ON A COUNTEREXAMPLE IN THE THEORY OF POLYNOMIALS HAVING CONCENTRATION AT LOW DEGREES

RICHARD S. VARGA AND IGOR E. PRITSKER

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Abstract: It is shown here that a 1989 conjecture of Rigler, Trimble, and Varga in the theory of polynomials having concentration at low degrees, is not true in general, and counterexamples are explicitly derived here. What is intriguing is that these counterexamples produce only small deviations from the conjecture mentioned above.

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1. Introduction.

There have been a number of recent papers, by Beauzamy and Enflo [2], Beauzamy [3], Rigler, Trimble, and Varga [6], and Varga [7], which are connected with polynomials and the classical Jensen inequality. To describe these results, let $p_m(z) = \sum_{j=0}^{m} a_j z^j$ be a complex polynomial ($\not\equiv 0$), let d be a real number in the interval $(0, 1)$, and let k be a nonnegative integer. Then (cf. [2, 3]), $p_m(z)$ is said to have *concentration d at degree k* if

$$\sum_{j=0}^{k} |a_j| \geq d \sum_{j=0}^{m} |a_j|. \tag{1.1}$$

The first result established in this area was

THEOREM A. (Beauzamy and Enflo [2], Beauzamy [3]). *Given any real number d in $(0, 1)$ and given any nonnegative integer k, let the real number $\tilde{C}_{d,k}$, depending only on d and* *k, be defined by*

$$\tilde{C}_{d,k} := \sup_{1<t<\infty} \left[t \log \left\{ \frac{2d}{(t-1)\left[\left(\frac{t+1}{t-1}\right)^{k+1} - 1\right]} \right\} \right]. \tag{1.2}$$

Then, for any polynomial $p_m(z) = \sum_{j=0}^{m} a_j z^j (\not\equiv 0)$ *which satisfies* (1.1),

$$\frac{1}{2\pi} \int_0^{2\pi} \log \left| p_m(e^{i\theta}) \right| d\theta - \log \left(\sum_{j=0}^{m} |a_j| \right) \geq \tilde{C}_{d,k}. \tag{1.3}$$

The important feature of Theorem A is that the lower bound, $\tilde{C}_{d,k}$ of (1.3), is *independent* of the degree of the polynomial $p(z)$. For our purposes here, we define the functional

$$J(p_m) := \frac{1}{2\pi} \int_0^{2\pi} \log \left| p_m\left(e^{i\theta}\right) \right| d\theta - \log \left(\sum_{j=0}^{m} |a_j| \right) \tag{1.4}$$

for any polynomial $p_m(z) = \sum_{j=0}^{m} a_j z^j (\not\equiv 0)$. Then, the best lower bound in (1.3), where $0 < d < 1$ and k is a nonnegative integer, is given by

$$C_{d,k} := \inf \left\{ J(p) : p(z) \text{ is any polynomial } (\not\equiv 0) \text{which satisfies } (1.1) \right\}. \tag{1.5}$$

While neither the "best" constant $C_{d,k}$ of (1.5), nor its associated extremal functions, are known in general, there is a special set of polynomials for which the associated constant of (1.5), and its extremal functions, are explicitly known from Theorem 2 of [6]. To describe these results, it is well known that any real polynomial, all of whose zeros lie in Re $z < 0$, are called *Hurwitz polynomials* (cf. Marden [5, p. 181]), and we define $\mathcal{H}$ as the set of all such Hurwitz polynomials. Then, we set

$$C_{d,k}^{\mathcal{H}} := \inf \left\{ J(p) : p(z) (\not\equiv 0) \text{ is in } \mathcal{H} \text{ and satisfies } (1.1) \right\}. \tag{1.6}$$

Obviously,

$$C_{d,k}^{\mathcal{H}} \geq C_{d,k} \text{ for all } 0 < d < 1 \text{ and all nonnegative integers } k. \tag{1.7}$$

It is shown in [6, Lemma 3] that, for any $d \in (0,1)$ and for any positive integer k, there is a unique positive integer n (dependent on d and k) which satisfies

$$\frac{1}{2^n} \sum_{j=1}^{k} \binom{n}{j} \leq d < \frac{1}{2^{n+1}} \sum_{j=0}^{k} \binom{n-1}{j}, \tag{1.8}$$

and, with this definition of n, the number ρ, defined by

$$\rho := \frac{\binom{n-1}{k}}{\sum_{j=0}^{k} \binom{n-1}{j} - d2^{n-1}} - 1, \tag{1.9}$$

satisfies $1 \leq \rho < \infty$. Then, the main result of [6, Theorem 2] is that the constant $C_{d,k}^{\mathcal{H}}$ of (1.6) is explicitly given by

$$C_{d,k}^{\mathcal{H}} = \log\left(\frac{\rho}{(\rho+1)2^{n-1}}\right). \tag{1.10}$$

Moreover, any $p(z) \in \mathcal{H}$, which satisfies (1.1), is an *extremal element*, i.e., $J(p) = C_{d,k}^{\mathcal{H}}$, if and only if

$$p(z) = (\rho + z)(1+z)^{n-1}, \tag{1.11}$$

where n and ρ satisfy (1.8) and (1.9), respectively.

