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Comparing Norms of Polynomials in One and Several Variables

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We study Nikolskii-type inequalities for the L_p norms of an algebraic polynomial in one variable, defined either by a contour integral or by an area integral over a Jordan domain in $\mathbb{C}$. Further, we generalize the one-dimensional results to the case of polynomials in several variables over product domains in $\mathbb{C}^m$, $m > 1$.

1. INTRODUCTION AND RESULTS

We are interested in the relations between the L_p norms of algebraic polynomials (with different p's, $0 < p \leq \infty$), defined both by contour and by area integrals. Apparently, one of the first results in this direction is due to Jackson [3], who proved that

$$\max_{|z|=1} |P_n(z)| \leq 2n^{1/p} \left(\int_0^{2\pi} |P_n(e^{i\theta})|^p \, d\theta \right)^{1/p}, \qquad 0 < p < \infty, \quad (1.1)$$

where $P_n(z)$ is an algebraic polynomial, $\deg P_n \leq n$. More generally, Nikolskii [6, p. 126] showed, for an algebraic polynomial $P(z_1, \ldots, z_n)$ in m variables ($m \in \mathbb{N}$), that

$$\left(\int_0^{2\pi} \cdots \int_0^{2\pi} |P(e^{i\theta_1}, \ldots, e^{i\theta_m})|^q \, d\theta_1 \cdots d\theta_m \right)^{1/q}$$
$$\leq 2^m \left(\prod_{j=1}^m n_j \right)^{1/p - 1/q} \left(\int_0^{2\pi} \cdots \int_0^{2\pi} |P(e^{i\theta_1}, \ldots, e^{i\theta_m})|^p \, d\theta_1 \cdots d\theta_m \right)^{1/p}, \quad (1.2)$$

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where n_j is the highest power of z_j in $P(z_1, \dots, z_m)$, $j = 1, \dots, m$, and where $0 < p < q \le \infty$, with the left-hand side of (1.2) being replaced by $\max_{|z_j| \le 1} |P(z_1, \dots, z_m)|$ for $q = \infty$. We also refer to Szegő and Zygmund [9] for a one-dimensional variant of (1.2). It has become customary to call the inequalities, comparing norms of a polynomial in different spaces (similarly to (1.2)), Nikolskii-type inequalities. One can find a survey of Nikolskii-type inequalities, with extensive references, in Milovanović *et al.* [5, Sect. 5.3].

Let G be a bounded Jordan domain in the complex plane $\mathbb{C}$, with the boundary ∂G being a simple closed Jordan curve. Then the unbounded Jordan domain $\Omega := \overline{\mathbb{C}} \setminus \overline{G}$ can be mapped conformally onto $D' := \{w: |w| > 1\}$ by the canonical conformal mapping $\Phi: \Omega \to D'$, normalized by $\Phi(\infty) = \infty$ and $\lim_{z \to \infty} \Phi(z)/z > 0$. The distance, from ∂G to the level curve of $\Phi(z)$ of the order $1 + 1/n$, defined as

$$d_n := \inf\left\{|z - \xi|: z \in \partial G, |\Phi(\xi)| = 1 + \frac{1}{n}\right\}, \qquad n \in \mathbb{N}, \quad (1.3)$$

plays an important role in the polynomial inequalities in the complex plane.

If the boundary of G is rectifiable, then we introduce

$$\|P_n\|_{L_p(\partial G)} = \left(\int_{\partial G} |P_n(z)|^p |dz|\right)^{1/p}, \qquad 0 < p < \infty, \quad (1.4)$$

and

$$\|P_n\|_{L_\infty(\partial G)} = \max_{z \in \partial G} |P_n(z)|. \quad (1.5)$$

Mamedhanov [4] proved the following generalization of (1.2) in the case of one variable.

