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POLYNOMIAL APPROXIMATION WITH VARYING WEIGHTS ON COMPACT SETS OF THE COMPLEX PLANE

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ABSTRACT. For a compact set E with connected complement, let $A(E)$ be the uniform algebra of functions continuous on E and analytic interior to E. We describe $A(E,W)$, the set of uniform limits on E of sequences of the weighted polynomials $\{W^n(z)P_n(z)\}_{n=0}^{\infty}$, as $n \to \infty$, where $W \in A(E)$ is a nonvanishing weight on E. If E has empty interior, then $A(E,W)$ is completely characterized by a zero set $Z_W \subset E$. However, if E is a closure of Jordan domain, the description of $A(E,W)$ also involves an inner function.

In both cases, we exhibit the role of the support of a certain extremal measure, which is the solution of a weighted logarithmic energy problem, played in the descriptions of $A(E,W)$.

1. INTRODUCTION

Let E be a compact set in the complex plane $\mathbb{C}$ with the connected complement $\overline{\mathbb{C}}\setminus E$. We denote the uniform algebra of continuous on E and analytic in the interior of E functions by $A(E)$ (see, e.g., [5, p. 25]). Clearly, the corresponding uniform norm for any $f \in A(E)$ is defined by

$$\|f\|_E := \max_{z\in E} |f(z)|. \tag{1.1}$$

Consider a weight function $W \in A(E)$ such that $W(z) \neq 0$ for any $z \in E$, and define the *weighted polynomials* $W^n(z)P_n(z)$, where $P_n(z)$ is an algebraic polynomial in z with complex coefficients, with $\deg P_n \le n$. Note that the power of weight varies with the degree of polynomial. We are interested in a description of the function set $A(E,W)$, consisting of the *uniform limits on E* of sequences of the weighted polynomials $\{W^n(z)P_n(z)\}_{n=0}^{\infty}$, as $n \to \infty$. It is well known that if $W(z) \equiv 1$ on E then $A(E,1) = A(E)$ by Mergelyan's theorem [5, p. 48]. In general, we have that $A(E,W) \subset A(E)$.

Our problem originated in the work of Lorentz [12] on incomplete polynomials on the real line. Surveys of results in this area, dealing with weighted approximation on the real line, can be found in [22] and [18, Ch. VI]. The most recent developments are in [8]–[10].

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The questions of density of the weighted polynomials in the set of analytic functions in a domain have been considered in [4], [15] and [16]. In particular, [16] contains a necessary and sufficient condition such that any analytic in a bounded open set function is uniformly approximable by the weighted polynomials $W^n(z)P_n(z)$ on *compact subsets.* However, the description of $A(E,W)$ seems to be much more complicated, in that no general necessary and sufficient condition is known (in terms of the weight $W(z)$), even for the real interval case, i.e., for $E=[a,b]\subset\mathbb{R}$.

We shall approach the above mentioned problems on $A(E,W)$, using ideas of the theories of uniform algebras and of weighted potentials.

The second section of this paper deals with certain general inclusions of $A(E,W)$ and some weighted potential background. We state the results on $A(E,W)$ for E having empty interior in Section 3. The corresponding results for E being the closure of the unit disk D or, more generally, of a Jordan domain G, are developed in Section 4. Section 5 contains all proofs. We conclude with remarks and open problems in Section 6.

2. Inclusion of $A(E,W)$ as a closed ideal and weighted potentials

We start with

Proposition 2.1. $A(E,W)$, *endowed with norm* (1.1)*, is a closed function algebra (not necessarily containing constants and separating points).*

We have already remarked that $A(E,W)\subset A(E)$. To make this inclusion more precise, let us introduce the algebra $[W(z),zW(z)]$ generated by the two functions $W(z)$ and $zW(z)$, which is the uniform closure of all polynomials in $W(z)$ and $zW(z)$ (with constant terms included) on E. Clearly, $[W(z),zW(z)]\subset A(E)$. Furthermore, since any weighted polynomial $W^n(z)P_n(z)$ is an element of $[W(z),zW(z)]$, then $A(E,W)\subset[W(z),zW(z)]$. Thus, we arrive at the following

Proposition 2.2. $A(E,W)\subset[W(z),zW(z)]\subset A(E)$.

