Weighted Polynomial Approximation in the Complex Plane*

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Abstract

Given a pair (G, W) of an open bounded set G in the complex plane and a weight function $W(z)$ which is analytic and different from zero in G, we consider the problem of the locally uniform approximation of any function $f(z)$, which is analytic in G, by weighted polynomials of the form $\{W^n(z)P_n(z)\}_{n=0}^{\infty}$, where $\deg P_n \leq n$. The main result of this paper is a necessary and sufficient condition for such an approximation to be valid. We also consider a number of applications of this result to various classical weights, which give explicit criteria for these weighted approximations.

Key words. weighted polynomials, weighted energy problem, logarithmic potential, balayage, modified Robin constant.

AMS subject classification. 30E10, 30C15, 31A15, 41A30.

*Version of 11/5/96. Received, etc.

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1 Introduction and General Result

In this paper, we will examine pairs of the form

$$(G, W) \tag{1.1}$$

where

$$\begin{cases} i) & G \text{ is an open bounded set, in the complex plane } \mathbb{C}, \text{ which can be represented as a finite or countable union of disjoint simply connected domains, i.e., } G = \bigcup_{\ell=1}^{\sigma} G_\ell \text{ (where } 1 \le \sigma \le \infty\text{);} \\ ii) & W(z), \text{ the weight function, is analytic in } G \text{ with } W(z) \neq 0 \text{ for any } z \in G. \end{cases} \tag{1.2}$$

We say that the pair (G, W) has the **approximation property** if,

$$\begin{cases} \text{for any } f(z) \text{ which is analytic in } G \text{ and for any compact subset } E \text{ of } G, \text{ there exists a sequence of polynomials } \{P_n(z)\}_{n=0}^{\infty}, \text{ with } \deg P_n \le n \text{ for all } n \ge 0, \text{ such that} \\ \lim_{n\to\infty} \|f - W^n P_n\|_E = 0, \end{cases} \tag{1.3}$$

where all norms throughout this paper are the uniform (Chebyshev) norms on the indicated sets.

Given a pair (G, W), as in (1.2), we state below our main result, Theorem 1.1, which gives a characterization, in terms of potential theory, for the pair (G, W) to have the approximation property. For notation, let $\mathcal{M}(E)$ be the space of all positive unit Borel measures on $\mathbb{C}$ which are supported on a compact set E, i.e., for any $\mu \in \mathcal{M}(E)$, we have $\mu(\mathbb{C}) = 1$ and $\operatorname{supp}\mu \subset E$. The logarithmic potential of a compactly supported measure μ is defined (cf. Tsuji [12, p. 53]) by

$$U^{\mu}(z) := \int \log \frac{1}{|z-t|}\, d\mu(t). \tag{1.4}$$

Theorem 1.1 *A pair (G, W), as in (1.2), has the approximation property (1.3) if and only if there exist a measure $\mu(G, W) \in \mathcal{M}(\partial G)$ and a constant $F(G, W)$ such that*

$$U^{\mu(G,W)}(z) - \log|W(z)| = F(G, W), \qquad \textit{for any } z \in G. \tag{1.5}$$

Remark 1.2 It is well known that any open set in the complex plane is a finite or countable union of disjoint domains, and this is more general than the assumption on the open set G in (1.2i). However, we note that the approximation property (1.3) *cannot hold*, even in the classical case where $W(z) \equiv 1$ for all $z \in G$, if $G = \bigcup_{\ell=1}^{\sigma} G_\ell$, when some G_ℓ is multiply connected (cf. Walsh [13, p. 25]). In this sense, our initial assumptions on G are quite general.

Remark 1.3 The condition that $W(z) \neq 0$ for all $z \in G$ cannot be dropped, for if $W(z_0) = 0$ for some $z_0 \in G_\ell$, where $G = \bigcup_{\ell=1}^{\sigma} G_\ell$, then the necessarily null sequence $\{W^n(z_0)P_n(z_0)\}_{n=0}^{\infty}$ trivially fails to converge to any $f(z)$, analytic in G, with $f(z_0) \neq 0$; whence, the approximation property fails. Even more decisive is the result, to be proved in Section 4, that if $W(z_0) = 0$ for some $z_0 \in G_\ell$, then the sequence $\{W^n(z)P_n(z)\}_{n=0}^{\infty}$ can converge, locally uniformly in G, to $f(z)$, only if $f(z) \equiv 0$ in G_ℓ. In this sense, the assumptions on $W(z)$ are also quite general.

Remark 1.4 The measure $\mu(G, W)$ of Theorem 1.1 is, in some cases, related to the solution of a minimal weighted energy problem, discussed in Section 3 (cf. Theorem 3.2).

Remark 1.5 In the case $W(z) \equiv 1$ of Theorem 1.1, the result, that the approximation property (1.3) holds, is a known classical result in complex approximation theory (cf. [13, p. 26]). This also follows from Theorem 1.1 because the measure $\mu(G, 1)$ exists by Theorems III.12 and III.14 of Tsuji [12], and is the classical equilibrium distribution measure (in the sense of logarithmic potential theory) for $\overline{G}$.

The topic of weighted approximation by $\{W^n(z)P_n(z)\}_{n=0}^{\infty}$, on the real line, has been extensively and thoroughly treated in the recent books of Saff and Totik [9] and Totik [11]. Here, we emphasize weighted approximation *in the complex plane*, which has received far less attention in the current approximation theory literature, with the exception of the recent papers by Borwein and Chen [1] and Pritsker and Varga [8].

We shall present in Section 2 a number of applications of Theorem 1.1 to special pairs (G, W). Section 3 is devoted to a weighted energy problem and weighted potentials; as will be clear, the major tools for our research come from potential theory. The proofs of all results and remarks on weighted approximation, stated in Sections 1 and 2, are given in Section 4. Finally, we conclude this paper with Section 5, where further remarks, open problems and a discussion of possible generalizations are given.

2 Applications

Finding the measure $\mu(G,W)$ of Theorem 1.1 or verifying its existence is a nontrivial problem in general. Since $U^{\mu(G,W)}(z)$ is harmonic in $\mathbb{C}\setminus \operatorname{supp}\mu(G,W)$ and, since it can be shown from (1.5), if $\log|W(z)|$ is continuous on $\overline{G}$ and if G is a *finite* union of $G_\ell, \ell = 1, 2, \ldots, \ell_0$, that $U^{\mu(G,W)}(z)$ is equal to $\log|W(z)| + F(G,W)$ on $\operatorname{supp}\mu(G,W)$, then $U^{\mu(G,W)}(z)$ can be found as the solution of the corresponding Dirichlet problems. The measure $\mu(G,W)$ can be recovered from its potential, using the Fourier method described in Section IV.2 of Saff and Totik [9]. This method has already been used successfully by the authors in [8] to study the approximation of analytic functions by the weighted polynomials $\{e^{-nz}P_n(z)\}_{n=0}^{\infty}$, i.e., when $W(z) := e^{-z}$, and it is also used in the proof of Theorem 2.7, given in Section 4.