THEOREM 1.1. *Let $G \subset \mathbb{C}$ be bounded by a rectifiable Jordan curve. Then*

$$\|P_n\|_{L_q(\partial G)} \le 2^{1-p/q} e^{1/p - 1/q} d_n^{1/q - 1/p} \|P_n\|_{L_p(\partial G)}, \qquad 0 < p < q \le \infty, \quad (1.6)$$

where $P_n(z)$ is an algebraic polynomial, $\deg P_n \le n$.

As the proof of Theorem 1.1 is short, we present it here, in Section 3. Further, we extend this result to the case of polynomials in many variables over the product domains. Suppose that $H := G_1 \times \cdots \times G_m \subset \mathbb{C}^m$, where each G_i is a Jordan domain with rectifiable boundary. For a polynomial

$P(z_1, \ldots, z_m)$ in m variables, define

$$\|P\|_{L_p(\partial H)} := \left(\int_{\partial G_1} \cdots \int_{\partial G_m} |P(z_1, \ldots, z_m)|^p |dz_m| \cdots |dz_1| \right)^{1/p}, \qquad 0 < p < \infty, \tag{1.7}$$

and

$$\|P\|_{L_\infty(\partial H)} := \max_{\substack{z_i \in \partial G_i \\ i=1,\ldots,m}} |P(z_1, \ldots, z_m)|. \tag{1.8}$$

THEOREM 1.2. *Let n_i be the highest power of z_i in $P(z_1, \ldots, z_m)$ and let d_{i,n_i} be the distance defined in* (1.3), *relative to a Jordan domain $G_i \subset \mathbb{C}$ with rectifiable boundary, $i = 1, \ldots, m$. Then*

$$\|P\|_{L_q(\partial H)} \le 2^{m(1-p/q)} e^{m(1/p-1/q)} \left(\prod_{i=1}^{m} d_{i,n_i} \right)^{1/q-1/p} \|P\|_{L_p(\partial H)}, \qquad 0 < p < q \le \infty, \tag{1.9}$$

where $H = G_1 \times \cdots \times G_m \subset \mathbb{C}^m$.

We continue with Nikolskii-type inequalities in L_p spaces, defined by the *area* measure σ in a Jordan domain $G \subset \mathbb{C}$; that is, we set

$$\|P_n\|_{L_p(G)} := \left(\int_G |P_n(z)|^p \, d\sigma(z) \right)^{1/p}, \qquad 0 < p < \infty, \tag{1.10}$$

and

$$\|P_n\|_{L_\infty(G)} := \max_{z \in G} |P_n(z)| \tag{1.11}$$

for a polynomial of a single variable $P_n(z)$. Clearly, (1.11) defines the same norm as (1.5) by the maximum modulus principle.

A Jordan domain $G \subset \mathbb{C}$ is called a quasidisk if its boundary is a quasicircle; that is, ∂G satisfies the following condition:

> There exists $K \ge 1$ such that $\operatorname{diam}(\tau(z_1, z_2)) \le K|z_1 - z_2|$, $z_1, z_2 \in \partial G$, where $\tau(z_1, z_2) \subset \partial G$ is the arc of smaller diameter connecting z_1 and z_2. (1.12)

We remark that quasicircles may not even be locally rectifiable and that (1.12) is valid for a very wide class of curves. Perhaps the only exceptions here are the curves with "cusplike" behavior. For more information on quasidisks and their geometry, we refer to Gehring [2] and Pommerenke [7, Chap. 5].