The next fact is rather simple but important.

Proposition 2.3. $A(E,W)$ *is a closed ideal of* $[W(z),zW(z)]$.

It turns out that in many cases $[W(z),zW(z)]=A(E)$, so that $A(E,W)$ becomes a closed ideal of $A(E)$ by Proposition 2.3. This situation is the most interesting for us, because then we can employ characterizations of the closed ideals of $A(E)$ for some types of the compact E. This is done in Sections 3 and 4.

Proposition 2.4. $[W(z),zW(z)]=A(E)$ *iff* $1/W(z)\in[W(z),zW(z)]$.

Unfortunately, we do not know any effectively verifiable necessary and sufficient condition on the weight $W(z)$, so that the equality $[W(z),zW(z)]=A(E)$ is valid. Nevertheless, a number of sufficient conditions can be given, guaranteeing that the two algebras $[W(z),zW(z)]$ and $A(E)$ coincide.

Proposition 2.5. *Each of the following conditions implies that* $[W(z),\ zW(z)]=A(E)$*:*

(a) *The point* $\zeta=0$ *belongs to the unbounded component of* $\overline{\mathbb{C}}\setminus W(E)$*;*
(b) E *is the closure of a Jordan domain or a Jordan arc, and* $W(z)$ *is one-to-one on* E*;*
(c) E *is a Jordan arc and* $W(z)$ *is of bounded variation on* E*;*

(d) *E is a Jordan arc and $W(z)$ is locally one-to-one on E;*
(e) *$E = \overline{G}$, where G is a Jordan domain bounded by an analytic curve, and $W'(z) \in A(\overline{G})$.*

Other sufficient conditions, implying the conclusion of Proposition 2.5, can be found in [1], [6], [21, §30] and [20]. Obviously, if the interior of E is empty, then $A(E) = C(E)$ by definition, where $C(E)$ is the algebra of all continuous functions on E.

A very important tool for analyzing the behavior of the weighted polynomials $W^n P_n$ is the theory of logarithmic potentials with external fields (cf. [18]). Assuming that E has positive logarithmic capacity (cf. [23, p. 55]), then

$$w(z) := \begin{cases} |W(z)|, & z \in E, \\ 0, & z \notin E, \end{cases} \tag{2.1}$$

is an *admissible weight* for the *weighted logarithmic energy problem* on E considered in Section I.1 of [18]. This enables us to use certain results of [18], which we summarize below for the convenience of the reader. Recall that the logarithmic potential of a compactly supported Borel measure μ is given by (cf. [23, p. 53])

$$U^{\mu}(z) := \int \log \frac{1}{|z-t|}\, d\mu(t). \tag{2.2}$$

Proposition 2.6. *There exists a positive unit Borel measure μ_w, with support $S_w := \operatorname{supp} \mu_w \subset \partial E$, such that for any polynomial $P_n(z), \deg P_n \le n$, we have*

$$|W^n(z)P_n(z)| \le \|W^n P_n\|_{S_w} \exp(n(F_w - U^{\mu_w}(z) + \log|W(z)|)), \quad z \in E, \tag{2.3}$$

where F_w is a constant.

Furthermore, the inequality

$$U^{\mu_w}(z) - \log|W(z)| \ge F_w \tag{2.4}$$

holds quasi-everywhere on E, and

$$U^{\mu_w}(z) - \log|W(z)| \le F_w, \text{ for any } z \in S_w. \tag{2.5}$$

By saying quasi-everywhere (q.e.), we mean that a property holds everywhere, with the exception of a set of zero logarithmic capacity. The measure μ_w is the solution of a weighted energy problem, corresponding to the weight $w(z)$ of (2.1) (see Section I.1 of [18]).