In contrast to the above procedure, we next consider a different method, dealing with specific weight functions, which allows us to deduce "explicit" expressions for the measure $\mu(G,W)$ of Theorem 1.1, and to treat some important cases of pairs (G,W). For simplicity, we assume throughout this section that G is given as in (1.2i), but with σ finite. We denote the unbounded component of $\overline{\mathbb{C}}\setminus\overline{G}$ by Ω. Let ν_1 and ν_2 be two unit positive Borel measures on $\mathbb{C}$ with compact supports satisfying

$$\operatorname{supp}\nu_1 \subset \overline{\mathbb{C}}\setminus G \text{ and } \operatorname{supp}\nu_2 \subset \overline{\mathbb{C}}\setminus G, \tag{2.1}$$

such that

$$\nu_1(\mathbb{C}) = \nu_2(\mathbb{C}) = 1. \tag{2.2}$$

For real numbers α and β, assume that $W(z)$, satisfying

$$\log|W(z)| = -\left(\alpha U^{\nu_1}(z) + \beta U^{\nu_2}(z)\right), \quad z \in G, \tag{2.3}$$

is analytic in G. Then, we state, as an application of Theorem 1.1, our next result as

Theorem 2.1 *Given any pair of real numbers α and β, given an open bounded set $G = \bigcup_{\ell=1}^{\sigma} G_\ell$ as in* (1.2i) *with σ finite, and given the weight function $W(z)$ of* (2.3)*, then the pair (G, W) has the approximation property* (1.3) *if and only if the measure*

$$\mu := (1+\alpha+\beta)\omega(\infty, \cdot, \Omega) - \alpha\hat{\nu}_1 - \beta\hat{\nu}_2 \tag{2.4}$$

is positive, where $\omega(\infty, \cdot, \Omega)$ is the harmonic measure at ∞ with respect to Ω; here, $\hat{\nu}_1$ and $\hat{\nu}_2$ are, respectively, the balayages of ν_1 and ν_2 from $\overline{\mathbb{C}} \backslash \overline{G}$ to $\overline{G}$.

Furthermore, if μ of (2.4) *is a positive measure, then* (cf. Theorem 1.1)

$$\mu(G, W) = \mu \quad \textit{and} \quad \operatorname{supp} \mu(G, W) \subset \partial G. \tag{2.5}$$

We point out that the harmonic measure $\omega(\infty, \cdot, \Omega)$ (cf. Nevanlinna [7] and Tsuji [12]) is the same as the equilibrium distribution measure for $\overline{G}$, in the sense of classical logarithmic potential theory [12]. For the notion of balayage of a measure, we refer the reader to Chapter IV of Landkof [5] or Section II.4 of Saff and Totik [9].

In the following series of subsections, we consider various classical weight functions and find their corresponding measures, associated with the weighted approximation problem in G by Theorem 1.1.

2.1 Incomplete Polynomials and Laurent Polynomials

With $\mathbb{N}_0$ and $\mathbb{N}$ denoting respectively the sets of nonnegative and positive integers, the *incomplete polynomials* of Lorentz [6] are a sequence of polynomials of the form

$$\left\{z^{m(i)}P_{n(i)}(z)\right\}_{i=0}^{\infty}, \quad \deg P_{n(i)} \le n(i),\ (m(i), n(i) \in \mathbb{N}_0), \tag{2.6}$$

where it is assumed that $\lim_{i\to\infty} \frac{m(i)}{n(i)} =: \alpha$, where $\alpha > 0$ is a real number. The question of the possibility of approximation by incomplete polynomials is closely connected to that of approximation by the weighted polynomials

$$\{z^{\alpha n}P_n(z)\}_{n=0}^{\infty}, \quad \deg P_n \le n. \tag{2.7}$$

The question of approximation by the incomplete polynomials of (2.6) was completely settled by Saff and Varga [10], and by v. Golitschek [3] on the interval $[0, 1]$ (see Totik [11] and Saff and Totik [9] for the associated history and later developments). We consider now the analogous problem in the complex plane. Since the weight $W(z) := z^{\alpha}$ in (2.7) is multiple valued in $\mathbb{C}$ if $\alpha \notin \mathbb{N}_0$, we then restrict ourselves to the slit domain $S_1 := \mathbb{C}\setminus(-\infty, 0]$ and the single valued branch of $W(z)$ in S_1 satisfying $W(1) = 1$.

For the related question of the approximation by the so-called Laurent polynomials

$$\left\{\frac{P_{n(i)}(z)}{z^{m(i)}}\right\}_{i=0}^{\infty}, \deg P_{n(i)} \le n(i),\ (m(i), n(i) \in \mathbb{N}_0), \tag{2.8}$$

where $\lim_{i\to\infty} \frac{m(i)}{n(i)} := \alpha, \alpha > 0$, we are similarly led to the question of the approximation by the weighted polynomials

$$\left\{z^{-\alpha n}P_n(z)\right\}_{n=0}^{\infty}, \deg P_n \le n, \tag{2.9}$$

with the only difference being in the sign in the exponent of the weight function. Thus, we can give a unified treatment of both problems by considering weighted approximation by $\{W^n(z)P_n(z)\}_{n=0}^{\infty}$, $\deg P_n \le n$, with

$$W(z) := z^{\alpha}, \quad z \in S_1 := \mathbb{C}\backslash(-\infty, 0], \tag{2.10}$$

where α is *any* fixed real number and where we choose, as before, the single valued branch of $W(z)$ in S_1 satisfying $W(1) = 1$.

Theorem 2.2 *Given an open set G as in* (1.2i) *with σ finite, such that $\overline{G} \subset S_1$, and given the weight function $W(z)$ of* (2.10)*, then the pair (G, W) has the approximation property* (1.3) *if and only if*

$$\mu = (1+\alpha)\omega(\infty, \cdot, \Omega) - \alpha\omega(0, \cdot, \Omega) \tag{2.11}$$

is a positive measure, where $\omega(\infty, \cdot, \Omega)$ and $\omega(0, \cdot, \Omega)$ are, respectively, the harmonic measures with respect to the unbounded component Ω of $\overline{\mathbb{C}}\backslash\overline{G}$, at $z = \infty$ and at $z = 0$.

In some cases, when the geometric shape of G is given explicitly, we can determine the explicit form of the measure of (2.11). This is especially easy to do for disks.

Corollary 2.3 *Given the disk* $D_r(a) := \{z \in \mathbb{C} : |z-a| < r\}$, *where* $a \in (0, +\infty)$ *and where* $\overline{D}_r(a) \subset S_1 = \mathbb{C} \backslash (-\infty, 0]$, *i.e.,* $r < a$, *and given the weight function of* (2.10), *then the pair* $(D_r(a), W)$ *has the approximation property* (1.3) *if and only if*

$$r \leq r_{\max}(a,\ \alpha) = \begin{cases} a,\ \ \alpha \in [-1, 0], \\ \dfrac{a}{|2\alpha+1|},\ \ \alpha \in (-\infty, -1) \cup (0, \infty). \end{cases} \tag{2.12}$$

Furthermore, if (2.12) *is satisfied, then the associated measure* $\mu(D_r(a), z^\alpha)$ (see Theorem 1.1) *is given by*

$$d\mu(D_r(a), z^\alpha) = \left(1 + \alpha - \alpha \frac{a^2 - r^2}{|z|^2}\right) \frac{ds}{2\pi r}, \tag{2.13}$$

where ds *is the arclength measure on the circle* $|z - a| = r$.