THEOREM 1.3. *Let $G \subset \mathbb{C}$ be a quasidisk and let $P_n(z)$ be a polynomial of a single variable, with* $\deg P_n \le n$. *Then*

$$\|P_n\|_{L_q(G)} \le 2^{1-p/q} C^{1/p-1/q} d_n^{2(1/q-1/p)} \|P_n\|_{L_p(G)}, \qquad 0 < p < q \le \infty, \tag{1.13}$$

where d_n is defined in (1.3) *and C is a constant, which depends only on G.*

To state an analogue of the above theorem for a product domain $H = G_1 \times \cdots \times G_m \subset \mathbb{C}^m$, where each G_i is a Jordan domain in $\mathbb{C}$ equipped with the area measure $\sigma(z_i)$, we introduce

$$\|P\|_{L_p(H)} := \left(\int_{G_1} \cdots \int_{G_m} |P(z_1, \ldots, z_m)|^p \, d\sigma(z_m) \cdots d\sigma(z_1) \right)^{1/p}, \qquad 0 < p < \infty, \tag{1.14}$$

and

$$\|P\|_{L_\infty(H)} := \max_{\substack{z_i \in G_i \\ i=1,\ldots,m}} |P(z_1, \ldots, z_m)| = \|P\|_{L_\infty(\partial H)} \tag{1.15}$$

for a polynomial $P(z_1, \ldots, z_m)$ in m variables.

THEOREM 1.4. *Assume that $H = G_1 \times \cdots \times G_m \subset \mathbb{C}^m$, where each $G_i \subset \mathbb{C}$ is a quasidisk, $i = 1, \ldots, m$. If $P(z_1, \ldots, z_m)$ is a polynomial in m variables, with degree n_i corresponding to z_i, $i = 1, \ldots, m$, then*

$$\|P\|_{L_q(H)} \le 2^{m(1-p/q)} \left(\prod_{i=1}^{m} C_i \right)^{1/p-1/q} \left(\prod_{i=1}^{m} d_{i,n_i} \right)^{2(1/q-1/p)} \|P\|_{L_p(H)}, \qquad 0 < p < q \le \infty, \tag{1.16}$$

where d_{i,n_i} is a distance of (1.3) *for G_i and where C_i is a constant from the one-dimensional inequality* (1.13), *which depends on G_i only, $i = 1, \ldots, m$.*

We now briefly discuss how to obtain the estimates of the distance d_n, defined in (1.3), knowing the geometry of G. In general, if the conformal mapping $\Phi\colon \overline{\Omega} \to \overline{D'}$ satisfies a Hölder continuity condition of the type

$$|\Phi(z) - \Phi(\zeta)| \le L|z - \zeta|^{1/\alpha}, \qquad z, \zeta \in \overline{\Omega}, 1 \le \alpha \le 2, \tag{1.17}$$

where $L > 0$ depends only on G, then

$$d_n = \inf\{|z - \zeta| : |\Phi(z)| = 1, |\Phi(\zeta)| = 1 + 1/n\} \geq \frac{1}{L^\alpha n^\alpha}. \quad (1.18)$$

The boundary continuity of conformal mappings is a very well studied subject, with a number of geometric conditions, sufficient for (1.17), known in the literature. A detailed account on this and related topics is contained in [7] (see also the references cited therein). We consider only three cases:

(i) if the boundary of G is a Dini-smooth Jordan curve [7, p. 48], then (1.17) holds with $\alpha = 1$ and

$$d_n \geq \frac{1}{Ln}. \quad (1.19)$$

(ii) if the boundary of G is formed by a finite number of Dini-smooth arcs and the largest exterior angle at ∂G has opening $\alpha\pi$, $1 \leq \alpha \leq 2$, then (1.17) holds [7, p. 52] and

$$d_n \geq \frac{1}{L^\alpha n^\alpha}. \quad (1.20)$$

(iii) if G is an arbitrary Jordan domain, then [10, p. 181]

$$d_n \geq \frac{\operatorname{diam}(G)}{4n^2}, \quad (1.21)$$

that is, (1.18) holds with $\alpha = 2$, being the "worst" case.

We remark that the exponent α is sharp in (i), (ii), and (iii).