It is important to remark that this result of (1.10) is valid, more generally, for real entire functions, of exponential order zero (cf. Boas [4, p. 29]), of the form

$$f(z) = \prod_{j=1}^{\infty}\left(1 - \frac{z}{z_j}\right), \text{ where } \sum_{j=1}^{\infty} \frac{1}{|z_j|} < \infty, \text{ and } \operatorname{Re} z_j < 0,$$

for all $j \geq 1$. These extensions can be found in [6].

Because repeated attempts at obtaining estimates of $C_{d,k}$, which were *smaller* than $C_{d,k}^{\mathcal{H}}$, had failed, the following conjecture of [6] was made in 1989:

(1.12) *Conjecture* [6] : $C_{d,k}^{\mathcal{H}} \stackrel{?}{=} C_{d,k}$, for any $d \in (0,1)$ and any positive integer k.

The main point of this paper is to show that the Conjecture of (1.12) is in general *false*, and explicit examples will be included.

2. Equivalent Conditions for the Conjecture of (1.12).

In this section, we derive an equivalent condition for the truth of the Conjecture of (1.12). The development here makes use of the results of Varga [7].

Consider any complex polynomial $p_m(z) = \sum_{j=0}^{m} a_j z^j (\not\equiv 0)$. If a_N is the first nonzero Taylor coefficient of $p_m(z)$, if $\Delta := \{z \in \mathbb{C} : |z| < 1\}$ and if $Z_\Delta(p_m)$ denotes the zeros of $p_m(z)$ in Δ (where multiple zeros are counted according to their multiplicity), then Jensen's formula is (cf. Ahlfors [1, p. 208])

$$\frac{1}{2\pi}\int_0^{2\pi} \log\left|p_m\left(e^{i\theta}\right)\right| d\theta = \log|a_N| + \sum_{z_j \in Z_\Delta(p_m)} \log\left(\frac{1}{|z_j|}\right). \tag{2.1}$$

With the functional $J(p)$ of (1.4), it follows from (2.1) that

$$J(p_m) = \log \left\{ \frac{|a_N|}{\left(\prod_{z_j \in Z_\Delta(p_m)} |z_j| \right) \cdot \sum_{j=0}^{m} |a_j|} \right\}. \tag{2.2}$$

It is convenient then to define the associated functional

$$K(p_m) := \frac{|a_N|}{\left(\prod_{z_j \in Z_\Delta(p_m)} |z_j| \right) \cdot \sum_{j=0}^{m} |a_j|}, \tag{2.3}$$

so that from (2.2),

$$J(p_m) = \log K(p_m). \tag{2.4}$$

It is evident that minimizing $J(p_m)$ over some set is equivalent to minimizing $K(p_m)$ over the same set. With N_0 and N denoting, respectively, the sets of nonnegative integers and positive integers, we define the numbers

$$\delta_k(p_m) := \sum_{j=0}^{k} |a_j| / \sum_{j=0}^{m} |a_j| \quad (k \in N_0), \tag{2.5}$$

so that $\delta_\ell(p_m) = 1$ for all $\ell \geq m$. We also define the ratios

$$L_k(p_m) := K(p_m)/\delta_k(p_m) \quad (k \in N_0), \tag{2.6}$$

where $L_k(p_m) := +\infty$ if $\delta_k(p_m) = 0$. ¿From (2.3) and (2.5), we see that $K(\gamma p_m) = K(p_m)$ and $\delta_k(\gamma p_m) = \delta_k(p_m)$ for any complex number $\gamma \neq 0$, so that

$$L_k(\gamma p_m) = L_k(p_m) \quad (\gamma \in \mathbb{C} \text{ with } \gamma \neq 0).$$