2.2 Jacobi and Jacobi-Type Weights

We continue along the same line by considering weighted approximation with Jacobi weights, i.e., we set

$$W(z) := (1-z)^\alpha (1+z)^\beta, \quad z \in S_2 := \mathbb{C} \backslash \{(-\infty, -1] \cup [1, \infty)\}, \tag{2.14}$$

where $\alpha, \beta \in \mathbb{R}$ are any numbers, and where we choose the branch of weight function in (2.14) such that $W(0) = 1$.

An analogue of Theorem 2.2 in this case is the following result:

Theorem 2.4 *Given an open set G as in* (1.2i) *with σ finite, such that $\overline{G} \subset S_2$, and given the weight function $W(z)$ of* (2.14)*, then the pair (G, W) has the approximation property* (1.3) *if and only if*

$$\mu = (1 + \alpha + \beta)\omega(\infty, \cdot, \Omega) - \alpha\omega(1, \cdot, \Omega) - \beta\omega(-1, \cdot, \Omega) \tag{2.15}$$

is a positive measure, where Ω is the unbounded component of $\overline{\mathbb{C}} \backslash \overline{G}$.

We next state a corollary of Theorem 2.4, which deals with the explicit formula for the radius of a *largest* disk $D_r(a)$, centered at $a \in (-1, 1)$, for which $(D_r(a), W)$ has the approximation property.

Corollary 2.5 *Given the disk $D_r(a) := \{z \in \mathbb{C} : |z - a| < r\}$, with $a \in (-1, 1)$ and with $\overline{D}_r(a) \subset S_2$, and given the Jacobi weight function $W(z)$ of* (2.14)*, then the pair $(D_r(a), W)$ has the approximation property* (1.3) *if and only if*

$$1 + \alpha + \beta - \alpha\frac{(1-a)^2 - r^2}{|z-1|^2} - \beta\frac{(1+a)^2 - r^2}{|z+1|^2} \geq 0 \ \text{ on } |z - a| = r. \tag{2.16}$$

In particular, if $\alpha \geq 0$ and $\beta \geq 0$, then the approximation property (1.3) *holds if and only if*

$$r \leq r_{\max}(a, \alpha, \beta) := \frac{\sqrt{[\alpha - \beta + a(1 + \alpha + \beta)]^2 + (1 - a^2)(1 + 2\alpha + 2\beta)} - |\alpha - \beta + a(1 + \alpha + \beta)|}{1 + 2\alpha + 2\beta}. \tag{2.17}$$

Furthermore, if (2.16) *is valid, then*

$$d\mu\left(D_r(a),(1-z)^\alpha(1+z)^\beta\right)=$$

(2.18)

$$=\left(1+\alpha+\beta-\alpha\frac{(1-a)^2-r^2}{|z-1|^2}-\beta\frac{(1+a)^2-r^2}{|z+1|^2}\right)\ \frac{ds}{2\pi r},$$

where ds is the arclength measure on $|z-a|=r$.

Both weight functions, introduced in (2.10) and (2.14), are special cases of the following Jacobi-type weight function

$$W(z):=\prod_{i=1}^{p}(z-t_i)^{\alpha_i}, \tag{2.19}$$

where $\{\alpha_i\}_{i=1}^p$ are real numbers and where $\{t_i\}_{i=1}^p\subset\mathbb{C}$ is a fixed set of distinct points. For a given open set G (as in (1.2i) with σ finite) such that $t_i\notin\overline{G},\ i=1,...,p$, we assume that there exist p cuts, connecting each t_i with ∞. Then, we can define a single valued branch of $W(z)$ in the p-slit complex plane which contains $\overline{G}$ in its interior. (It is not possible to specify in advance those cuts, as they necessarily depend on each preassigned open set G.)

Theorem 2.6 *The pair* (G,W)*, defined in the previous paragraph, has the approximation property* (1.3) *if and only if*

$$\mu=\left(1+\sum_{i=1}^{p}\alpha_i\right)\omega(\infty,\cdot,\Omega)-\sum_{i=1}^{p}\alpha_i\omega(t_i,\cdot,\Omega) \tag{2.20}$$

is a positive measure, where Ω *is the unbounded component of* $\overline{\mathbb{C}}\backslash\overline{G}$.

Furthermore, if $G=D_r(a):=\{z\in\mathbb{C}:|z-a|<r\}$ *where* $a\in\mathbb{C}$*, then the pair* $(D_r(a),W)$ *has the approximation property* (1.3) *if and only if*

(2.21) $$1+\sum_{i=1}^{p}\alpha_i-\sum_{i=1}^{p}\alpha_i\frac{|t_i-a|^2-r^2}{|z-t_i|^2}\geq 0,\quad |z-a|=r.$$

2.3 Exponential Weights

Let

(2.22) $$W(z):=e^{-z^m},\quad m\in\mathbb{N}.$$

The special case $m=1$ of the weight function (2.22) was considered in [8]. To avoid technical complications, we shall study only the weighted approximation, with respect to the weight function $W(z)=e^{-z^m}$ in disks centered at the origin. Our next result generalizes Theorems 3.8 and 4.3 of [8].

Theorem 2.7 *Given* $D_r(0):=\{z\in\mathbb{C}:|z|<r\}$ *and given the weight function* $W(z)$ *of* (2.22)*, then the pair* $(D_r(0),W)$ *has the approximation property* (1.3) *if and only if*

(2.23) $$r\leq r_{\max}(m):=(2m)^{-1/m},\quad m\in\mathbb{N}.$$

Moreover, if (2.23) *holds, then*

(2.24) $$d\mu\left(D_r(0),e^{-z^m}\right)=(1-2mr^m\cos m\theta)\,\frac{d\theta}{2\pi},$$

where $d\theta$ *is the angular measure on* $|z|=r$ *and where* $z=re^{i\theta}$*.*

3 The Weighted Energy Problem

We establish an important connection between the weighted approximation of analytic functions in the complex plane and the weighted potential theory developed during the last two decades (see the excellent and thorough exposition of [9]). In particular, we show that, under certain conditions, the measure $\mu(G, W)$ of Theorem 1.1 is related to the solution of a minimal weighted energy problem, which enables us to use the powerful tools of [9] and to provide a basis for a complete analysis of the problem of weighted polynomial approximation.

Let G be an arbitrary bounded open set in the complex plane and let $W(z) \not\equiv 0$ be a weight function which is analytic in G and continuous on $\overline{G}$. From W, we define the function

$$w(z) := |W(z)|, \quad z \in \overline{G}, \tag{3.1}$$

so that w is continuous on $\overline{G}$ and is *admissible* in the sense of [9, Section I.1]. Following [9], we set

$$Q(z) := -\log |W(z)|, \quad z \in \overline{G}, \tag{3.2}$$

so that

$$w(z) = e^{-Q(z)}, \quad z \in \overline{G}. \tag{3.3}$$

With $\mathcal{M}(\overline{G})$ denoting the class of all positive Borel measures μ on $\mathbb{C}$ such that $\mu(\mathbb{C}) = 1$ and $\operatorname{supp} \mu \subset \overline{G}$, consider the following weighted energy problem (cf. [9, Section I.1]):

For the weighted energy integral

$$I_w(\mu) := \iint \log \frac{1}{|z-t|w(z)w(t)}\, d\mu(z)d\mu(t), \quad \mu \in \mathcal{M}(\overline{G}), \tag{3.4}$$

find

$$V_w := \inf_{\mu \in \mathcal{M}(\overline{G})} I_w(\mu), \tag{3.5}$$

and identify the extremal measure $\mu_w \in \mathcal{M}(\overline{G})$ for which the infimum in (3.5) is attained.