It is easy to see from (1.19) that if ∂G is Dini-smooth, then the exponents of n in (1.6) and (1.13) are the same as those in the case $G = D$. But the latter are known to be sharp, which is directly verified by using $Q_n(z) := 1 + z + \cdots + z^n$ in D (cf. [9, p. 236]). Similarly, we can show the sharpness of the *exponents of* n_i in (1.9) and (1.16), if each G_i is bounded by a Dini-smooth Jordan curve, $i = 1, \ldots, m$, by considering

$$Q(z_1, \ldots, z_m) := \prod_{i=1}^{m} Q_{n_i}(z_i).$$

However, the *constants* (like $2^{1-p/q}e^{1/p-1/q}$ in (1.6)) can be definitely improved, but the best values are not known even in the classical case $G = D$.

The next section contains lemmas, which are necessary for the proofs of the results given in Section 3.

2. LEMMAS

One of our tools is the well-known Bernstein−Walsh lemma, stated below (cf. [1, p. 27]).

LEMMA 2.1. *If $P_n(z)$ is a polynomial of a single variable*, $\deg P_n \le n$, *and G is a Jordan domain, then*

$$|P_n(z)| \le |\Phi(z)|^n \|P_n\|_{L_\infty(G)}, \qquad z \in \Omega, \tag{2.1}$$

where $\Phi\colon \Omega \to D'$ is a conformal mapping of $\Omega = \overline{\mathbb{C}} \setminus \overline{G}$, defined in the Introduction.

Proof. Observe that $f(z) := P_n(z)/\Phi^n(z)$ is analytic in Ω. It follows by the maximum modulus principle that

$$\left|\frac{P_n(z)}{\Phi^n(z)}\right| \le \max_{t\in\partial\Omega}\left|\frac{P_n(t)}{\Phi^n(t)}\right| = \|P_n\|_{L_\infty(G)}, \qquad z \in \Omega. \qquad \blacksquare$$

LEMMA 2.2 (Szegő [8]). *Let P_n and G be as in Lemma* 2.1. *Then*

$$\|P_n'\|_{L_\infty(G)} \le \frac{e}{d_n}\|P_n\|_{L_\infty(G)}, \tag{2.2}$$

with d_n defined in (1.3).

Proof. Differentiating the Cauchy formula for $P_n(z)$, we obtain

$$P_n'(z) = \frac{1}{2\pi i}\int_{|t-z|=d_n} \frac{P_n(t)}{(t-z)^2}\,dt, \qquad z \in \partial G. \tag{2.3}$$

It follows from the definition of d_n in (1.3) and from (2.1) that, for any $z \in \partial G$,

$$\max_{|t-z|=d_n} |P_n(t)| \le \left(1+\frac{1}{n}\right)^n \|P_n\|_{L_\infty(G)} \le e\|P_n\|_{L_\infty(G)}. \tag{2.4}$$

Therefore, by (2.3) and (2.4),

$$|P_n'(z)| \le \frac{1}{2\pi}\int_{|t-z|=d_n} \frac{|P_n(t)|}{|t-z|^2}|dt| \le \frac{e}{d_n}\|P_n\|_{L_\infty(G)}. \qquad \blacksquare$$

LEMMA 2.3. *If $G \subset \mathbb{C}$ is a quasidisk, then, for any pair of points $z_1, z_2 \in \overline{G}$, there exists an arc $\gamma(z_1, z_2) \subset \overline{G}$ connecting them, such that*

$$|\gamma(z_1, z_2)| \le A|z_1 - z_2|, \tag{2.5}$$

where A depends only on G and $|\gamma(z_1, z_2)|$ is the length of $\gamma(z_1, z_2)$.

Proof. If $z_1, z_2 \in G$, then (2.5) follows directly from [2, pp. 36–37], where $\gamma(z_1, z_2) \subset G$ is the *hyperbolic segment* connecting z_1 and z_2 in G. Furthermore, if z_1 or z_2 is on the boundary of G, then (2.5) still holds by a standard limiting argument, because A does not depend on z_1 or z_2. ∎

LEMMA 2.4. *If $G \subset \mathbb{C}$ is a quasidisk, then there exists $\delta > 0$ such that, for any $z \in \partial G$, we have*

$$\sigma(\{t : |z - t| < r, t \in G\}) \geq Br^2, \qquad 0 < r < \delta, \tag{2.6}$$

where σ is the area measure and $B = \arcsin(1/2K)$, with $K \geq 1$ as in the definition of the quasidisk in (1.12).