In addition, it also follows from (2.6) that

$$L_k(p_m) = L_{k+s}(z^s p_m) \quad (k, s \in N_0).$$

Note that the $\delta_k(p_m)$'s are nondecreasing in k from (2.5), and similarly, that the $L_k(p_m)$'s are nonincreasing in k from (2.6). Thus, because we are interested in minimizing $K(p_m)$ over all polynomials for which $\delta_k(p_m) \geq d$, we may assume that $p_m(z)$ is a monic polynomial with $p_m(0) \neq 0$, and this allows us to express $p_m(z)$ as

$$p_m(z) = \prod_{i=1}^{m} (\zeta_i + z), \tag{2.7}$$

where the complex numbers $\{\zeta_i\}_{i=1}^m$ are ordered by modulus, i.e.,

$$0 < |\zeta_1| \le |\zeta_2| \le \cdots \le |\zeta_r| < 1 \le |\zeta_{r+1}| \le |\zeta_{r+2}| \le \cdots \le |\zeta_m|; \tag{2.8}$$

here, r is a nonnegative integer satisfying $0 \le r \le m$.

¿From the constructions in [7], we have the following result.

THEOREM B. ([7]). *For any complex monic polynomial* $p_m(z) = \prod_{i=1}^m (\zeta_i + z)$, *where the* $\{\zeta_j\}_{j=1}^m$ *satisfy* (2.8), *there exists a polynomial* $\hat{p}_m(z) := (\hat{\rho} + z)(1+z)^{\hat{m}-1}$, *(where* $\hat{\rho}$ *is a real number with* $\hat{\rho} \ge 1$ *and where* $\hat{m}$ *is a positive integer with* $\hat{m} \le m$*), such that*

$$L_k(p_m) \ge L_k(\hat{p}_m) \quad (k \in N_0). \tag{2.9}$$

In particular, if the numbers $\{\zeta_j\}_{j=1}^m$ *do not all lie on some ray from the origin, then*

$$L_k(p_m) > L_k(\hat{p}_m) \quad (k \in N), \text{ with } L_0(p_m) \ge L_0(\hat{p}_m) = 1. \tag{2.10}$$

Note that the polynomials, of the special form $(\rho + z)(1+z)^{n-1}$, have already played a role in both (1.11) and in Theorem B. These are of course Hurwitz polynomials, and it is convenient to give them the special name of

DEFINITION 1. Any real polynomial of the form $(\rho + z)(1+z)^{n-1}$, where $1 \le \rho < \infty$ and where $n \in N$, is called an *h^*-polynomial.*

Some useful properties of h^*-polynomials are known from [7, eq. (3.41)]:

$$\left\{\begin{array}{ll} \text{(i)} & L_k\left((\rho+z)(1+z)^{n-1}\right) \\ & \text{is a strictly increasing function of } \rho \text{ on the interval} \\ & [1,\infty), \text{ for any } k \in N, \\ & \text{while } L_0\left((\rho+z)(1+z)^{n-1}\right) \equiv 1, \\ & \text{and similarly,} \\ \text{(ii)} & \delta_k\left((\rho+z)(1+z)^{n-1}\right) \\ & \text{is a strictly increasing function of } \rho \text{ on the interval} \\ & [1,\infty), \text{ for any integer k with } 0 \le k < n, \\ & \text{while } \delta_n\left((\rho+z)(1+z)^{n-1}\right) \equiv 1. \end{array}\right. \tag{2.11}$$

We also note that as $L_k\left((\rho+z)(1+z)^{n-1}\right) = L_k((1+\frac{z}{\rho})(1+z)^{n-1})$, then letting ρ increase to $+\infty$ in $(1+\frac{z}{\rho})(1+z)^{n-1}$, produces the h^*-polynomial $(1+z)^{n-1}$ whose degree is one less than that of $(\rho+z)(1+z)^{n-1}$. We call the process of *increasing* ρ in the h^*-polynomial $(\rho+z)(1+z)^{n-1}$ as a *lifting* of the polynomial $(\rho+z)(1+z)^{n-1}$, and the process of decreasing

ρ as a *lowering* of the polynomial $(\rho+z)(1+z)^{n-1}$ (which may correspondingly increase the degree of the associated h^*-polynomial).