The following is a special case of Theorems I.1.3 and IV.1.10 (a) of [9].

Theorem 3.1 *For the function $w(z)$ defined by* (3.1)*,*

(a) V_w *of* (3.5) *is finite;*

(b) *there exists a unique $\mu_w \in \mathcal{M}(\overline{G})$ such that $I_w(\mu_w) = V_w$ and* $\operatorname{supp} \mu_w \subset \partial G$*;*

(c) $U^{\mu_w}(z) + Q(z) \geq F_w$*, for quasi every $z \in \overline{G}$, where $F_w := V_w - \int Q(t)d\mu_w(t)$;*

(d) $U^{\mu_w}(z) + Q(z) \leq F_w,\ z \in \operatorname{supp} \mu_w$.

By saying in (c) that a property holds quasi everywhere (q.e.), we mean that it holds everywhere, with the possible exception of a set of zero logarithmic capacity (cf. [9, Sec. I.1]).

The next theorem is also a special case of the results in [9], which is stated in a form convenient for use in our proofs in the next section.

Theorem 3.2 *Let* $G = \cup_{\ell=1}^{\ell_0} G_\ell$ *be the finite union of disjoint Jordan domains* $\{G_\ell\}_{\ell=1}^{\ell_0}$ *and let* $W(z)$ *be analytic in* G *and continuous in* $\overline{G}$ *with* $W(z) \neq 0$ *for any* $z \in \overline{G}$. *Suppose that the measure* $\mu(G, W) \in \mathcal{M}(\overline{G})$ *of* Theorem 1.1 *exists, i.e.,*

$$U^{\mu(G,W)}(z) - \log|W(z)| = F(G, W), \quad z \in G, \tag{3.6}$$

where $\operatorname{supp}\mu(G, W) \subset \partial G$ *and where* $F(G, W)$ *is a constant. Then,* $\mu(G, W)$ *is the solution of weighted energy problem* (3.5) *for the function of* (3.1), *i.e.,*

$$\mu_w = \mu(G, W) \tag{3.7}$$

and

$$F_w = F(G, W). \tag{3.8}$$

Proof. Using the continuity of potential in the *fine topology* (cf. Section I.5 of [9]) and the continuity of $\log|W(z)|$ on $\overline{G}$, we obtain by Corollary I.5.6 of [9] that (3.6) also holds for any $z \in \partial G_\ell$, $\ell = 1, 2, \cdots, \ell_0$. Thus,

$$U^{\mu(G,W)}(z) - \log|W(z)| = F(G, W), \quad z \in \overline{G}. \tag{3.9}$$

Integrating (3.9) over $\overline{G}$ with respect to the measure $\mu(G, W)$, it follows immediately that $\mu(G, W)$ has finite logarithmic energy (cf. [9, Section I.1]). Then, we obtain the desired results of (3.7) and (3.8) by using (3.2), (3.9), and Theorem I.3.3 of [9]. □

4 Proofs

Proof of Theorem 1.1 Assuming that the measure $\mu(G,W)$, satisfying the conditions of Theorem 1.1, exists, we first prove that the pair (G,W) has the approximation property (1.3). To show this, recall that $G=\bigcup_{\ell=1}^{\sigma} G_\ell$ is a bounded open set where $\{G_\ell\}_{\ell=1}^{\sigma}$ are disjoint simply connected domains, and consider the Jordan domains $G_{\ell,m}\subset G_\ell$, $m\in\mathbb{N}$, which exhaust the domain G_ℓ, for each ℓ with $1\le\ell\le\sigma$. A convenient way to define the sequence $\{G_{\ell,m}\}_{m=1}^{\infty}$ is to set

$$G_{\ell,m}:=\left\{z\in\mathbb{C}:|\varphi_\ell(z)|<1-\frac{1}{2m}\right\},\quad m\in\mathbb{N}, \tag{4.1}$$

where $\varphi_\ell : G_\ell\to D:=\{w\in\mathbb{C}:|w|<1\}$ is a canonical conformal map of domain G_ℓ onto the open unit disk D, where $1\le\ell\le\sigma$. Thus, each $G_{\ell,m}$ is bounded by the analytic Jordan curve

$$\Gamma_{\ell,m}:=\left\{z\in\mathbb{C}:|\varphi_\ell(z)|=1-\frac{1}{2m}\right\},$$

which is a level curve of φ_ℓ. Let $f(z)$ be an arbitrary function which is analytic in G, and let $E\subset G$ be an arbitrary compact set. Because E is compact, it is clear that E is contained in the finite union of $G_{\ell,m}, \ell=1,2,\ldots,\ell_0$, for some $\ell_0\in\mathbb{N}$, provided that m is large enough. Set $H_m:=\bigcup_{\ell=1}^{\ell_0} G_{\ell,m}$ and $\Gamma_m:=\bigcup_{\ell=1}^{\ell_0}\Gamma_{\ell,m}$. Then, $\Gamma_m=\partial H_m$ for all $m\in\mathbb{N}$, and also $E\subset H_m$ for all sufficiently large $m\in\mathbb{N}$.

Introducing the domain $\Omega_m:=\overline{\mathbb{C}}\backslash\overline{H}_m$, $m\in\mathbb{N}$, we observe, for the balayage μ_m of $\mu(G,W)$, out of Ω_m to $\partial\Omega_m=\partial H_m$, that for each $m\in\mathbb{N}$, the following statements are true (cf. Theorem II.4.4 of [9]):

$$U^{\mu_m}(z) = U^{\mu(G,W)}(z) + c_m, \quad z \in \overline{H}_m, \tag{4.2}$$

and

$$U^{\mu_m}(z) \leq U^{\mu(G,W)}(z) + c_m, \quad z \in \mathbb{C}, \tag{4.3}$$

where $\mu_m(\mathbb{C}) = 1$, $\operatorname{supp} \mu_m \subset \partial H_m$ and $c_m > 0$. (We remark that equality in (4.2) holds on $\partial\Omega_m$ since each point of $\partial\Omega_m$ is regular (cf. [12, Theorem I.11]).) As (1.5) holds by hypothesis for any $z \in G$ and as $\overline{H}_m \subset G$, then (4.2) and (1.5) give

$$U^{\mu_m}(z) - \log|W(z)| = F(G,W) + c_m, \quad z \in \overline{H}_m, \tag{4.4}$$

i.e.,

$$\mu(H_m, W) = \mu_m, \tag{4.5}$$

and

$$F(H_m, W) = F(G,W) + c_m =: F_m, \tag{4.6}$$

for any $m \in \mathbb{N}$.