Proof. Let $U(z, r)$ be a connected component of $\{t : |z - t| < r, t \in G\}$, such that z is on the boundary of $U(z, r)$, $0 < r < \delta$, with $\delta > 0$ sufficiently small. There exists a circular arc $\gamma(z, r) \subset \partial U(z, r)$, with endpoints $z_1, z_2 \in \partial G$, such that $z \in \tau(z_1, z_2)$, where $\tau(z_1, z_2) \subset \partial G$ is the arc of smaller diameter, connecting z_1 and z_2. Thus, by (1.12),

$$r = |z_1 - z| \leq \operatorname{diam}(\tau(z_1, z_2)) \leq K|z_1 - z_2|$$

or

$$|z_1 - z_2| \geq \frac{r}{K}.$$

The last inequality easily implies an estimate for the length of $\gamma(z, r)$ from below:

$$|\gamma(z, r)| \geq 2r \arcsin \frac{1}{2K}, \qquad z \in G, 0 < r < \delta. \tag{2.7}$$

Using (2.7) and polar coordinates, we obtain

$$\begin{aligned} &\sigma(\{t : |z - t| < r, t \in G\}) \\ &\quad \geq \sigma(U(z, r)) = \int_0^r \int_0^{\theta(u)} u \, d\theta \, du = \int_0^r u\theta(u) \, du \\ &\quad \geq \int_0^r |\gamma(z, u)| \, du \geq \int_0^r 2u \arcsin \frac{1}{2K} \, du = r^2 \arcsin \frac{1}{2K}. \end{aligned}$$ ∎

Next, we consider a general idea on how to proceed from one-dimensional Nikolskii-type inequalities to multidimensional ones, over the product domains in $\mathbb{C}^m$, $m > 1$. Let $f : E \to \mathbb{C}$ be a continuous function on a compact set $E = E_1 \times \cdots \times E_m \subset \mathbb{C}^m$, where each $E_i \subset \mathbb{C}$ is a compactum, $i = 1, \ldots, m$. If μ_i is a finite positive Borel measure supported on

E_i, $i = 1, \dots, m$, then we define

$$\|f\|_{L_p(E,\mu)} := \left(\int_{E_1} \cdots \int_{E_m} |f(z_1, \dots, z_m)|^p \, d\mu_m(z_m) \cdots d\mu_1(z_1) \right)^{1/p}, \qquad 0 < p < \infty, \quad (2.8)$$

and

$$\|f\|_{L_\infty(E,\mu)} := \max_E |f|. \quad (2.9)$$

LEMMA 2.5. *Suppose that, for any i, $1 \le i \le n$,*

$$\max_{z_i \in E_i} |f(z_1, \dots, z_m)| \le M_i \left(\int_{E_i} |f(z_1, \dots, z_m)|^p \, d\mu_i(z_i) \right)^{1/p}, \qquad 0 < p < \infty, \quad (2.10)$$

where M_i is independent of the variables $z_j \in E_j$, $j \ne i$. Then we have

$$\|f\|_{L_q(E,\mu)} \le \left(\prod_{i=1}^{m} M_i \right)^{1-p/q} \|f\|_{L_p(E,\mu)}, \qquad 0 < p < q \le \infty. \quad (2.11)$$

Proof. First, observe that, for any $j \ne k$, $1 \le j, k \le m$,

$$\max_{z_j \in E_j} \int_{E_k} |f(z_1, \dots, z_m)|^p \, d\mu_k(z_k) \le \int_{E_k} \max_{z_j \in E_j} |f(z_1, \dots, z_m)|^p \, d\mu_k(z_k). \quad (2.12)$$