It follows from (2.3) and (2.4) that the norm of a weighted polynomial $W^n P_n$ essentially "lives" on S_w, i.e.,

$$|W^n(z)P_n(z)| \le \|W^n P_n\|_{S_w} \tag{2.6}$$

for quasi-every $z \in E$. We can replace, in fact, the words "quasi-everywhere on E" in (2.4) and (2.6) by "everywhere on E", by requiring a certain regularity for the set E. In particular, the following is valid (see Corollary III.2.6 of [18]).

Proposition 2.7. *Suppose that for every point $z_0 \in E$, the set $\{z : |z - z_0| < \delta,\ z \in E\}$ has positive capacity for any $\delta > 0$. Then*

$$\|W^n P_n\|_E = \|W^n P_n\|_{S_w} \tag{2.7}$$

for any polynomial $P_n,\ \deg P_n \le n$.

Consequently, the Shilov boundary (cf. [21, pp. 36–39]*) of $A(E, W)$ is contained in S_w.*

One may consult Section III.2 of [18] on further details, regarding the supremum norm of weighted polynomials.

3. Weighted approximation on sets with empty interior

Let E be a compact set with connected complement and empty interior. Obviously, $A(E) = C(E)$ in this case. We characterize $A(E, W)$ in terms of a certain zero set.

Theorem 3.1. *Suppose that E has a connected complement and an empty interior, and that $W \in C(E)$ is a nonvanishing weight on E. Assume that $[W(z), zW(z)] = C(E)$.*

Then, there exists a closed set $Z_W \subset E$ such that

$$f \in A(E, W) \text{ if and only if } f \in C(E) \text{ and } f|_{Z_W} \equiv 0.$$

It is clear that $A(E, W) = C(E)$ if and only if the set Z_W is empty. This is true, for example, for $W(z) \equiv 1$ on E.

Theorem 3.1 generalizes a recent result of Kuijlaars (see Theorem 3 of [10]), related to polynomial approximation with varying weights on the real line. However, it has a new part even in the latter case, allowing us to consider the *complex valued weights* $W(z)$ on subsets of the real line.

A description of the set Z_W in terms of the weight $W(z)$ is unknown in general. We can only show that Z_W must contain the complement of S_w (see Proposition 2.6) in E.

Theorem 3.2. *Let E be an arbitrary compact set with the connected complement $\overline{\mathbb{C}} \setminus E$ and let $W \in A(E)$ be a nonvanishing weight on E. Suppose that for every point $z_0 \in E$, the set $\{z : |z - z_0| < \delta,\ z \in E\}$ has positive logarithmic capacity for any $\delta > 0$. Assume further that $\overline{\mathbb{C}} \setminus S_w$ is connected and $[W(z), zW(z)] = C(S_w)$ on S_w.*

If $f \in A(E, W)$, then $f(z) = 0$ for any $z \in \overline{E \setminus S_w}$. In particular, if E has empty interior, then $\overline{E \setminus S_w} \subset Z_W$.

The proof of Theorem 3.2 is based on an idea of Kuijlaars (see Theorem 2 and its proof in [10]).

If E is a compact subset of the real line and the weight $W(z)$ is real valued, then condition (a) of Proposition 2.5 is clearly satisfied, so that $[W(z), zW(z)] = C(E)$. Therefore, the conclusion of Theorem 3.1 is valid, and coincides with that of Theorem 3 of [10]. Furthermore, if for any point in E, the intersection of its arbitrary neighborhood with E has positive logarithmic capacity, then $\overline{E \setminus S_w} \subset Z_W$. Since $[W(z), zW(z)] = C(S_w)$ on S_w by Proposition 2.5(a), Theorem 3.2 essentially reduces to Theorem 2 of [10] in this case, which in turn contains an earlier result of Theorem 4.1 of [22].