Fixing a sufficiently large $m \in \mathbb{N}$ so that $E \subset H_m$, consider the function

$$v(z) := U^{\mu_m}(z) - U^{\mu(G,W)}(z), \quad z \in \overline{\mathbb{C}}, \tag{4.7}$$

which is subharmonic in Ω_m with $v(\infty) = 0$, and satisfies, by (4.3), the inequality

$$v(z) \leq c_m, \quad z \in \overline{\mathbb{C}}. \tag{4.8}$$

Observe that if we have equality in (4.8) for some $z_0 \in \Omega_m$ then, by the maximum principle for subharmonic functions and (4.2), this gives

$$v(z) \equiv c_m > 0, \quad z \in \Omega_m,$$

which is in contradiction with the fact that $v(\infty) = 0$. Thus, it follows from (4.7) that

$$U^{\mu_m}(z) < U^{\mu(G,W)}(z) + c_m, \quad z \in \Omega_m.$$

Adding $-\log|W(z)|$ to both sides of the above inequality and, using (1.5) and (4.6), we obtain that

$$U^{\mu_m}(z) - \log|W(z)| < F_m, \quad z \in G \cap \Omega_m. \tag{4.9}$$

To construct a sequence of weighted polynomials which is uniformly convergent to $f(z)$ on E, we interpolate the analytic function $W^{-n}(z)f(z)$ by the polynomial $P_n(z)$ (of degree $\leq n$) at the $(n+1)$-th weighted Fekete points $\left\{z_k^{(n+1)}\right\}_{k=1}^{n+1} \subset \Gamma_m,\ n \in \mathbb{N}$, corresponding to the function $w(z)$ of (3.1) on $\overline{H}_m$. (For details on weighted Fekete points, see Section III.1 of [9].) Introducing the Fekete polynomials, associated with $w(z)$, by

$$\omega_{n+1}(z) := \prod_{k=1}^{n+1} \left(z - z_k^{(n+1)}\right), \tag{4.10}$$

and setting $L_m := \bigcup_{\ell=1}^{\ell_0} L_{\ell,m}$, where each $L_{\ell,m} \subset G_\ell \backslash \overline{G}_{\ell,m}$, $\ell = 1, 2, \cdots, \ell_0$, is a rectifiable Jordan curve containing $\overline{G}_{\ell,m}$ in its interior, we obtain by the Hermite interpolation formula (cf. [13, p. 50])

$$W^{-n}(z)f(z) - P_n(z) = \frac{\omega_{n+1}(z)}{2\pi i}\int_{L_m} \frac{W^{-n}(t)f(t)dt}{(t-z)\omega_{n+1}(t)}, \quad z \in E. \tag{4.11}$$

Multiplying (4.11) by $W^n(z)$ gives

$$f(z) - W^n(z)P_n(z) = \frac{W^n(z)\omega_{n+1}(z)}{2\pi i}\int_{L_m} \frac{f(t)dt}{(t-z)W^n(t)\omega_{n+1}(t)}, \quad z \in E. \tag{4.12}$$

Using Theorem III.1.8 of [9], (4.4)-(4.6) and Theorem 3.2, we have that

$$\lim_{n\to\infty} |\omega_n(z)|^{1/n} = \exp\{-U^{\mu_m}(z)\} \tag{4.13}$$

holds locally uniformly in $\mathbb{C}\backslash\Gamma_m$. Consequently, we obtain by (4.4)-(4.6) that

$$\lim_{n\to\infty} |\omega_{n+1}(z)W^n(z)|^{1/n} = e^{-F_m}, \tag{4.14}$$

uniformly on E. Also, by (4.9) and the compactness of L_m,

$$\min_{z\in L_m} \lim_{n\to\infty} |\omega_{n+1}(z)W^n(z)|^{1/n} > e^{-F_m}, \tag{4.15}$$

since $U^{\mu_m}(z) - \log|W(z)|$ is harmonic in $G \bigcap \Omega_m$. Thus from (4.12), on using (4.14) and (4.15), it follows that

$$\limsup_{n\to\infty} \|f - W^n P_n\|_E^{1/n} \le \limsup_{n\to\infty} \frac{\|W^n\omega_{n+1}\|_E^{1/n}}{\min_{z\in L_m} |W^n(z)\omega_{n+1}(z)|^{1/n}} < 1.$$

Hence, the sequence $\{W^n(z)P_n(z)\}_{n=0}^{\infty}$ converges to $f(z)$, uniformly on E, which completes the first part of the proof of Theorem 1.1.

Now, suppose that a pair (G, W), satisfying the condition of (1.2), has the approximation property (1.3). To show that the measure $\mu(G, W)$, satisfying the conditions of Theorem 1.1, exists, we consider a sequence of polynomials $\{P_n(z)\}_{n=0}^{\infty}$ such that $W^n(z)P_n(z)$ converges to $f(z) \equiv 1$, locally uniformly in G. We may assume, without loss of generality, that $\deg P_n = n$. Otherwise, one may define a new sequence of polynomials

$$\tilde{P}_n(z) := P_n(z) + a_n z^n, \quad n \in \mathbb{N},$$

in such a way that $W^n(z)\tilde{P}_n(z)$ also converges to $f(z) \equiv 1$, locally uniformly in G with $a_n \neq 0$ for any $n \in \mathbb{N}$, by choosing $a_n > 0$ to be sufficiently small.

Let $a_n \neq 0$ be the leading coefficient of $P_n(z)$ and let

(4.16) $$\nu_n := \frac{1}{n} \sum_{P_n(z_i)=0} \delta_{z_i}$$

be the *normalized zero counting measure* for $P_n(z)$, where δ_{z_i} is a unit point mass at z_i. We count all zeros of $P_n(z)$ in (4.16), according to their multiplicities, so that

(4.17) $$\nu_n(\mathbb{C}) = 1, \quad n \in \mathbb{N},$$

i.e., these measures are unit positive Borel measures. Hence, as $W^n(z)P_n(z) \to 1$ locally uniformly in G,

$$\frac{1}{n}\log|a_n| - U^{\nu_n}(z) + \log|W(z)| = \frac{1}{n}\log|W^n(z)P_n(z)| \to 0, \quad \text{as } n \to \infty, \tag{4.18}$$

locally uniformly in G.

If $\hat{\nu}_n$ is the balayage of ν_n out of the open set $\overline{\mathbb{C}}\backslash\overline{G}$ to $\overline{G}$ (note that the part of ν_n supported on $\overline{G}$ is kept fixed), then

$$U^{\hat{\nu}_n}(z) = U^{\nu_n}(z) + b_n, \quad z \in G, \tag{4.19}$$

where $b_n \geq 0$, $\operatorname{supp}\hat{\nu}_n \subset \overline{G}$ and $\hat{\nu}_n(\mathbb{C}) = \nu_n(\mathbb{C}) = 1$ (cf. Theorem II.4.7 of [9]).