Suppose that the strict inequalities of (2.10) hold in Theorem B for all $k \in N$. Then, on lifting the h^*-polynomial $(\hat{\rho}+z)(1+z)^{\hat{m}-1}$, it follows from (2.11(i)) that there is a *unique* h^*-polynomial $\overset{\square}{p}_m(z)$, which has the form

$$\overset{\square}{p}_m(z) := (\overset{\square}{\rho}+z)(1+z)^{\overset{\square}{m}-1} \quad (\text{with } 1 \leq \overset{\square}{\rho} < \infty \text{ and } 1 \leq \overset{\square}{m} \leq \hat{m}), \tag{2.12}$$

such that

$$L_j(p_m) \geq L_j\left(\overset{\square}{p}_m\right) \quad (j \in N_0), \text{ where equality holds for some } j \geq 1. \tag{2.13}$$

(Note that lifting any h^*-polynomial $q(z)$ leaves $L_0(q) \equiv 1$ unchanged, so that equality in (2.13) is sought only for some $j \geq 1$.) If, on the other hand, equality already holds in (2.9) of Theorem B for some $k \in N$, we simply set $\overset{\square}{p}_m(z) := \hat{p}_m(z)$.

Then, as in Theorem B, for any $p_m(z) = \prod_{i=1}^{m}(\zeta_i+z)$ where the $\{\zeta_j\}_{j=1}^m$ satisfy (2.8), we define the set

$$E(p_m) := \left\{k \in N : L_k(p_m) = L_k(\overset{\square}{p}_m)\right\}. \tag{2.14}$$

By construction, $E(p_m) \neq \emptyset$, and card $E(p_m)$ denotes the number of elements in $E(p_m)$.

This brings us to the following

<u>DEFINITION 2.</u> Given two polynomials $q(z)$ and $r(z)$, and given a nonnegative integer k, we say that

$$r(z) \text{ is } k\text{-better than } q(z) \text{ if } \delta_k(r) = \delta_k(q) \text{ and } K(q) \geq K(r), \tag{2.15}$$

and

$$r(z) \text{ is strictly } k\text{-better than } q(z) \text{ if } \delta_k(r) = \delta_k(q) \text{ and } K(q) > K(r). \tag{2.16}$$

The motivation for Definition 2 is clear: Given a d with $0 < d < 1$ and given a $k \in N_0$, we are ultimately interested in determining the constant $\Gamma_{d,k}$, defined by

$$\Gamma_{d,k} := \inf\left\{K(p_m) : \delta_k(p_m) \geq d\right\}.$$

If $d = \delta_k(r) = \delta_k(q)$, and if $r(z)$ is strictly k-better than $q(z)$, then $K(q) > K(r)$, which certainly implies that $r(z)$ yields a *superior* upper bound for $\Gamma_{d,k}$, than does $q(z)$.

Our main theoretical result is the following theorem, whose proof will be given in Section 5. (We use ∂q to denote the exact degree of a complex polynomial q.)

THEOREM 1. *Let* $p_m(z) = \prod_{i=1}^{m}(\zeta_i + z)$, *where the* $\{\zeta_i\}_{i=1}^m$ *satisfy* (2.8). *If card* $E(p_m) = \infty$, *then for each* $k \in N_0$, *there is an* h^**-polynomial* $q_k(z)$, *with* $\partial q_k \le \partial p_m$, *such that*

$$q_k(z) \text{ is } k\text{-better than } p_m(z). \tag{2.17}$$

If card $E(p_m) \neq \infty$, *then for each* $k \in E(p_m)$, *there is a unique* h^**-polynomial* $\tilde{q}_k(z)$, *with* $\partial \tilde{q}_k \le \partial p_m$, *such that*

$$p_m(z) \text{ is strictly } k\text{-better than } \tilde{q}_k(z). \tag{2.18}$$

An important theoretical consequence of Theorem 1, for our purposes here, is

Corollary 2. *The Conjecture of* (1.12) *is false if there is some polynomial* $p_m(z)$, *satisfying the hypotheses of* Theorem B, *for which card* $E(p_m) \neq \infty$.

In the next section, we explicitly give an example of a polynomial $p_m(z)$ for which card $E(p_m) \neq \infty$, thereby producing a counterexample to the Conjecture of (1.12).

3. A Counterexample via Corollary 2.

To obtain an explicit counterexample to the Conjecture of (1.12), consider the monic polynomial

$$p_5(z) := (1+z)^3\{6 - 3z + z^2)\} = 6 + 15z + 10z^2 + z^5, \tag{3.1}$$

which is lacunary, as the coefficients of z^3 and z^4 are zero. As the zeros of $6 - 3z + z^2$ are $\left(3 \pm i\sqrt{15}\right)/2$, then $p_5(z)$ has no zeros in Δ. Also, it is evident that the zeros of $p_5(z)$ do not all lie on a ray, nor do they all lie in Re $z < 0$.