Using this idea, we estimate by (2.10):

$$\begin{aligned}
\|f\|^p_{L_\infty(E,\mu)} &= \max_{z_1 \in E_1} \cdots \max_{z_m \in E_m} |f(z_1, \dots, z_m)|^p \\
&\le \max_{z_1 \in E_1} \cdots \max_{z_{m-1} \in E_{m-1}} M_m^p \int_{E_m} |f(z_1, \dots, z_m)|^p \, d\mu_m(z_m) \\
&\le M_m^p \max_{z_1 \in E_1} \cdots \max_{z_{m-2} \in E_{m-2}} \int_{E_m} \max_{z_{m-1} \in E_{m-1}} |f(z_1, \dots, z_m)|^p \, d\mu_m(z_m) \\
&\le (M_m M_{m-1})^p \max_{z_1 \in E_1} \cdots \max_{z_{m-2} \in E_{m-2}} \int_{E_m} \int_{E_{m-1}} |f(z_1, \dots, z_m)|^p \, d\mu_{m-1} \\
&\qquad (z_{m-1}) \, d\mu_m(z_m) \\
&\le \cdots \le \left(\prod_{i=1}^{m} M_i \right)^p \|f\|^p_{L_p(E,\mu)}, \qquad 0 < p < \infty.
\end{aligned}$$

It follows that

$$\|f\|_{L_\infty(E,\mu)} \le \left(\prod_{i=1}^{m} M_i\right)\|f\|_{L_p(E,\mu)}, \qquad 0 < p < \infty.$$

Applying the above inequality, we obtain, for $0 < p < q \le \infty$,

$$\begin{aligned}\|f\|^q_{L_q(E,\mu)} &\le \int_E |f|^q \, d\mu = \int_E |f|^{q-p}|f|^p \, d\mu \\ &\le \|f\|^{q-p}_{L_\infty(E,\mu)}\|f\|^p_{L_p(E,\mu)} \le \left(\prod_{i=1}^{m} M_i\right)^{q-p}\|f\|^q_{L_p(E,\mu)}.\end{aligned}$$ ∎

3. PROOFS OF THE RESULTS

Proof of Theorem 1.1. Let $z_0 \in \partial G$ be such that

$$|P_n(z_0)| = \|P_n\|_{L_\infty(\partial G)}. \tag{3.1}$$

Using (2.2), we have, for any $z \in \partial G$,

$$\begin{aligned}|P_n(z_0)| - |P_n(z)| &\le \left|\int_{\gamma(z,z_0)} P_n'(t)\, dt\right| \\ &\le \int_{\gamma(z,z_0)} |P_n'(t)|\,|dt| \le |\gamma(z,z_0)|\frac{e}{d_n}\|P_n\|_{L_\infty(\partial G)}, \end{aligned}\tag{3.2}$$

where $|\gamma(z,z_0)|$ is the length of the arc $\gamma(z,z_0) \subset \partial G$ connecting z_0 and z. Clearly, if $|\gamma(z,z_0)| \le d_n/(2e)$, then $|P_n(z)| \ge \|P_n\|_{L_\infty(\partial G)}/2$ by (3.1) and (3.2). This gives the following estimate:

$$\left|\left\{z \in \partial G: |P_n(z)| \ge \frac{1}{2}\|P_n\|_{L_\infty(\partial G)}\right\}\right| \ge \frac{d_n}{e},$$

which implies in turn that

$$\|P_n\|_{L_p(\partial G)} = \left(\int_{\partial G} |P_n(z)|^p |dz|\right)^{1/p} \ge \left(\frac{d_n}{e}\right)^{1/p} \frac{\|P_n\|_{L_\infty(\partial G)}}{2}.$$