By Helley's Theorem (Theorem 0.1.2 of [9]), we have that the sequence $\{\hat{\nu}_n\}_{n=1}^{\infty}$ contains a weak* convergent subsequence, so that

$$\hat{\nu}_{n_j} \xrightarrow{*} \mu, \quad \text{as } j \to \infty, \tag{4.20}$$

where μ is a positive Borel measure. One can immediately see, by the locally uniform convergence, in G, of W^nP_n to unity, that

$$\mu(\mathbb{C}) = 1 \text{ and } \operatorname{supp}\mu \subset \partial G. \tag{4.21}$$

Furthermore, by (4.20),

$$\lim_{j\to\infty} U^{\hat{\nu}_{n_j}}(z) = U^{\mu}(z), \quad z \in G. \tag{4.22}$$

It follows from (4.18) and (4.19) that

$$U^{\hat{\nu}_n}(z) - \log|W(z)| + b_n - \frac{1}{n}\log|a_n| \to 0, \quad \text{as } n \to \infty, \tag{4.23}$$

for any $z \in G$. Consequently, $b_{n_j} - \frac{1}{n_j}\log|a_{n_j}|$ converges to a finite limit by (4.22). On defining

$$F := \lim_{j\to\infty}\left(\frac{1}{n_j}\log|a_{n_j}| - b_{n_j}\right),$$

we obtain by (4.22) and (4.23) that

$$U^{\mu}(z) - \log|W(z)| = F, \quad z \in G.$$

Finally, from the above equation and (4.21), we see that (1.5) of Theorem 1.1 is satisfied with

$$\mu(G, W) = \mu,$$

and with

$$F(G, W) = F.$$

This completes the proof. □

Proof of Remark 1.3. To prove the second statement in Remark 1.3, suppose then that $W(z_0) = 0$ with $z_0 \in G_\ell$, where $W(z) \not\equiv 0$ in G_ℓ, and suppose, given an analytic function $f(z)$ in G, that polynomials $\{P_n(z)\}_{n=0}^{\infty}$ can be found such that $\{W^n(z)P_n(z)\}_{n=0}^{\infty}$ converges to $f(z)$, locally uniformly in G. As $W(z) \not\equiv 0$, we can choose $R > 0$ such that $D_R(z_0) := \{z \in \mathbb{C} : |z - z_0| < R\}$ satisfies $\overline{D}_R(z_0) \subset G_\ell$ and that

$$M := \min_{|z-z_0|=R} |W(z)| > 0. \tag{4.24}$$

Then, by the locally uniform convergence of $\{W^n(z)P_n(z)\}_{n=0}^{\infty}$ to $f(z)$,

$$\|P_n\|_{\overline{D}_R(z_0)} = \|P_n\|_{\partial D_R(z_0)} \leq \|W^n P_n\|_{\partial D_R(z_0)} \|W^{-n}\|_{\partial D_R(z_0)} \leq \frac{\|f\|_{\partial D_R(z_0)} + 1}{M^n}, \tag{4.25}$$

for all $n \in \mathbb{N}$ sufficiently large. Since $W(z_0) = 0$, we can find an $r \in (0, R)$ such that

$$m := \|W\|_{\overline{D}_r(z_0)} < M. \tag{4.26}$$

Using (4.25) and (4.26), we obtain

$$\|W^n P_n\|_{\overline{D}_r(z_0)} \leq \|W\|^n_{\overline{D}_r(z_0)} \|P_n\|_{\overline{D}_r(z_0)} \leq \left(\frac{m}{M}\right)^n \left(\|f\|_{\partial D_R(z_0)} + 1\right) \to 0, \quad \text{as } n \to \infty.$$

But because of the locally uniform approximation of $f(z)$ by $\{W^n(z)P_n(z)\}_{n=0}^{\infty}$, it follows that $f(z) \equiv 0$ for any $z \in \overline{D}_r(z_0)$ which implies, by the uniqueness theorem, that $f(z) \equiv 0$ in G_ℓ. $\square$

Proof of Theorem 2.1. First, we recall, by the results of Section IV.2 of [5] (see also Theorem II.4.7 of [9]), that the following are valid:

$$U^{\hat{\nu}_1}(z) = U^{\nu_1}(z) + \int g_\Omega(t, \infty) d\nu_1(t), \quad z \in G, \tag{4.27}$$

and

$$U^{\hat{\nu}_2}(z) = U^{\nu_2}(z) + \int g_\Omega(t, \infty) d\nu_2(t), \quad z \in G, \tag{4.28}$$

where $g_\Omega(t, \infty)$ is the Green function for Ω with pole at ∞. Using (2.3), (4.27), (4.28) and Frostman's theorem [12, p. 60], it follows, for the measure μ defined in (2.4) and for $z \in G$, that

$$
\begin{aligned}
&U^{\mu}(z) - \log|W(z)| = \\
&(1+\alpha+\beta)\,U^{\omega(\infty,\cdot,\Omega)}(z) - \alpha U^{\hat{\nu}_1}(z) - \beta U^{\hat{\nu}_2}(z) - \log|W(z)| = \\
&(1+\alpha+\beta)\log\frac{1}{\mathrm{cap}\overline{G}} - \alpha\int g_{\Omega}(t,\infty)d\nu_1(t) - \beta\int g_{\Omega}(t,\infty)d\nu_2(t),
\end{aligned}
\tag{4.29}
$$

where cap $\overline{G}$ denotes the logarithmic capacity of $\overline{G}$ (cf. [12, p. 55]).

Observe that for μ defined by (2.4), we have

$$
\operatorname{supp}\mu \subset \partial G \text{ and } \mu(\mathbb{C}) = 1. \tag{4.30}
$$

Thus, if μ is a positive measure, then Theorem 1.1 implies, by (4.29) and (4.30), that the pair (G, W) with $W(z)$ defined by (2.3) has the approximation property, with

$$
\mu(G, W) = \mu \tag{4.31}
$$

and

$$
F(G,W) = (1+\alpha+\beta)\log\frac{1}{\mathrm{cap}\overline{G}} - \alpha\int g_{\Omega}(t,\infty)d\nu_1(t) - \beta\int g_{\Omega}(t,\infty)d\nu_2(t). \tag{4.32}
$$

Suppose now that the pair (G, W) with $W(z)$ defined by (2.3) has the approximation property (1.3). Then by Theorem 1.1, there exists a positive Borel measure $\mu(G, W)$ with

$$
\operatorname{supp}\mu(G,W) \subset \partial G \text{ and } \mu(G,W)(\mathbb{C}) = 1, \tag{4.33}
$$

such that

$$U^{\mu(G,W)}(z) - \log|W(z)| = F(G,W), \quad z \in G. \tag{4.34}$$

It follows from (4.29) and (4.34) that

$$U^{\mu(G,W)}(z) = U^{\mu}(z) + c, \quad z \in G, \tag{4.35}$$

where c is a constant. Since potentials are continuous in the *fine topology* (see Section I.5 of [9]) and since the boundary of each G_ℓ, $\ell = 1, \ldots, \sigma$, in the fine topology is the same as the Euclidean boundary (see Corollary I.5.6 of [9]), then (4.35) also holds for any $z \in \partial G$. Thus,

$$u(z) := U^{\mu(G,W)}(z) - U^{\mu}(z) = c, \quad z \in \overline{G}. \tag{4.36}$$

Observe that $u(z)$ is harmonic in Ω (including $z = \infty$) with $u(\infty) = 0$, and that $u(z) \equiv c$ on $\partial\Omega \subset \partial G$. Therefore,

$$u(z) \equiv 0, \quad z \in \Omega \cup \overline{G}, \tag{4.37}$$

by the minimum-maximum principle for harmonic functions and the continuity of $u(z)$ in the fine topology. Applying a similar argument to the bounded components of $\overline{\mathbb{C}}\setminus\overline{G}$, we obtain from (4.37) that

$$u(z) \equiv 0, \quad z \in \overline{\mathbb{C}}. \tag{4.38}$$

Assume that $\alpha > 0$ and $\beta < 0$ (the other cases are treated similarly). Then we have from (4.38) that

$$U^{\mu(G,W)-\beta\omega(\infty,\cdot,\Omega)+\alpha\hat{\nu}_1}(z) = U^{(1+\alpha)\omega(\infty,\cdot,\Omega)-\beta\hat{\nu}_2}(z), \quad z \in \mathbb{C}, \tag{4.39}$$

where we deal with the potentials of the *positive* measures on both sides of (4.39). It follows from Theorem II.2.1 of [9] that

$$\mu(G,W) - \beta\omega(\infty,\cdot,\Omega) + \alpha\hat{\nu}_1 = (1+\alpha)\omega(\infty,\cdot,\Omega) - \beta\hat{\nu}_2. \tag{4.40}$$