It can be verified (using the rotation, reflection, and reduction methods of [7]) that the h^*-polynomial of Theorem B, namely, $\hat{p}_5(z)$, is given by

$$\hat{p}_5(z) := (\hat{\rho} + z)(1+z)^3, \text{ where } \hat{\rho} := (12\sqrt{6} - 6)/23 \text{ satisfies } 1 < \hat{\rho} < \infty. \tag{3.2}$$

Thus, as the zeros of $p_5(z)$ do not lie on a ray, we have from (2.10) of Theorem B that

$$L_k(p_5) > L_k(\hat{p}_5) \quad (k \in N), \text{ while } L_0(p_5) = 1 = L_0(\hat{p}_5). \tag{3.3}$$

The inequalities of (3.3) can be directly verified in Table 1 below. (All numbers in Tables 1 and 2 have been truncated to six decimal digits.)

On lifting $\hat{p}_5(z)$ of (3.2), a unique h^*-polynomial $\overset{\square}{p}_5(z)$ can be determined for which

$$L_j(p_5) \ge L_j(\overset{\square}{p}_5) \quad (j = 0, 1, \cdots), \text{ with equality holding for some } j \ge 1, \tag{3.4}$$

Table 1:

p	$L_0(p)$	$L_1(p)$	$L_2(p)$	$L_3(p)$	$L_4(p)$	$L_j(p), j \geq 5$
p_5	1	0.285 714	0.193 548	0.193 548	0.193 548	0.187 500
$\hat{p}_5$	1	0.200 675	0.091 469	0.067 194	0.063 030	0.063 030
$\overset{\square}{p}_5$	1	0.285 714	0.181 818	0.166 666	0.166 666	0.166 666

Table 2:

p	$\delta_0(p) = K(p)$	$\delta_1(p)$	$\delta_2(p)$	$\delta_3(p)$	$\delta_4(p)$	$\delta_5(p)$
p_5	0.187 500	0.656 250	0.968 750	0.968 750	0.968 750	1
$\hat{p}_5$	0.063 030	0.314 091	0.689 091	0.938 030	1	1
$\overset{\square}{p}_5$	0.166 666	0.583 333	0.916 666	1	1	1
$\tilde{q}$	0.203 125	0.656 250	0.953 125	1	1	1
$\overset{*}{q}$	0.218 750	0.687 500	0.968 750	1	1	1

and this h^*-polynomial $\overset{\square}{p}_5(z)$ is explicitly given by

$$\overset{\square}{p}_5(z) := (2+z)(1+z)^2 = 2 + 5z + 4z^2 + z^3. \tag{3.5}$$

The associated values of the $L_j(\overset{\square}{p}_5)$'s are also included in Table 1. We see from Table 1 that equality holds in (3.4) only for $j = 1$, so that (cf. (2.14))

$$E(p_5) = \{1\}, \text{ and card } E(p_5) \neq \infty. \tag{3.6}$$

This means, from (2.18) of Theorem 1, that there is an h^*-polynomial $\tilde{q}(z)$ for which

$$\delta_1(\tilde{q}) = \delta_1(p_5), \text{ but } K(\tilde{q}) > K(p_5), \tag{3.7}$$

and $\tilde{q}(z)$ is in fact given by

$$\tilde{q}(z) := \left(\frac{13}{3} + z\right)(1+z)^2 = \frac{1}{3}\left\{13 + 29z + 19z^2 + 3z^3\right\}. \tag{3.8}$$

The associated values of $\delta_j(\cdot)$ are included in Table 2, where we see that $p_5(z)$ is strictly 1-better than $\tilde{q}(z)$, i.e.,

$$\delta_1(\tilde{q}) = 0.656\ 250 = \delta_1(p_5), \text{ and } K(\tilde{q}) = 0.203\ 125 > K(p_5) = 0.187\ 500, \tag{3.9}$$

which is an explicit counterexample to the Conjecture of (1.12).

While Theorem 1, in the case that card $E(p_m) \neq \infty$, gives that each element of $E(p_m)$ gives rise to a counterexample of the Conjecture in (1.12), it is the case that elements *not* in $E(p_m)$ may also give rise to such counterexamples. To see this, we have from (3.6) that $2 \notin E(p_5)$, but on lifting $\overset{\square}{p}_5(z)$, to the unique h^*-polynomial $\overset{*}{q}(z)$ so that $\delta_2(\overset{*}{q}) = \delta_2(p_5)$, we obtain in this case that

$$\overset{*}{q}(z) := (7+z)(1+z)^2, \tag{3.10}$$

we have, from the last line of Table 2, that $p_5(z)$ is strictly 2-better than $\overset{*}{q}(z)$, i.e.,

$$\delta_2(\overset{*}{q}) = 0.968\ 750 = \delta_2(p_5), \text{ and } K(\overset{*}{q}) = 0.218\ 750 > K(p_5) = 0.187\ 500. \tag{3.11}$$

Thus, this also gives a counterexample to the Conjecture of (1.12).