Thus,

$$\|P_n\|_{L_\infty(\partial G)} \le 2e^{1/p}d_n^{-1/p}\|P_n\|_{L_p(\partial G)}, \qquad 0 < p < \infty. \tag{3.3}$$

For any q such that $0 < p < q \le \infty$, we use (3.3) to obtain

$$\|P_n\|_{L_q(\partial G)} = \left(\int_{\partial G} |P_n(t)|^q |dt|\right)^{1/q}$$
$$\le \|P_n\|_{L_\infty(\partial G)}^{(q-p)/q} \left(\int_{\partial G} |P_n(t)|^p |dt|\right)^{1/q}$$
$$= \|P_n\|_{L_\infty(\partial G)}^{1-p/q} \|P_n\|_{L_p(\partial G)}^{p/q} \le 2^{1-p/q} e^{1/p-1/q} d_n^{1/q-1/p} \|P_n\|_{L_p(\partial G)}.$$

∎

Proof of Theorem 1.2. The statement of Theorem 1.2 follows immediately from (3.3) (or (1.6) with $q = \infty$) and Lemma 2.5. ∎

Proof of Theorem 1.3. Choose $z_0 \in \partial G$ so that

$$|P_n(z_0)| = \|P\|_{L_\infty(G)}. \tag{3.4}$$

By Lemma 2.3, we can find an arc $\gamma(z, z_0) \subset \overline{G}$ connecting z_0 with any $z \in G$, such that

$$|\gamma(z, z_0)| \le A|z - z_0|, \tag{3.5}$$

where A depends only on G and where $|\gamma(z, z_0)|$ is the length of $\gamma(z, z_0)$. Using (2.2) and (3.5), we obtain, for any $z \in G$, that

$$|P_n(z_0)| - |P_n(z)| \le \left|\int_{\gamma(z, z_0)} P_n'(t)\, dt\right| \le \int_{\gamma(z, z_0)} |P_n'(t)|\,|dt|$$
$$\le |\gamma(z, z_0)| \frac{e}{d_n} \|P_n\|_{L_\infty(G)} \le \frac{eA}{d_n} |z - z_0|\, \|P_n\|_{L_\infty(G)}.$$

It follows from the above estimate and (3.4) that

$$|P_n(z)| \ge \frac{1}{2}\|P_n\|_{L_\infty(G)}, \qquad |z - z_0| \le \frac{d_n}{2eA}, \qquad z \in G. \tag{3.6}$$

Hence, Lemma 2.4 and (3.6) give

$$\sigma\left(\left\{z \in G\colon |P_n(z)| \ge \frac{1}{2}\|P_n\|_{L_\infty(G)}\right\}\right) \ge B\left(\frac{d_n}{2eA}\right)^2,$$

so that

$$\|P_n\|_{L_p(G)} = \left(\int_G |P_n(z)|^p \, d\sigma(z)\right)^{1/p} \geq \left(B\left(\frac{d_n}{2eA}\right)^2\right)^{1/p} \frac{\|P_n\|_{L_\infty(G)}}{2}.$$

Finally, we obtain

$$\|P_n\|_{L_\infty(G)} \leq 2C^{1/p} d_n^{-2/p} \|P_n\|_{L_p(G)}, \qquad 0 < p < \infty, \tag{3.7}$$

where C depends only on G. Applying the same idea as in the proof of Theorem 1.1, we have by (3.7), for any q with $0 < p < q \leq \infty$,

$$\begin{aligned} \|P_n\|_{L_q(G)} &\leq \|P_n\|_{L_\infty(G)}^{1-p/q} \|P_n\|_{L_p(G)}^{p/q} \\ &\leq 2^{1-p/q} C^{1/p-1/q} d_n^{2(1/q-1/p)} \|P_n\|_{L_p(G)}. \end{aligned}$$ ∎

Proof of Theorem 1.4. Follows by the combination of (3.7) (or (1.13) with $q = \infty$) and Lemma 2.5. ∎

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