We mention two examples here just for illustrative purposes. A number of additional examples, with their complete discussions, is in [22], [18, Ch. VI], and [8]-[9].

Example 3.3 (Incomplete Polynomials)**.** Let $E = [0, 1]$ and $W(x) = x^{\theta/(1-\theta)}$, where $0 < \theta < 1$. It is known that $S_w = [\theta^2, 1]$ and that $A([0, 1], x^{\theta/(1-\theta)})$ consists of all continuous functions on $[0, 1]$, vanishing on $[0, \theta^2]$ (see [19] and [18, Sect. VI.1]). Thus, the set Z_W of Theorem 3.1 is just $[0, \theta^2] = \overline{E \setminus S_w}$ in this case.

However, it is not always true that $Z_W = \overline{E \setminus S_w}$, as the next example shows.

Example 3.4 (Exponential Weights). Let $E = [-2, 2]$ and $W(x) = \exp(-\gamma_\alpha |x|^\alpha)$, where $\alpha > 0$ and

$$\gamma_\alpha = \frac{\Gamma(\alpha/2)\ \Gamma(1/2)}{2\Gamma(\alpha/2 + 1/2)}.$$

It is known that $S_w = [-1, 1]$ and that

$$A([-2,2], \exp(-\gamma_\alpha |x|^\alpha)) = \begin{cases} \{f \in C([-2,2]) : f|_{[-2,-1]\cup[1,2]} \equiv 0\} & \text{for } \alpha \geq 1, \\ \{f \in C([-2,2]) : f|_{[-2,-1]\cup[1,2]\cup\{0\}} \equiv 0\} & \text{for } 0 < \alpha < 1, \end{cases} \tag{3.1}$$

where the range $\alpha > 1$ was studied in [13], and the case of the remaining interval $0 < \alpha \leq 1$ was covered by the results of [14]. In our context, (3.1) means that $Z_W = [-2, -1] \cup [1, 2] = \overline{E \backslash S_w}$ for $\alpha \geq 1$ and that $Z_W = [-2, -1] \cup [1, 2] \cup \{0\} = \overline{E \backslash S_w} \cup \{0\}$ for $0 < \alpha < 1$.

4. Weighted approximation on the unit disk and on Jordan domains

The first result of this section is a consequence of the well-known description of closed ideals of $A(\overline{D})$, where D is the unit disk, due to Beurling (unpublished) and Rudin [17] (see also [7, pp. 82-87] for a discussion). Recall that g is an *inner function* if it is analytic in D, with $\|g\|_{\overline{D}} \leq 1$, and $|g(e^{i\theta})| = 1$ almost everywhere on the unit circle (cf. [7, p. 62]). By the factorization theorem, every inner function can be uniquely expressed in the form

$$g(z) = B(z)S(z), \quad z \in D, \tag{4.1}$$

where $B(z)$ is a Blaschke product and $S(z)$ is a *singular function*, i.e.,

$$S(z) := \exp\left(-\int \frac{e^{i\theta} + z}{e^{i\theta} - z}\, d\nu_s(\theta)\right), \quad z \in D, \tag{4.2}$$

with ν_s being a positive measure on the unit circle, singular with respect to $d\theta$ (see [7, pp. 63-67]).

Theorem 4.1. *Let a nonvanishing weight $W \in A(\overline{D})$ be such that $[W(z),\ zW(z)] = A(\overline{D})$. Assume that $A(\overline{D}, W)$ contains a not identically zero function.*

Then there exist a closed set $H_W \subset \partial D$ of Lebesgue measure zero and an inner function g_W satisfying

(i) *every accumulation point of the zeros of its Blaschke product is in H_W,*

(ii) *the measure ν_s of its singular function is supported on H_W;*

such that

$f \in A(\overline{D}, W)$ if and only if $f = g_W h$, where $h \in A(\overline{D})$ and $h|_{H_W} \equiv 0$.