4. Counterexamples via Lacunary Polynomials.

In this section, we obtain an *infinity* of counterexamples to the Conjecture of (1.12), by direct calculations with lacunary polynomials. To this end, consider the monic polynomial

$$r_{m+2}(z) := (1+z)^m \left\{A_m + B_m z + z^2\right\} =: \sum_{j=0}^{m+2} c_{m+2,j} z^j \quad (m \geq 3), \tag{4.1}$$

where it follows that

$$c_{m+2,j} = A_m \cdot \binom{m}{j} + B_m \cdot \binom{m}{j-1} + \binom{m}{j-2} \quad (j = 0, 1, \cdots, m+2); \tag{4.2}$$

here, as usual, $\binom{m}{k} := 0$ for $k < 0$ or $k > m$. From (4.2), we have that

$$c_{m+2,m} = 0 = c_{m+2,m+1} \text{ if and only if } A_m = \binom{m+1}{2} \text{ and } B_m = -m. \tag{4.3}$$

Thus, fixing $A_m := \binom{m+1}{2}$ and $B_m := -m$, then $r_{m+2}(z)$ of (4.1) is a lacunary polynomial. Also, as the zeros of $A_m + B_m z + z^2$ are $\{m \pm i(m^2+2m)^{1/2}\}/2$, then $r_{m+2}(z)$ has no zeros in Δ, the zeros do not all lie on a ray, nor do they all lie in Re $z < 0$.

With the choices of (4.3), it is readily verified that

$$c_{m+2,j} > 0 \text{ for } j = 0, 1, \cdots, m-1; c_{m+2,m} = 0 = c_{m+2,m+1} \text{ and } c_{m+2,m+2} = 1. \tag{4.4}$$

Thus, the nonnegativity of these coefficients $\{c_{m+2,j}\}_{j=0}^{m+2}$ implies that

$$\sum_{j=0}^{m+2} |c_{m+2,j}| = \sum_{j=0}^{m+2} c_{m+2,j} = h_{m+2}(1) = 2^{m-1}(m^2 - m + 2), \tag{4.5}$$

the last equality making use of (4.3). Moreover, using the last three equations of (4.4), we see that

$$\delta_{m-1}(r_{m+2}) = \frac{2^{m-1}(m^2-m+2)-1}{2^{m-1}(m^2-m+2)}, \tag{4.6}$$

and, as r_{m+2} has no zeros in Δ, then

$$K(r_{m+2}) = \frac{|c_{m+2,0}|}{\sum_{j=0}^{m+2}|c_{m+2,j}|} = \frac{A_m}{2^{m-1}(m^2-m+2)} = \frac{m(m+1)}{2^m(m^2-m+2)}. \tag{4.7}$$

We now show, for *each* $m \geq 3$, that the values of (4.6) and (4.7) give a counterexample to the Conjecture of (1.12). Specifically, we seek an h^*-polynomial, say $q_m(z;\rho_m) = (\rho_m + z)(1+z)^{m-1}$, with $1 \leq \rho_m < \infty$, such that

$$\delta_{m-1}(r_{m+2}) = \delta_{m-1}(q_m). \tag{4.8}$$

It is easily verified that

$$\delta_{m-1}(q_m) = \frac{2^{m-1}(\rho_m+1)-1}{2^{m-1}(\rho_m+1)}, \text{ and } K(q_m) = \frac{\rho_m}{2^{m-1}(\rho_m+1)}, \tag{4.9}$$

and, because the expressions in (4.6) and (4.9) are similar, it follows that (4.8) is satisfied for

$$\rho_m := m^2-m+1. \tag{4.10}$$

With the value of ρ_m in (4.10), we then have

$$K(q_m) = \frac{m^2-m+1}{2^{m-1}(m^2-m+2)}, \tag{4.11}$$

and a short calculation shows that

$$\frac{K(q_m)}{K(r_{m+2})} = \frac{2(m^2-m+1)}{m(m+1)} > 1 \text{ if and only if } (m-2)(m-1) > 0. \tag{4.12}$$

As the final condition in (4.12) is satisfied for *each* $m \geq 3$, we have, from (4.8) and (4.12), that

$$r_{m+2}(z) \text{ is strictly } (m-1) - \text{better than } q_m(z),$$

which gives a counterexample to the Conjecture of (1.12) for *each* $m \geq 3$. Also, we note that the ratio of the K's in (4.12) is *monotone increasing* in m for $m \geq 3$, with

$$\lim_{m\to\infty} \frac{K(q_m)}{K(r_{m+2})} = 2. \tag{4.13}$$

5. Proofs of Theorem 1 and Corollary 2.

For the proofs of these results, we begin with

Lemma 1. *Let* $p_m(z)$ *be as in* Theorem B. *Then, card* $E(p_m) = \infty$ *if and only if* $m \in E(p_m)$.

