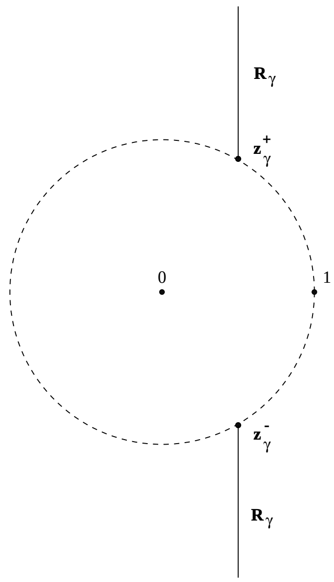

Figure 1. The set $\mathbb{C}\backslash\mathbb{R}_\gamma$.

values determined by analytic continuation. These functions are also analytic and single-valued on $\mathbb{C}\backslash\mathbb{R}_\gamma$. With these definitions, we then set

$$w_\gamma(z) := \frac{4\gamma\left(\frac{1-\gamma}{\gamma}\right)^{1-\gamma} z e^{\hat{g}_\gamma(z)}}{(1+z+\hat{g}_\gamma(z))^{2\gamma}(1-z+\hat{g}_\gamma(z))^{2(1-\gamma)}} \quad (0<\gamma<1), \tag{2.20}$$

and it follows that $w_\gamma(z)$ is also analytic and single-valued on $\mathbb{C}\backslash\mathbb{R}_\gamma$. For the omitted cases $\gamma=0$ and $\gamma=1$, it can be verified that $w_1(z)=\lim_{\gamma\to1} w_\gamma(z)$ and $w_0(z)=\lim_{\gamma\to0} w_\gamma(z)$ satisfy

$$\left.\begin{array}{ll} w_1(z) = ze^{1-z} & \text{for Re } z<1, \text{ and} \\ w_0(z) = ze^{1+z} & \text{for Re } z>-1. \end{array}\right\} \tag{2.21}$$

(Again, for those familiar with [15], the function $w_\gamma(z)$ of (2.20) is exactly the function $w_\sigma(z)$ in [15, eq. (2.5)].)

It is known (see [15, Lemma 4.1]) that $w_\gamma(z)$ is univalent in $|z|<1$, for any γ with $0\le\gamma\le1$, and this allows us to define the domain

$$G_\gamma := \left\{z\in\mathbb{C} : |w_\gamma(z)|<1 \text{ and } |z|<1\right\}. \tag{2.22}$$

Its boundary, ∂G_γ, is a well-defined Jordan curve which lies interior to the unit disk, except for its points $z_\gamma^{\pm}$ of (2.17). This is shown in Figure 2. We call G_γ an *extended Szegő domain*, as the special case $\gamma = 1$ corresponds to a domain originally treated by Szegő in [16] in 1924.

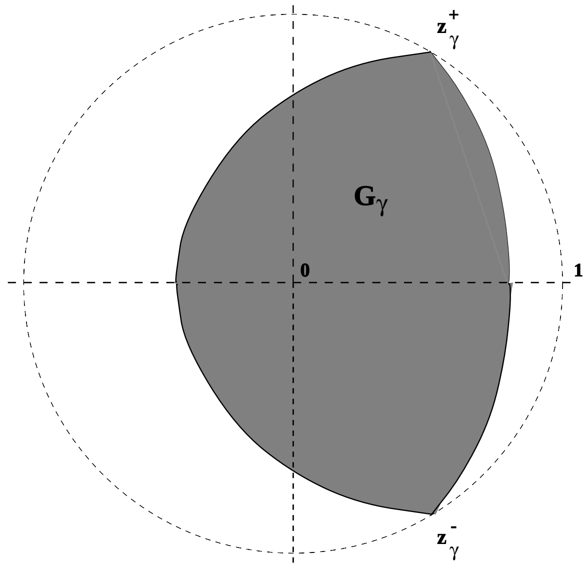

Figure 2. The set G_γ.

We now have all the necessary definitions for the statement of our next result.

Theorem 5. *For any γ with $0 \le \gamma \le 1$, let G_γ be the domain of* (2.22), *and let $W(z) = e^{-z}$ be the weight function of* (2.13). *Then, the triple $(G_\gamma, e^{-z}, \gamma)$ has the rational approximation property* (1.3).

To conclude this section, we note that, except for the final result of Theorem 5, all preceding results stated in Sections 1 and 2 are of the "if and only if" type, i.e., these results are by definition *sharp*. The result of Theorem 5, however, leaves open the possibility that for a given γ with $0 \le \gamma \le 1$, there could be a *larger* domain H, with $G_\gamma \subset H$, such that the triple (H, e^{-z}, γ) has the rational approximation property (1.3), but we strongly doubt this.

Also of general interest is the extension of the results of this paper to triples (G, W, γ) of (1.1), where one has the *sharpened rational approximation property*, that is, for any $f(z)$, analytic in G and continuous in $\overline{G}$, there is a sequence of rational functions $\{R_{m_i,n_i}\}_{i=0}^{\infty}$ satisfying (1.3i), such that

$$\lim_{i\to\infty} \left\| f - W^{m_i+n_i} R_{m_i,n_i} \right\|_{\overline{G}} = 0.$$

For the essentially polynomial case of $\gamma = 0$ and $W(z) := e^{-z}$, this is treated in part in [10, Theorem 3.2]. Some general results in weighted polynomial approximation on compact sets are obtained in [9]. To our knowledge, general results on the sharpened rational approximation property have not as yet been treated in the literature.

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