The case of a Jordan domain G can be reduced to that of the unit disk, using a canonical conformal mapping $\phi : G \to D$ and its inverse $\psi := \phi^{-1}$. We state the corresponding result for completeness.

Theorem 4.2. *Let a nonvanishing weight $W \in A(\overline{G})$, where G is a Jordan domain, be such that $[W(z), zW(z)] = A(\overline{G})$. Assume that $A(\overline{G}, W)$ contains a not identically zero function.*

Then there exist a closed set $H_W \subset \partial D$ and an inner function g_W, as in Theorem 4.1, such that

$$f \in A(\overline{G}, W) \text{ if and only if } f = (g_W \circ \phi)h,$$

where $h \in A(\overline{G})$ and $h|_{\psi(H_W)} \equiv 0$.

It follows from Theorems 4.1 and 4.2 that $A(\overline{D}, W) = A(\overline{D})$ or $A(\overline{G}, W) = A(\overline{G})$ if and only if $g_W \equiv 1$ and H_W is empty.

Example 4.3 (Exponential weight on the Szegő domain)**.** Let $W(z) = e^{-z}$ and G be the Szegő domain

$$\{z : |ze^{1-z}| < 1 \text{ and } |z| < 1\}, \tag{4.3}$$

which is bounded by a piecewise analytic curve (the *Szegő curve*) with the only corner point $z = 1$ (see [15] for more information and the graph of G). We remark that the function $\phi(z) = ze^{1-z}$ maps G conformally onto D. It follows from Proposition 3.1 of [15] that $f_0(z) := (z-1)e^{-z}$ belongs to $A(\overline{G}, e^{-z})$. Thus, either $H_W = \{1\}$ or H_W is empty. Clearly, $g_W \equiv 1$ and we obtain from Theorem 4.2 that if $f \in A(\overline{G})$ and $f(1) = 0$, then $f \in A(\overline{G}, e^{-z})$. This implies the result of Theorem 3.2 of [15], in particular. It was also conjectured in [15] that $H_W = \{1\}$ (in the present notation), but this problem remains open.

Our next goal is to exhibit the role of the set S_w (see Proposition 2.6) in the case of weighted approximation on Jordan domains. Since $W \in A(\overline{G})$ is analytic in G, then $S_w \subset \partial G$ by Theorem IV.1.10(a) of [18] and (2.1). The following result shows that $S_w = \partial G$ is necessary for nontrivial weighted approximation on $\overline{G}$.

Theorem 4.4. *Let G be a Jordan domain and let $W \in A(\overline{G})$ be a nonvanishing weight. Assume that S_w is a proper subset of ∂G and that $[W(z),\ zW(z)] = C(S_w)$ on S_w. Then $A(\overline{G}, W)$ contains the identically zero function only.*

We construct below a specific example of the weight $W(z)$, satisfying the conditions of Theorem 4.4.

Example 4.5. Let G be a Jordan domain, as before, and let γ be a proper closed subarc of ∂G. Define the conformal mapping $\Phi : \overline{\mathbb{C}}\backslash\gamma \to \{w : |w| > 1\}$, normalized by $\Phi(\infty) = \infty$ and $\lim_{z\to\infty} \Phi(z)/z > 0$. We extend Φ to γ, using boundary limits *from inside of* G, so that $\Phi \in A(\overline{G})$ is one-to-one on $\overline{G}$. Thus, we can consider approximation by weighted polynomials on $\overline{G}$, with $W(z) = 1/\Phi(z),\ z \in \overline{G}$.