Proof. If $m \in E(p_m)$, then from (2.14), $L_m(p_m) = L_m\left(\overset{\square}{p}_m\right)$, where, if ∂q denotes the exact degree of a complex polynomial $q(z)$, then Theorem B and the construction above give that $\partial \overset{\square}{p}_m \leq \partial p_m = m$. Hence (cf. (2.5) and (2.6)),

$$L_j(p_m) = K(p_m) = L_j\left(\overset{\square}{p}_m\right) = K\left(\overset{\square}{p}_m\right) \text{ for all } j \geq m,$$

so that all integers $j \geq m$ are elements of $E(p_m)$, and consequently, card $E(p_m) = \infty$. If $m \notin E(p_m)$, it follows from (2.13) and (2.14) that

$$L_j(p_m) = K(p_m) > L_j\left(\overset{\square}{p}_m\right) = K\left(\overset{\square}{p}_m\right) \text{ for all integers } j \geq m,$$

i.e., $j \notin E(p_m)$ for all $j \geq m$. Hence, $E(p_m)$ can have, by definition, only a finite number of elements, and card $E(p_m) \neq \infty$. ■

Lemma 2. *Let* $p_m(z)$ *be as in* Theorem B *and assume that card* $E(p_m) = \infty$. *Then, for each* $j \in N_0$, *there is a unique* h^**-polynomial* $q_j(z) = (\rho_j + z)(1+z)^{m_j - 1}$ *which is* j*-better than* $p_m(z)$. *More precisely, if* $1 \leq j < m$ *and* $j \notin E(p_m)$, *then* $q_j(z)$ *is strictly* j*-better than* $p_m(z)$.

Proof. ¿From Lemma 1, card $E(p_m) = \infty$ implies that $m \in E(p_m)$; whence, $K(p_m) = K(\overset{\square}{p}_m)$. Thus from (2.13) and (2.6),

$$L_j(p_m) = \frac{K(p_m)}{\delta_j(p_m)} \geq L_j(\overset{\square}{p}_m) = \frac{K(\overset{\square}{p}_m)}{\delta_j(\overset{\square}{p}_m)} \quad (j \in N_0), \text{ with } K(p_m) = K(\overset{\square}{p}_m), \tag{5.1}$$

from which it follows that

$$\delta_j(\overset{\square}{p}_m) \geq \delta_j(p_m) \quad (j \in N_0). \tag{5.2}$$

If $\delta_j(\overset{\square}{p}_m) = \delta_j(p_m)$, i.e., $j \in E(p_m)$, then the h^*-polynomial $\overset{\square}{p}_m(z)$ is, by definition, j-better than $p_m(z)$. If j, with $1 \leq j < m$, is such that $j \notin E(p_m)$, we similarly deduce that

$$\delta_j(\overset{\square}{p}_m) > \delta_j(p_m).$$

Then, by the process of lowering, there is a unique h^*-polynomial $g_j(z) = (\rho_j + z)(1+z)^{m_j - 1}$ such that $\delta_j(g_j) = \delta_j(p_m)$, but in the process of lowering $\overset{\square}{p}_m(z)$, the monotonicity in (2.11)(ii) gives that $K(g_j) < K(\overset{\square}{p}_m) = K(p_m)$, so that the h^*-polynomial $g_j(z)$ is strictly j-better than $p_m(z)$. ■

What is of interest in Lemma 2 is that if card $E(p_m) = \infty$, then for *each* $j \in N_0$, one can find an associated h^*-polynomial which is at least j-better than $p_m(z)$, which is a *global-type result*. As a consequence, we remark that the Conjecture of (1.12) is valid if and only if card $E(p_m) = \infty$ for every $p_m(z)$, of the form (2.7), for *arbitrary* $m \in N$. As we have seen from Sections 3 and 4, this Conjecture fails in general to be true.