Thus,

$$\mu = (1+\alpha+\beta)\omega(\infty,\cdot,\Omega) - \alpha\hat{\nu}_1 - \beta\hat{\nu}_2 = \mu(G,W) \tag{4.41}$$

is a positive measure. □

Proof of Theorem 2.2. It is clear that for $W(z) = z^\alpha$ we have

$$\log|W(z)| = -\alpha \log\frac{1}{|z|} = -\alpha U^{\delta_0}(z), \quad z \in \mathbb{C}\setminus\{0\}, \tag{4.42}$$

where δ_0 is the unit point mass at $z=0$ and α is any real number. Since the balayage $\hat{\delta}_0$ of δ_0 out of Ω to $\overline{G}$ is given by (cf. [5, p. 222])

$$\hat{\delta}_0 = \omega(0,\cdot,\Omega), \tag{4.43}$$

then Theorem 2.2 is an immediate consequence of Theorem 2.1 with $\beta = 0$. □

Proof of Corollary 2.3. First, we explicitly find the measure μ of (2.11) for $\Omega = \overline{\mathbb{C}}\setminus\overline{D}_r(a)$. Introducing the conformal mapping of Ω onto the exterior of the unit disk $D' = \{w \in \mathbb{C} : |w| > 1\}$,

$$w = \Phi(z) := \frac{r^2 - (\overline{z_0 - a})(z-a)}{r(z-z_0)}, \quad z \in \Omega, \tag{4.44}$$

where $\Phi(z_0) = \infty$ for $z_0 \in \Omega$, we obtain that

$$\omega(z_0, B, \Omega) = m\left(\Phi(B \cap \partial\Omega)\right) \tag{4.45}$$

for any Borel set $B \subset \mathbb{C}$, where

$$dm = d\theta/(2\pi) \text{ on } \{w \in \mathbb{C} : |w| = 1\}$$

(see [7, p. 37]). It follows from (4.45) that

$$\frac{d\omega(z_0, \cdot, \Omega)}{ds}(z) = \frac{1}{2\pi}\ |\Phi'(z)| = \frac{|z_0 - a|^2 - r^2}{2\pi r |z - z_0|^2}, \quad |z-a| = r, \tag{4.46}$$

where ds is the arclength on $|z-a| = r$. As is well known,

$$\frac{d\omega(\infty, \cdot, \Omega)}{ds}(z) = \frac{1}{2\pi r}, \quad |z-a| = r, \tag{4.47}$$

which gives, for μ of (2.11) that

$$\frac{d\mu}{ds}(z) = \frac{1}{2\pi r}\left(1 + \alpha - \alpha\ \frac{a^2 - r^2}{|z|^2}\right), \quad |z-a| = r. \tag{4.48}$$

Since μ is a positive measure if and only if the density function on the right of (4.48) is positive for all z such that $|z-a| = r$, then the validity of the approximation property is equivalent to

$$1+\alpha-\alpha\,\frac{a^2-r^2}{(a-r)^2}\geq 0, \qquad \text{if } \alpha>0, \tag{4.49}$$

or to

$$1+\alpha-\alpha\,\frac{a^2-r^2}{(a+r)^2}\geq 0, \qquad \text{if } \alpha<0. \tag{4.50}$$

Solving the above inequalities for r under the condition $r<a$, we arrive at (2.12) and (2.13).

□

Proof of Theorem 2.4. Following the proof of Theorem 2.2, we write for $W(z)$ of (2.14):

$$\log|W(z)| = -\alpha\log\frac{1}{|z-1|}-\beta\log\frac{1}{|z+1|} = -\alpha U^{\delta_1}(z)-\beta U^{\delta_{-1}}(z), \quad z\in\mathbb{C}\setminus\{-1,1\}, \tag{4.51}$$

where δ_1 and δ_{-1} are the unit point masses at $z=1$ and $z=-1$, respectively. For their balayages to $\overline{G}$, we have [5, p. 222]:

$$\hat{\delta}_1=\omega(1,\cdot,\Omega) \text{ and } \hat{\delta}_{-1}=\omega(-1,\cdot,\Omega). \tag{4.52}$$

Thus, Theorem 2.4 follows from Theorem 2.1. □

Proof of Corollary 2.5. Using the same notations as in the proof of Corollary 2.3, we obtain for μ, given by (2.15), that, from (4.46) and (4.47),

$$\frac{d\mu}{ds}(z)=\frac{1}{2\pi r}\left(1+\alpha+\beta-\alpha\,\frac{(1-a)^2-r^2}{|z-1|^2}\;-\beta\,\frac{(1+a)^2-r^2}{|z+1|^2}\right), \tag{4.53}$$

where $|z-a|=r$ and ds is the arclength on $|z-a|=r$. Thus, the possibility of weighted approximation is equivalent, by Theorem 2.4, to the positivity of the density function in (4.53). It can be verified by elementary methods that, for $\alpha \geq 0$ and $\beta \geq 0$, the latter is equivalent to

$$\frac{d\mu}{ds}(a-r) \geq 0 \text{ and } \frac{d\mu}{ds}(a+r) \geq 0 \tag{4.54}$$

holding simultaneously. Substituting $z=a-r$ and $z=a+r$ into (4.53) and solving (4.54) for r gives the desired results of (2.17). □

Proof of Theorem 2.6. For the Jacobi-type weight function of (2.19), we have

$$\log|W(z)| = \sum_{i=1}^{p} \alpha_i \log|z-t_i| = -\sum_{i=1}^{p} \alpha_i U^{\delta_{t_i}}(z), \quad z \in \mathbb{C}\backslash\{t_i\}_{i=1}^{p}. \tag{4.55}$$

By the properties of balayage of a unit mass (cf. [5, p. 222]) from Ω to $\overline{G}$, we have

$$\hat{\delta}_{t_i} = \omega(t_i, \cdot, \Omega), \quad i=1,\ldots,p. \tag{4.56}$$

On defining

$$\nu_1 := \left(\sum_{\alpha_i \geq 0} \alpha_i \delta_{t_i}\right) \Big/ \left(\sum_{\alpha_i \geq 0} \alpha_i\right) \tag{4.57}$$

and

$$\nu_2 := \left(\sum_{\alpha_i < 0} |\alpha_i| \delta_{t_i}\right) \Big/ \left(\sum_{\alpha_i < 0} |\alpha_i|\right), \tag{4.58}$$

we observe that ν_1 and ν_2 are the unit positive measures with compact support in Ω, such that

$$\log|W(z)| = -\left(\alpha U^{\nu_1}(z) + \beta U^{\nu_2}(z)\right), \quad z \in \mathbb{C}\setminus\{t_i\}_{i=1}^p, \tag{4.59}$$

where $\alpha := \sum_{\alpha_i \geq 0} \alpha_i$ and $\beta := \sum_{\alpha_i < 0} \alpha_i$. Using (4.56), it follows from (4.57) and (4.58) (see Section II.4 of [9]) that

$$\hat{\nu}_1 = \left(\sum_{\alpha_i \geq 0} \alpha_i \omega(t_i, \cdot, \Omega)\right) \Big/ \sum_{\alpha_i \geq 0} \alpha_i \tag{4.60}$$

and

$$\hat{\nu}_2 = \left(\sum_{\alpha_i < 0} |\alpha_i| \omega(t_i, \cdot, \Omega)\right) \Big/ \sum_{\alpha_i < 0} |\alpha_i| . \tag{4.61}$$