Denote the Green function of the domain $\overline{\mathbb{C}}\backslash\gamma$, with pole at ∞, by $g(z, \infty)$ [23, p. 14], and the classical equilibrium distribution of γ (in the sense of logarithmic potential theory) by μ_γ [23, p. 55]. Then,

$$g(z, \infty) = \log \frac{1}{\text{cap}(\gamma)} - U^{\mu_\gamma}(z), \quad z \in \mathbb{C},$$

where $\text{cap}(\gamma)$ is the logarithmic capacity of γ (see [18, Sect. I.4]). Furthermore, since $g(z, \infty) = \log|\Phi(z)|,\ z \in \mathbb{C}$ (cf. [23, p. 18]), we obtain

$$U^{\mu_\gamma}(z) - \log|1/\Phi(z)| = \log \frac{1}{\text{cap}(\gamma)}, \quad z \in \mathbb{C}. \tag{4.4}$$

It follows from (4.4) and Theorem I.3.3 of [18] that μ_γ is the solution of a weighted energy problem (cf. [18, Sec. I.1]) for the weight $w(z) = 1/|\Phi(z)|$ on $\overline{G}$. Hence, we have shown that $S_w = \text{supp}\ \mu_\gamma = \gamma$.

Observe that the weighted polynomials $W^n(z)P_n(z)$, with $W(z) = 1/\Phi(z)$, $z \in \overline{G}$, can approximate an arbitrary analytic function in G on compact subsets of G by Theorem 1.1 of [16] and (4.4). On the other hand, Proposition 2.5(b) and Theorem 4.4 indicates that they can only approximate the identically zero function uniformly on $\overline{G}$.

5. Proofs

Proof of Proposition 2.1. We have to show that $A(E,W)$ is closed under addition, multiplication by constants and by functions of $A(E,W)$, and under uniform limits. Suppose that $W^nP_n \to f \in A(E,W)$ and $W^nQ_n \to g \in A(E,W)$ uniformly on E, as $n \to \infty$. Then $W^n(P_n + Q_n) \to (f+g)$, as $n \to \infty$, so that $(f+g) \in A(E,W)$. If $\alpha \in \mathbb{C}$ then $W^n\alpha P_n \to \alpha f$, as $n \to \infty$, and $\alpha f \in A(E,W)$. Observe that

$$\begin{aligned}&\|fg - W^{2n}P_nQ_n\|_E \le \|fg - fW^nQ_n\|_E \ + \ \|fW^nQ_n - W^{2n}P_nQ_n\|_E \le \\ &\|f\|_E\ \|g - W^nQ_n\|_E + \|W^nQ_n\|_E\ \|f - W^nP_n\|_E \to 0,\end{aligned}$$

as $n \to \infty$, i.e., $fg \in A(E,W)$. Applying the standard diagonalization argument, we see that $A(E,W)$ is closed in norm (1.1). □

Proof of Proposition 2.3. Assume that $f \in A(E,W)$ and $W^nP_n \to f$ uniformly on E, as $n \to \infty$. Then, for any pair of nonnegative integers k and ℓ such that $k \ge \ell$, we have

$$\begin{aligned}&\|f(z)W^k(z)z^\ell - W^{n+k}(z)z^\ell P_n(z)\|_E \le \\ &\|W^k(z)z^\ell\|_E\ \|f - W^nP_n\|_E \to 0, \qquad \text{as } n \to \infty,\end{aligned}$$

which gives that $f(z)W^k(z)z^\ell \in A(E,W)$. Since $A(E,W)$ is closed under addition and multiplication by constants (by Proposition 2.1), then the product of f and any polynomial in $W(z)$ and $zW(z)$ belongs to $A(E,W)$. Thus, if $g \in [W(z),\ zW(z)]$ then $fg \in A(E,W)$ follows immediately, because $A(E,W)$ is closed in the uniform norm on E (cf. Proposition 2.1). The proof is now complete in view of Propositions 2.1 and 2.2. □

Proof of Proposition 2.4. Obviously, if $[W(z), zW(z)] = A(E)$ then $1/W(z) \in A(E)$ $= [W(z), zW(z)]$.