Lemma 3. *Let $p_m(z)$ be as in* Theorem B, *and assume that card $E(p_m) \neq \infty$. Then for each $k \in E(p_m)$, there is a unique h^*-polynomial $r_k(z) = (\rho_k + z)(1+z)^{m_k-1}$ such that*

$$\delta_k(r_k) = \delta_k(p_m), \ \ \textit{but } K(r_k) > K(p_m), \tag{5.3}$$

i.e., $p_m(z)$ is strictly k-better than the h^*-polynomial $r_k(z)$.

Proof. Since, by hypothesis, card $E(p_m) \neq \infty$, then $m \neq E(p_m)$ from Lemma 1; whence,

$$K(p_m) > K(\overset{\square}{p}_m). \tag{5.4}$$

On the other hand, for any $k \in E(p_m)$, we have

$$L_k(p_m) = L_k(\overset{\square}{p}_m), \tag{5.5}$$

so that, with (5.4),

$$\delta_k(p_m) > \delta_k(\overset{\square}{p}_m). \tag{5.6}$$

Now, on lifting the h^*-polynomial $\overset{\square}{p}_m(z) = (\overset{\square}{\rho} + z)(1+z)^{\overset{\square}{m}-1}$ until equality holds in (5.6), thereby forming the h^*-polynomial $r_k(z) := (\rho_k + z)(1+z)^{m_k-1}$ (where $\partial r_k \leq \partial \overset{\square}{p}_m \leq \partial p_m = m$), we have (cf. (2.11))

$$\delta_k(r_k) = \delta_k(p_m), \ \text{ and } K(r_k) > K(\overset{\square}{p}_m). \tag{5.7}$$

Suppose, to the contrary of the desired second inequality of (5.3), that

$$K(r_k) \leq K(p_m).$$

On dividing the above expression by $\delta_k(r_k) = \delta_k(p_m)$ (cf. (5.7)), this yields

$$L_k(r_k) \leq L_k(p_m) = L_k(\overset{\square}{p}_k), \tag{5.8}$$

the last equality following from (5.5). But, as the lifting of $\overset{\square}{p}_k(z)$ to $r_k(z)$ strictly increases $L_k(\overset{\square}{p}_m)$ (cf. (2.11)(i)), then

$$L_k(\overset{\square}{p}_m) < L_k(r_k),$$

which contradicts (5.8). ■

It is clear that Lemmas 1-3 give the result of Theorem 1. Finally, to deduce Corollary 2, it can be verified that *each* h^*-polynomial $(\rho+z)(1+z)^{m-1}$, where $1 \le \rho < \infty$ and $n \in N$, gives rise to a unique set of constants (cf. (1.10)) $\left\{C^{\mathcal{H}}_{d_k,k}\right\}_{k=0}^{m}$. Thus, if there is a polynomial $p_m(z)$ for which card $E(p_m) \neq \infty$, then from Lemma 3, for each $k \in E(p_m), p_m(z)$ is strictly k-better than the h^*-polynomial $r_k(z)$,. and this implies (cf. (1.10)) that

$$C^{\mathcal{H}}_{d,k} = \log K(r_k) > \log K(p_m) \geq C_{d,k},$$

proving that the Conjecture of (1.12) is false.

6. Final Remarks.

What is apparent from the previous two sections is that not only do infinitely many counterexamples exist for the Conjecture of (1.12), but also that the improvements in (smaller) K-values, via non-h^*-polynomials are *relatively small.* For example, for the two counterexamples given in Section 3, we see from Table 2 that

$$\text{(6.1)} \qquad \frac{K(\tilde{q})}{K(p_5)} = \frac{13}{12} = 1.083\ 333\cdots, \text{ and } \frac{K(\overset{*}{q})}{K(p_5)} = \frac{7}{6} = 1.166\ 666\ldots,$$

while from (4.13), the associated ratios do not exceed 2. It would be interesting to determine if these ratios are *always* bounded at most by 2. It is also worth mentioning that each counterexample given here, to the Conjecture of (1.12), involved constructing a lacunary polynomial which was k-better than some h^*-polynomial of the form $(\rho+z)(1+z)^{m-1}$ with $\rho > 1$, i.e., *no* counterexamples were found for the special h^*-polynomials $(1+z)^m, m \in N$. We find this very intriguing!

Finally, lacunary polynomials have entered the construction of all our counterexamples, and it is interesting to speculate if extremal functions (if they exist) associated with the constants $C_{d,k}$, are also lacunary.

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Richard S. Varga
Igor E. Pritsker
Institute for Computational Mathematics
Department of Mathematics and Computer Science
Kent State University
Kent, Ohio 44242 (U.S.A.)
e-mail: varga@mcs.kent.edu
e-mail: pritsker@mcs.kent.edu