Applying Theorem 2.1, we obtain that the approximation property holds for the pair (G, W) if and only if

$$\mu = (1 + \alpha + \beta)\,\omega(\infty, \cdot, \Omega) - \alpha\hat{\nu}_1 - \beta\hat{\nu}_2 = \left(1 + \sum_{i=1}^p \alpha_i\right) \omega(\infty, \cdot, \Omega) - \sum_{i=1}^p \alpha_i \omega(t_i, \cdot, \Omega)$$

is a positive measure.

Furthermore, calculating the density function of μ for $\Omega = \overline{\mathbb{C}}\setminus\overline{D}_r(a)$ by (4.46), (4.47) and (2.20), we get

$$\frac{d\mu}{ds}(z) = \frac{1}{2\pi r}\left(1 + \sum_{i=1}^p \alpha_i - \sum_{i=1}^p \alpha_i \frac{|t_i - a| - r^2}{|z - t_i|^2}\right), \quad |z - a| = r, \tag{4.62}$$

which implies the desired result of (2.21). $\square$

Proof of Theorem 2.7. We essentially follow the proof of Theorem 4.3 of [8] (corresponding to $m = 1$), generalizing it to the case of arbitrary $m = 1, 2, \ldots$. We know by Theorem 1.1 that the approximation property for $D_r(0)$, with respect to the exponential weight $W(z)$ of (2.22), holds if and only if there exists a positive unit Borel measure $\mu_r := \mu(D_r(0), e^{-z^m})$, with $\operatorname{supp}\mu_r \subset \partial D_r(0)$, such that

$$U^{\mu_r}(z) + \operatorname{Re}\{z^m\} = F_r, \quad z \in D_r(0), \tag{4.63}$$

where $F_r := F(D_r(0), e^{-z^m})$ is a constant. For $z = 0$, (4.63) gives, from (1.4), that

$$F_r = U^{\mu_r}(0) = \int \log \frac{1}{|t|}\, d\mu_r(t) = \log \frac{1}{r}. \tag{4.64}$$

By the continuity of potentials in the fine topology and the fact that the boundary of $D_r(0)$ in the fine topology coincides with usual boundary $\partial D_r(0)$, we conclude that (4.63) holds for any $z \in \partial D_r(0)$. Thus, (4.63) is equivalent to

$$U^{\mu_r}(z) + \operatorname{Re}\{z^m\} = \log \frac{1}{r}, \quad |z| \le r. \tag{4.65}$$

Consider the function $U^{\mu_r}(z) + \log \dfrac{|z|}{r}$, which is harmonic in $\Omega := \overline{\mathbb{C}} \backslash \overline{D}_r(0)$, and whose boundary values satisfy

$$U^{\mu_r}(z) + \log \frac{|z|}{r} = \log \frac{1}{r} - \operatorname{Re}\{z^m\}, \quad |z| = r, \tag{4.66}$$

by (4.65) and Theorem II.3.5 of [9]. Solving the associated Dirichlet problem in (4.66) for

Ω, we find that

$$U^{\mu_r}(z) + \log \frac{|z|}{r} = \log \frac{1}{r} - r^{2m} \mathrm{Re} \left\{ \frac{1}{z^m} \right\}, \quad |z| \geq r,$$

or

$$U^{\mu_r}(z) = \log \frac{1}{|z|} - r^{2m} \mathrm{Re} \left\{ \frac{1}{z^m} \right\}, \quad |z| \geq r. \tag{4.67}$$

On the other hand, we have from (4.65) that

$$U^{\mu_r}(z) = \log \frac{1}{r} - \mathrm{Re}\{z^m\}, \quad |z| \leq r. \tag{4.68}$$

We can now find μ_r explicitly (cf. Theorem II.1.5 of [9]) from

$$d\mu_r(\theta) = -\frac{1}{2\pi} \left(\frac{\partial U^{\mu_r}}{\partial n_+}(\theta) + \frac{\partial U^{\mu_r}}{\partial n_-}(\theta) \right) r d\theta,$$

where $d\theta$ is the angular measure on $|z| = r$, and where n_+ and n_- are respectively the inner and the outer normals to the circle $|z| = r$. A direct calculation using (4.67) and (4.68) gives that

$$d\mu_r(\theta) = -\frac{1}{2\pi} \left(mr^{m-1} \mathrm{Re}\, e^{im\theta} + \left(-\frac{1}{r} + mr^{m-1} \mathrm{Re}\, e^{-im\theta} \right) \right) r d\theta,$$

and, after simplifying, the above becomes

$$d\mu_r(\theta) = \frac{1}{2\pi} (1 - 2mr^m \cos m\theta)\, d\theta. \tag{4.69}$$

Clearly, $\mu_r(\theta)$ is a unit measure on $|z| = r$ satisfying (4.65) for any $r > 0$. However, $\mu_r(\theta)$ is a positive measure if and only if (2.23) holds for r. □

5 Further Remarks and Open Problems

Theorem 1.1 gives a rather complete answer to the question on weighted approximation by $W^n(z)P_n(z)$ in open sets of the complex plane. It is then very natural to consider the uniform approximation by such weighted polynomials on *compact* sets, aiming at an analogue (generalization) of Mergelyan's theorem (see [13, p. 367]). Let $E \subset \mathbb{C}$ be a compact set with connected complement $\overline{\mathbb{C}} \setminus E$. We denote the set of all functions, analytic interior to E and continuous on E, by $A(E)$. Let $W \in A(E)$, with $W(z) \neq 0$ for any $z \in E$.

Problem. *Give a necessary and sufficient condition for the pair (E, W) to have the following approximation property:*

For any $f \in A(E)$ there exists polynomials $\{P_n(z)\}_{n=0}^{\infty}$, with $\deg P_n \leq n$, such that

$$\lim_{n \to \infty} \|f - W^n P_n\|_E = 0. \tag{5.1}$$

Obviously, the classical uniform approximation by polynomials (Mergelyan's theorem) corresponds to $W(z) \equiv 1,\ z \in E$. We observe that (1.5) of Theorem 1.1, holding with $G = \text{Int}E$, is a necessary condition for (5.1). Let us also remark that this problem is open even in the case when E is a subset of the real line, such as an interval (see [9, 11] for background and general results, and Kuijlaars [4] for the recent progress in this area).

An even more general approach is to consider the approximation problem in (5.1) with polynomials replaced by rational functions. Certain results, concerning such weighted rational approximation have been obtained by Borwein and Chen [1] in the complex plane, and by Borwein, Rakhmanov and Saff [2] on the real line, for particular weights.

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