Assume that $1/W(z) \in [W(z), zW(z)]$. It follows that $z \in [W(z),\ zW(z)]$ and, consequently, every polynomial in z is in $[W(z),\ zW(z)]$. Since $[W(z), zW(z)]$ is uniformly closed on E by definition, then $A(E) \subset [W(z),\ zW(z)]$ by Mergelyan's theorem [5, p. 48]. Thus, Proposition 2.2 implies at once that $A(E) = [W(z), zW(z)]$. □

Proof of Proposition 2.5. First, we remark that $W(z)$ and $zW(z)$ together separate points of any set E.

(a) Observe that $W(E)$, the image of E in ζ-plane under the mapping $\zeta = W(z)$, is compact. By assumption, function $1/\zeta$ is analytic on the polynomial convex hull of $W(E)$ and can be uniformly approximated there by polynomials in ζ (by Mergelyan's theorem). Returning to z-plane, we obtain that $1/W(z)$ is uniformly approximable on E by polynomials in $W(z)$. It follows that $[W(z), zW(z)] = A(E)$ by Proposition 2.4.

(b) The mapping $\zeta = W(z)$ can be continued as a homeomorphism between z-plane and ζ-plane (cf. [11, p. 535]). Since $W(z)$ doesn't vanish on E, then $\zeta = 0$ belongs to the domain $\overline{\mathbb{C}}\backslash W(E) = W(\overline{\mathbb{C}}\backslash E)$, which contains $\zeta = \infty$. Hence, (b) follows from (a).

(c) If $E = [0,1]$ then (c) is a direct consequence of Theorem 2 of [2]. For E being a Jordan arc, we consider a homeomorphic parametrization of E by $\tau : [0,1] \to E$. Since $W \circ \tau(x)$ is of bounded variation on $[0,1]$, we have, as before, that $[W \circ \tau(x), \tau(x)(W \circ \tau)(x)] = C([0,1])$. Clearly, τ induces an isometric isomorphism between $C([0,1])$ and $C(E)$. Thus, the result follows after returning to E with the help of τ^{-1}.

(d) is implied by Theorem 1 of [1] for $E = [0,1]$. The case of a Jordan arc can be reduced to that of the interval as in the proof of (c).

(e) First, assume that $E = \overline{D}$. Then (e) follows at once from [24, p. 135] (see also [3]). It is well known that the conformal mapping $\phi : G \to D$ extends as a diffeomorphism between $\overline{G}$ and $\overline{D}$ (with nonvanishing derivatives of ϕ and $\psi := \phi^{-1}$), because G is bounded by an analytic Jordan curve. Using ϕ, the result for $E = \overline{G}$ is a consequence of [24, p. 135] too. □

Proof of Proposition 2.6. Since $W(z)$ is a continuous nonvanishing function on E and $w(z)$ of (2.1) is so too, then the existence of μ_w and inequalities (2.4)-(2.5) follow from Theorem I.1.3 of [18]. Moreover, $W(z)$ is analytic in the interior of E, which implies that $S_w \subset \partial E$ by Theorem IV.1.10(a) of [18] and (2.1). The inequality (2.3) is a direct consequence of Theorem III.2.1 of [18]. □

Proof of Theorem 3.1. We have that $[W(z), zW(z)] = C(E)$ by the assumption of the theorem. Thus, $A(E,W)$ is a closed ideal of $C(E)$ (cf. Proposition 2.3), which is known to be described by its zero set (see [21, p. 32]). □

Proof of Theorem 3.2. We essentially follow the proof of Theorem 2 of [10]. Suppose that there exist $f_0 \in A(E,W)$ and $z_0 \in E \backslash S_w$ such that $f_0(z_0) \neq 0$ and $W^n P_n \to f_0$ uniformly on E, as $n \to \infty$.

It is clear that $f_0|_{S_w} \in A(S_w, W)$. Recall that $S_w \subset \partial E$ by Proposition 2.6, i.e., S_w has empty interior. Applying Theorem 3.1, with E replaced by S_w, we obtain that $A(S_w, W)$ is described by the zero set $Z_W^* \subset S_w$. Observe that multiplying $A(S_w, W)$ by $(z - z_0)W(z)$, we obtain a closed ideal of $[W(z), zW(z)] = C(S_w)$ (cf. Proposition 2.3), which consists of all functions, uniformly approximable on S_w by the weighted polynomials $W^n(z)Q_n(z)$ such that $Q_n(z_0) = 0$, as $n \to \infty$. On the other hand, the zero set of the ideal $(z - z_0)W(z)A(S_w, W)$ coincides with that of $A(S_w, W)$. It follows that $(z - z_0)W(z)A(S_w, W) = A(S_w, W)$ (see [21, p. 32]) and that $f_0|_{S_w} \in (z - z_0)W(z)A(S_w, W)$.

Thus, there exists a sequence of the weighted polynomials $\{W^n Q_n\}_{n=0}^{\infty}$, with $Q_n(z_0) = 0$, uniformly convergent to f_0 on S_w, as $n \to \infty$. Since $W^n(z)(P_n(z) - Q_n(z))$ converges to zero uniformly on S_w and converges to $f_0(z_0) \neq 0$ for $z = z_0 \in E \backslash S_w$, as $n \to \infty$, then we obtain a direct contradiction with (2.7) for some sufficiently large n.

Consequently, if $f \in A(E,W)$ then $f(z) = 0$ for any $z \in E \backslash S_w$. Furthermore, the same is true for any $z \in \overline{E \backslash S_w}$ by the continuity of $f(z)$. □

Proof of Theorem 4.1. Since $[W(z),\ zW(z)] = A(\overline{D})$ by the assumption of the theorem, then $A(\overline{D}, W)$ is a closed ideal of $A(\overline{D})$ by Proposition 2.3. The result now follows from the description of nontrivial closed ideals of the disk algebra (see [17] and [7, pp. 82-87]). □

Proof of Theorem 4.2. First, we obtain that $A(\overline{G}, W)$ is a closed ideal of $A(\overline{G})$ (cf. Proposition 2.3). It is well known that the conformal mapping ϕ extends to a homeomorphism between $\overline{G}$ and $\overline{D}$, defining an isometric isomorphism between the algebras $A(\overline{G})$ and $A(\overline{D})$, and their closed ideals. Thus, we apply the result of [17] to the isomorphic image of $A(\overline{G}, W)$ in $A(\overline{D})$, as in the proof of Theorem 4.1, and return to $A(\overline{G}, W)$ with the help of the inverse conformal mapping ψ. □

Proof of Theorem 4.4. Since G is a Jordan domain, then the set $\{z : |z - z_0| < \delta,\ z \in \overline{G}\}$ has positive logarithmic capacity for any $z_0 \in \overline{G}$ and $\delta > 0$. It is clear that S_w is contained in some Jordan arc, as a proper closed subset of ∂G, so that $\overline{\mathbb{C}} \backslash S_w$ is connected. Observe that all conditions of Theorem 3.2 are satisfied in this case, which yields that any function $f \in A(\overline{G}, W)$ must vanish on $\overline{(\overline{G} \backslash S_w)} = \overline{G}$. □

6. Further remarks

One of the main assumptions of the theorems in Sections 3 and 4 is that $[W(z), zW(z)] = A(E)$. This equality is valid for a wide classes of sets E and weights $W \in A(E)$, as described in Proposition 2.5. On the other hand, it is clear that our results may be extended further. The main ingredients of such extentions are:

(i) a proof of the equality $[W(z), zW(z)] = A(E)$;
(ii) a description of the closed ideals of $A(E)$.

Nevertheless, we believe that a problem of much greater interest and difficulty is to uncover a more explicit relation between the zero set Z_W (or H_W and the inner function g_W) and the weight $W(z)$. A considerable progress has been achieved on this problem for real valued weights on the real line (see [22], [18, Ch. VI] and [8]-[9]), but a general description of the set Z_W through $W(z)$ is unknown even in the latter case.

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